On Metric Dimension of Subdivided Honeycomb Network and Aztec Diamond Network

Xiujun Zhang\textsuperscript{1}, Muhammad Bilal\textsuperscript{2}, Atiq Rehman\textsuperscript{2}, Muhammad Hussain\textsuperscript{2}, and Zhiqiang Zhang\textsuperscript{3}

\textsuperscript{1}Chengdu University, Chengdu
\textsuperscript{2}COMSATS University Islamabad, Lahore Campus.
\textsuperscript{3}Cheng Du University

September 12, 2022

Abstract

This paper investigates the metric dimensions of the polygonal networks, particularly, of subdivided honeycomb network as well as Aztec diamond network. The polygon is any two-dimensional shape formed by straight lines. Triangles, quadrilaterals, pentagons and hexagons are all representations of polygons. For instance, hexagons help us in many models to construct honeycomb network (HCN (n)), where n is the number of hexagons from a central point to the borderline of the network. Subdivided honeycomb network (SHCN (n)) is obtained by adding additional vertices on each edge of HCN (n). An Aztec diamond network (AZ N (n)) of order n is a lattice comprises of unit squares with center (a, b) satisfying $|a| + |b| \leq n$. In this work, our main aim is to establish the results to show that the metric dimensions of SHCN (n) and AZ N (n) are 2 and 3 for $n = 1$ and $n \geq 2$, respectively. In the end, some open problems are listed with regard to metric dimensions for k -subdivisions of SHCN (n) and AZ N (n).
This paper investigates the metric dimensions of the polygonal networks, particularly, of subdivided honeycomb network as well as Aztec diamond network. The polygon is any two-dimensional shape formed by straight lines. Triangles, quadrilaterals, pentagons and hexagons are all representations of polygons. For instance, hexagons help us in many models to construct honeycomb network $\text{HCN}(n)$, where $n$ is the number of hexagons from a central point to the borderline of the network. Subdivided honeycomb network $\text{SHCN}(n)$ is obtained by adding additional vertices on each edge of $\text{HCN}(n)$. An Aztec diamond network $\text{AZN}(n)$ of order $n$ is a lattice comprises of unit squares with center $(a, b)$ satisfying $|a| + |b| \leq n$. In this work, our main aim is to establish the results to show that the metric dimensions of $\text{SHCN}(n)$ and $\text{AZN}(n)$ are 2 and 3 for $n = 1$ and $n \geq 2$, respectively. In the end, some open problems are listed with regard to metric dimensions for $k$-subdivisions of $\text{SHCN}(n)$ and $\text{AZN}(n)$.

**Keywords** — Polygonal network, Honeycomb network, Aztec diamond network, Resolving set, Metric dimension, Chemical graphs.

**Abbreviations:** HCN($n$), honeycomb network; SHCN($n$), subdivided honeycomb network; AZN($n$), Aztec diamond network.
1 | INTRODUCTION

Graph theory is used as an important mean for modeling real world problems including physico-chemical property testing [7]. Inspired from the problem to evaluate the position of an individual across the defined network precisely, Slater [13] presented the concept of metric dimension of a graph, where the metric generators are referred to as locating sets. Coming after Slater’s concept, Harary and Melter [6] extended the work on metric dimension by defining metric generators as resolving sets. It has many applications in different fields of life, for example image processing, network theory, pattern recognition, optimization and robot navigation etc. Throughout the graph, a traveling point can be identified after measuring length between the point and sound stations that have been precisely located in the graph.

Mathematical illustration of various chemical structures is of vital importance for the chemists to discover drug. A labeled graph is used to describe the composition of a chemical compound, where the vertex is labeled for atom while the edge specifies bond type [2, 3]. In a navigation network, a robot which needs to find its current position during navigation in space is modeled by a graph. It may send signals in order to measure its distance from each among a set of defined destinations. Here, the problem is to measure the minimum number of destinations with their locations, as the robots can decide their positions likely. The set of nodes representing destinations and number of destinations are known as the metric basis and the metric dimension of the graph, respectively.

The metric dimension is formally initiated after considering a connected graph \( G = (V, E) \), carrying set \( V \) of vertices/nodes and set \( E \) of edges. Let \( v_1, v_2 \in V \) be two distinct vertices, then the length of shortest \((v_1 v_2)\)-path denotes the distance for them that is symbolized by \( d(v_1, v_2) \). A set \( N_k(v) = \{u | d(u, v) = k \} \) is the \( k \)-neighborhood of vertex \( v \in V \), where \( k \) is a positive integer. If \( M = \{m_1, m_2, ..., m_k\} \) is the ordered subset of vertices and \( v \in G \), then \( r(v|M) = \{d(v, m_1), d(v, m_2), ..., d(v, m_k)\} \) is called the code of \( v \) in relation with \( M \). If there exist separate codes for two vertices in \( G \), then \( M \) is the resolving set [6] (or locating set [14]) for \( G \). The minimum cardinality of a resolving (or locating) set refers as metric dimension of \( G \) symbolized by \( dim(G) \). While a locating (or resolving) set with minimum number of vertices is called basis for \( G \) [1].

F. Simon Raj et al. studied metric dimensions of different chemical networks as well as star of David network \( SDN(n) \) in [12], [11]. If the vertices of a connected graph are changed, then the metric dimension of the graph will also be changed and becomes infinite when numbers of vertices are infinite, and it is called unbounded metric dimension. Similarly metric dimension remains finite when changing in number of vertices is finite and is called bounded metric dimension. Finally, if the metric dimension remains same for all number of vertices in a connected graph \( G \), then it is called a constant metric dimension [8]. The metric dimension of path graph is 1 in [3]; cycles have metric dimension 2 for every \( n \geq 3 \). Rooted product of two graphs \( F \) and \( J \) is stated as take \( u = |V(F)| \) copies of \( J \), and for each vertex \( u_j \) of \( F \), identify \( u_j \) with the root node of the \( j^{th} \) copy of \( J \). Godsil et al. [5] the rooted product of Harary graphs \( H_m(n, n) \), Jahangir graphs, antiprism \( A_n \) and generalized Petersen graphs \( P(n, 2) \) by path and cycle would be calculated as well metric dimensions of line graph of certain families of graphs would be determined. It is also of interest to determined the rooted product of graphs and then find out the metric dimension of rooted product of graphs by path and cycles.

Manuel et al. [10] determined the constant metric dimension of honeycomb networks. Moreover, in [15, 16, 17] determined the metric dimensions and edge metric dimension for honeycomb, Hex Derived and hexagonal networks. After gaining some idea of Manuel, metric dimension of subdivided of honeycomb network would be determined.
2 | HONEYCOMB NETWORK

In this section, firstly, the structural introduction of $HCN(n)$ and $SHCN(n)$ is given. Secondly, we have established some results and showed that the metric dimensions of $SHCN(n)$ for $n = 1$ and $n \geq 2$ are 2 and 3, respectively.

There is a range of designs in which polygons play role to construct a honeycomb network $HCN(n)$, where $n$ denotes the number of hexagons from center to the boundary of network. For given $HCN(1)$, we will have to add a layer of six hexagons to exterior boundary of $HCN(1)$ in order to construct $HCN(2)$. Consequently, after coating $HCN(n-1)$ with $6(n-1)$ hexagons, we get $HCN(n)$. While $SHCN(n)$ is obtained by adding additional vertices on each edge of $HCN(n)$. Honeycomb network is very useful in navigation, computer graphics, image processing and cell phone.

The diagrams in Figure 1 is the example of $SHCN(2)$ with 1 subdivision.

![Honeycomb and Subdivided honeycomb network SHCN(2) with 1 subdivision.](image1)

**FIGURE 1** Honeycomb and Subdivided honeycomb network $SHCN(2)$ with 1 subdivision.

**Theorem 1** If $G \cong SHCN(1)$, then the metric dimension of $G$ is 2 with 1 subdivision.

**Proof** $SHC(1)$ is a cycle with 12 vertices as it is not path so its metric dimension is not 1 [2] and as it is $C_{12}$ so have metric dimension 2.

**Theorem 2** If $G \cong SHCN(n)$, then $G$ has metric dimension greater than 2 for $n \geq 2$ with one subdivision.

**Proof** Here, we have to show that $G$ does not have any resolving set $M$ with two vertices. On contrary, suppose that $G$ has metric dimension equal to 2.

For $M = \{a_1, a_2\}$, it implies that $r(v_1 | M) = r(u_1 | M)$. Hence, $M$ is not resolving set for the graph.

For $M = \{v_1, v_{8n-3}\}$, it implies that $r(c_1 | M) = r(u_{i+2} | M)$. Hence, $M$ is not resolving set.

For $M = \{d_1, d_2\}$, it implies that $r(t_1 | M) = r(u_{i+2} | M)$. Hence, $M$ is not resolving set.

For $M = \{c_1, c_2\}$, it implies that $r(t_2 | M) = r(t_1 | M)$. Hence, $M$ is not resolving set.

For $M = \{u_1, u_{8n-3}\}$, it implies that $r(u_1 | M) = r(v_{i+2} | M)$. Hence, $M$ is not resolving set.

For $M = \{t_1, t_2\}$, it implies that $r(c_3 | M) = r(c_2 | M)$. Hence, $M$ is not resolving set.

For $M = \{v_1, u_1\}$, it implies that $r(d_1 | M) = r(u_4 | M)$. Hence, $M$ is not resolving set.

For $M = \{v_1, a_1\}$, it implies that $r(a_1 | M) = r(v_4 | M)$. Hence, $M$ is not resolving set.

For $M = \{u_1, a_1\}$, it implies that $r(d_1 | M) = r(u_4 | M)$. Hence, $M$ is not resolving set.

For $M = \{v_1, c_1\}$, it implies that $r(u_3 | M) = r(v_1 | M)$. Hence, $M$ is not resolving set.

For $M = \{u_1, c_1\}$, it implies that $r(u_{12} | M) = r(t_8 | M)$. Hence, $M$ is not resolving set.

Therefor, with two vertices, there is no resolving set $M$ for $SHCN(n)$, $n \geq 2$, so its metric dimension is greater than 2.
Theorem 3  If $G \cong SHCN(n)$, $n \geq 2$, then the metric dimension of $G$ is 3 with one subdivision.

Proof  The $SHCN(n)$ with one vertex between every two vertices has vertex set,

$$V(SHCN(n)) = \{v_i : 1 \leq i \leq 8n - 3\} \cup \{a_i : 1 \leq i \leq 2n\} \cup \{a'_i : 1 \leq i \leq 2n - j + 1, 1 \leq j \leq n - 1\} \cup \{d_i : 1 \leq i \leq 2n - j + 1, 1 \leq j \leq n - 1\} \cup \{d'_i : 1 \leq i \leq 8n - 4j - 3, 1 \leq j \leq n - 1\} \cup \{u_i : 1 \leq i \leq 8n - 3\} \cup \{t_i : 1 \leq i \leq 8n - 4j - 3, 1 \leq j \leq n - 1\}.$$

Now let $M = \{v_1, u_1, v_{8n-3}\}$ be the resolving set for the above graph.

$$r(v_i|M) = (i - 1, i + 1, 8n - i - 3), 1 \leq i \leq 8n - 3,$$

$$r(a_i|M) = (4i - 3, 4i - 3, 8n - 4i + 1), 1 \leq i \leq 2n,$$

$$r(a'_i|M) = (4(i + j) - 5, 4(i + j) - 3, 8n - 4i - 1), 1 \leq i \leq 2n - j + 1, 1 \leq j \leq n - 1,$$

$$r(d_i|M) = (4i + j - 3, 4i + j - 5, 8n - 4i + 1), 1 \leq i \leq 2n - j + 1, 1 \leq j \leq n - 1,$$

$$r(d'_i|M) = (4j + i - 1, 4j + i + 1, 8n - i - 3), 1 \leq i \leq 8n - 4j - 3, 1 \leq j \leq n - 1,$$

$$r(u_i|M) = (i + 1, i - 1, 8n - i - 1), 1 \leq i \leq 8n - 3,$$

$$r(t_i|M) = (4j + i + 1, 4j + i - 1, 8n - i - 1), 1 \leq i \leq 8n - 4j - 3, 1 \leq j \leq n - 1.$$

Let $v_l$ and $v_p$ are two distinct vertices from $V(SHCN(n))$, then $r(v_l|M) = r(v_p|M) \Rightarrow (l - 1, l + 1, 8n - l - 3) = (p - 1, p + 1, 8n - p - 3) \Rightarrow l = p$, $l = p$ which is contradiction.

Let $a'_l$ and $a'_p$ are two distinct vertices from $V(SHCN(n))$, then $r(a'_l|M) = r(a'_p|M) \Rightarrow (4(l + j) - 5, 4(l + j) - 3, 8n - 4l + 1) = (4(p + j) - 5, 4(p + j) - 3, 8n - 4p + 1) \Rightarrow l = p$, $l = p$ which is contradiction.

Let $d'_l$ and $d'_p$ are two distinct vertices from $V(SHCN(n))$, then $r(d'_l|M) = r(d'_p|M) \Rightarrow (4(l + j) - 3, 4(l + j) - 5, 8n - 4l + 1) = (4(p + j) - 3, 4(p + j) - 5, 8n - 4p + 1) \Rightarrow l = p$, $l = p$ which is contradiction.

Let $u_l$ and $u_p$ are two distinct vertices from $V(SHCN(n))$, then $r(u_l|M) = r(u_p|M) \Rightarrow (l + 1, l - 1, 8n - l - 1) = (p + 1, p - 1, 8n - p - 1) \Rightarrow l = p$, $l = p$ which is contradiction.

Similarly now we will consider different two vertices from opposite sides we will get again contradiction.

Let $v_l$ and $a_p$ are two distinct vertices from $V(SHCN(n))$, then $r(v_l|M) = r(a_p|M) \Rightarrow (l - 1, l + 1, 8n - l - 3) = (4p - 3, 4p - 3, 8n - 4p + 1) \Rightarrow l = 2(2p - 1), l = 4(p - 1), l = 4(p - 1)$ which is contradiction.

Let $v_l$ and $a'_p$ are two distinct vertices from $V(SHCN(n))$, then $r(v_l|M) = r(a'_p|M) \Rightarrow (l - 1, l + 1, 8n - l - 3) = (4(p + j) - 5, 4(p + j) - 3, 8n - 4p - 1) \Rightarrow l = 4(p + j - 1), l = 4(p + j - 1), l = 2(2p - 1)$ which is contradiction.

Let $v_l$ and $d'_p$ are two distinct vertices from $V(SHCN(n))$, then $r(v_l|M) = r(d'_p|M) \Rightarrow (l - 1, l + 1, 8n - l - 3) = (4j + l - 1, 4j + l + 1, 8n - p - 3) \Rightarrow l = 4j + p, l = 4j + p, l = p$ which is contradiction.

Let $a'_l$ and $d'_p$ are two distinct vertices from $V(SHCN(n))$, then $r(a'_l|M) = r(d'_p|M) \Rightarrow (4(j + l) - 5, 4(j + l) - 3, 8n - 4l - 1) = (4(p + j) - 3, 4(p + j) - 5, 8n - 4p + 1) \Rightarrow l = 4p + 2, l = 4p - 2, l = 4p - 2$ which is contradiction.
Let \( s_i^j \) and \( u_p \) are two distinct vertices from \( V(\text{SHCN}(n)) \), then \( r(s_i^j | M) = r(u_p | M) \Rightarrow (4(j + l) - 5, 4(j + l) - 3, 8n - 4l - 1) = (p + 1, p - 1, 8n - p - 1) \Rightarrow l = \frac{p - 4j + l + 6}{4}, \quad l = \frac{p - 4j + l + 2}{4}, \quad l = \frac{p}{4} \) which is contradiction.

Let \( s_i^j \) and \( t_p^j \) are two distinct vertices from \( V(\text{SHCN}(n)) \), then \( r(s_i^j | M) = r(t_p^j | M) \Rightarrow (4(j + l) - 5, 4(j + l) - 3, 8n - 4l - 1) = (4j + p + 1, 4j + p - 1, 8n - p - 1) \Rightarrow l = \frac{p + 2}{4}, \quad l = \frac{p + 6}{4}, \quad l = \frac{p + 4}{4} \) which is contradiction.

Let \( d_i^j \) and \( c_p \) are two distinct vertices from \( V(\text{SHCN}(n)) \), then \( r(d_i^j | M) = r(c_p | M) \Rightarrow (4(j + l) - 3, 4(j + l) - 5, 8n - 4l + 1) = (4j + p - 1, 4j + p + 1, 8n - p - 3) \Rightarrow l = \frac{p^2}{4}, \quad l = \frac{p^2 + 6}{4}, \quad l = \frac{p^2 + 4}{4} \) which is contradiction.

Let \( c_i^j \) and \( t_p^j \) are two distinct vertices from \( V(\text{SHCN}(n)) \), then \( r(c_i^j | M) = r(t_p^j | M) \Rightarrow (4j + l - 1, 4j + l + 1, 8n - l - 3) = (4j + p - 1, 4j + p + 1, 8n - p - 1) \Rightarrow l = p + 2, \quad l = p - 2, \quad l = p - 2 \) which is contradiction.

If we consider the following options of two vertices and continue the process in this way we will get contradiction, \( s_i^j \) and \( t_p^j \), \( c_i^j \) and \( d_p^j \), \( a_i^j \) and \( u_p \), \( v_i \) and \( u_p \), \( v_i \) and \( t_p^j \), \( c_i^j \) and \( v_i \) and \( v_p \), \( a_i^j \) and \( c_i^j \) and \( v_i \) and \( v_p \), \( a_i^j \).

So every vertex has distinct representation with respect to \( M \), so \( M \) is a resolving set for \( \text{SHCN}(n), n \geq 2 \).

The following Figure 2 represents the justification for the above theorem 2 and theorem 3.

**FIGURE 2** Structure of Honeycomb network \( \text{SHCN}(n) \)

### 3 | AZTEC DIAMOND NETWORK

In this section, firstly, the structural introduction of \( \text{AZN}(n) \) is given. Secondly, we have established some results and showed that the metric dimensions of \( \text{AZN}(n) \) for \( n = 1 \) and \( n \geq 2 \) are 2 and 3, respectively.

The area derived from staircase shapes of height \( n \) when glued together by the straight edges is known as aztec diamond network \( \text{AZN}(n) \) of order \( n \). So we can define it as a lattice comprises of unit squares centered at \((a, b)\) such
that $|a| + |b| \leq n$. $AZN(n)$ with order $n$ is composed of $2n(n+1)$ number of unit squares. An $AZN(n)$ with different proportions is portrayed and further studied in [4],[9]. The diagrams in following Figure 3 depict $AZN(1)$, $AZN(2)$ and $AZN(3)$, respectively.

**Figure 3** Structure of Aztec diamond networks $AZN(1)$, $AZN(2)$ and $AZN(3)$ (from left to right), respectively.

**Theorem 4** If $G \cong AZN(1)$, then the metric dimension of $G$ is 2.

**Proof** An $AZN(n)$ has vertex set

$$V(AZN(n)) = \{v_i : 1 \leq i \leq 3\} \cup \{a_i : 1 \leq i \leq 3\} \cup \{u_i : 1 \leq i \leq 3\}.$$

Now let $Q = \{v_1, u_1\}$ be the resolving set for the above graph.

$$r(v_i | Q) = (i - 1, i + 1), 1 \leq i \leq 3$$
$$r(a_i | Q) = (i, i), 1 \leq i \leq 3$$
$$r(u_i | Q) = (i + 1, i - 1), 1 \leq i \leq 3$$

Let $v_l$ and $v_p$ be two distinct vertices from $V(AZN(n))$, then $r(v_l | Q) = r(v_p | Q) \Rightarrow (l - 1, l + 1) = (p - 1, p + 1) \Rightarrow l = p, l = p$ which is contradiction.

Let $a_l$ and $a_p$ be two distinct vertices from $V(AZN(n))$, then $r(a_l | Q) = r(a_p | Q) \Rightarrow (l, l) = (p, p) \Rightarrow l = p, l = p$ which is contradiction.

Let $u_l$ and $u_p$ are two distinct vertices from $V(AZN(n))$, then $r(u_l | Q) = r(u_p | Q) \Rightarrow (l + 1, l - 1) = (p + 1, p - 1) \Rightarrow l = p, l = p$ which is contradiction.

Let $v_l$ and $a_p$ are two distinct vertices from $V(AZN(n))$, then $r(v_l | Q) = r(a_p | Q) \Rightarrow (l - 1, l + 1) = (p, p) \Rightarrow l = p + 1, l = p - 1$ which is contradiction.
Let \( v_l \) and \( u_p \) be two distinct vertices from \( V(AZ\ N(n)) \), then \( r(v_l|Q) = r(u_p|Q) \Rightarrow (l, l + 1) = (p + 1, p - 1) \Rightarrow l = p + 2, \quad l = p - 2 \) which is contradiction.

Let \( a_i \) and \( u_p \) are two distinct vertices from \( V(AZ\ N(n)) \), then \( r(a_i|Q) = r(u_p|Q) \Rightarrow (l, l) = (p + 1, p - 1) \Rightarrow l = p + 1, \quad l = p - 1 \) which is contradiction.

Hence, every vertex has distinct representation with respect to \( Q \), so \( Q \) is a resolving set for \( AZ\ N(n) \) for \( n = 1 \).

**Theorem 5** If \( G \cong AZ\ N(n) \), then \( G \) has metric dimension greater than 2 for \( n \geq 2 \).

**Proof** Suppose on contrary \( AZ\ N(n) \) has \( D \) as its resolving set with cardinality 2. Let \( D = \{v_1, v_{2n+1}\} \) be a resolving set. Here \( r(c^j|D) = r(a_{i+1}|D), \quad i = 1, 2, ..., 2n, \) which is contradiction.

Let \( D = \{u_1, u_{2n+1}\} \) be a resolving set. Here \( r(c^j|D) = r(a_{i+1}|D), \quad i = 1, 2, ..., 2n, \) which is contradiction.

Let \( D = \{a_i, a_{2n+1}\} \) be a resolving set. Here \( r(c^j|D) = r(a_{i+1}|D) \) which is contradiction.

Let \( D = \{a_i, a_j\} \) be a resolving set. Here \( r(v_1|D) = r(a_{j+1}|D) \) which is contradiction.

Thus, there is no resolving set with two basis elements. It implies that \( dim(G) > 2 \) for \( n \geq 2 \).

**Theorem 6** If \( G \cong AZ\ N(n) \), \( n \geq 2 \), then the metric dimension of \( G \) is 3.

**Proof** The \( AZ\ N(n) \) has vertex set

\[
V(AZ(n)) = \{v_i : 1 \leq i \leq 2n + 1\} \cup \{u_i : 1 \leq i \leq 2n + 1\} \cup \{a_i : 1 \leq i \leq 2n + 1\} \cup \{c^j_i : 1 \leq i \leq 2n - 2j + 1, 1 \leq j \leq n - 1\} \cup \{p^j_i : 1 \leq i \leq 2n - 2j + 1, 1 \leq j \leq n - 1\}.
\]

Now let \( D = \{v_1, u_1, v_{2n+1}\} \) be the resolving set for the above graph.

\[
r(v_1|D) = (i - 1, i + 2n - i + 1), 1 \leq i \leq 2n + 1
r(u_1|D) = (i + 1, i - 1, 2n - i + 3), 1 \leq i \leq 2n + 1
r(a_i|D) = (i, i, 2n - i + 2), 1 \leq i \leq 2n + 1
r(c^j_i|D) = (2j + i - 1, 2j + i + 1, 2n - i + 1), 1 \leq i \leq 2n - 2j + 1
r(p^j_i|D) = (2j + i + 1, 2j + i - 1, 2n - i + 3), 1 \leq i \leq 2n - 2j + 1
\]

Let \( v_x \) and \( v_y \) are two distinct vertices from \( V(AZ\ N(n)) \), then \( r(v_x|D) = r(v_y|D) \Rightarrow (x - 1, x + 1, 2n - x + 1) = (y - 1, y + 1, 2n - y + 1) \Rightarrow x = y, \quad x = y, \quad x = y \) which is contradiction.

Let \( c^j_x \) and \( c^j_y \) are two distinct vertices from \( V(AZ\ N(n)) \), then \( r(c^j_x|D) = r(c^j_y|D) \Rightarrow (2j + x - 1, 2j + x + 1, 2n - x + 1) = (2j + y - 1, 2j + y + 1, 2n - y + 1) \Rightarrow x = y, \quad x = y, \quad x = y \) which is contradiction.
Let $p'_i$ and $p'_j$ are two distinct vertices from $V(AZN(n))$, then $r(p'_i | D) = r(p'_j | D) = (2j + x + 1, 2j + x - 1, 2n - y + 3) = (2j + y + 1, 2j + y - 1, 2n - y + 3) \Rightarrow x = y, \ x = y, \ x = y$ which is contradiction.

Now we will consider those vertices which are from opposite sides, as follow,

Let $v_x$ and $u_y$ are two distinct vertices from $V(AZN(n))$, then $r(v_x | D) = r(u_y | D) = (x - 1, x + 1, 2n - x + 1) = (y + 1, y - 1, 2n - y + 3) \Rightarrow x = y + 2, \ x = y - 2, \ x = y - 2$ which is contradiction.

Let $v_x$ and $p'_y$ are two distinct vertices from $V(AZN(n))$, then $r(v_x | D) = r(p'_y | D) = (x - 1, x + 1, 2n - x + 1) = (2j + y + 1, 2j + y - 1, 2n - y + 3) \Rightarrow x = 2j + y + 2, \ x = 2j + y - 2, \ x = y - 2$ which is contradiction.

Let $u_x$ and $a_y$ are two distinct vertices from $V(AZN(n))$, then $r(u_x | D) = r(a_y | D) = (x = 1, x - 1, 2n - x + 3) = (y, y, 2n - y + 2) \Rightarrow x = y - 1, \ x = y + 1, \ x = y + 1$ which is contradiction.

Let $u_x$ and $c'_y$ are two distinct vertices from $V(AZN(n))$, then $r(u_x | D) = r(c'_y | D) = (x + 1, x - 1, 2n - x + 3) = (2j + y - 1, 2j + y + 1, 2n - y + 1) \Rightarrow x = 2j + y - 2, \ x = 2j + y + 2, \ x = y + 2$ which is contradiction.

Let $c'_y$ and $p'_x$ are two distinct vertices from $V(AZN(n))$, then $r(c'_y | D) = r(p'_x | D) = (2j + x - 1, 2j + x + 1, 2n - x + 1) = (2j + y + 1, 2j + y - 1, 2n - y + 3) \Rightarrow x = y + 2, \ x = y - 2, \ x = y - 2$ which is contradiction.

Under this process by choosing any other options of two vertices we will get contradiction as above.

Hence, every vertex has distinct representation with respect to $D$, so $D$ is a resolving set for $AZN(n); n \geq 2$.

The following Figure 4 represents the justification for the above theorem 5 and theorem 6.
Theorem 7 If $G \cong \text{SAZ}(n)$, $n \geq 2$, then the metric dimension of $G$ is 3.

Proof The $\text{SAZ}(n)$ has vertex set

$V(\text{SAZ}(n)) = \{v_i : 1 \leq i \leq 4n+1\} \cup \{u_i : 1 \leq i \leq 4n+1\} \cup \{c_i : 1 \leq i \leq 4n+1\} \cup \{d_i : 1 \leq i \leq 2n+1\} \cup \{f_i : 1 \leq i \leq 2n+1\} \cup \{a_j : 1 \leq i \leq 2n-2j+1\} \cup \{b_j : 1 \leq i \leq 4n-4j+1\} \cup \{g_j : 1 \leq i \leq 2n-2j+1\} \cup \{h_j : 1 \leq i \leq 4n-4j+1\} \cup \{l_j : 1 \leq i \leq n-1\}$

Now let $D = \{v_1, u_1, v_{2n+1}\}$ be the resolving set for the above graph. Then

$r(v_i) = (i-1, i+3, 4n-i+1), 1 \leq i \leq 4n+1$

$r(u_i) = (i+3, i-1, 4n-i+5), 1 \leq i \leq 4n+1$

$r(c_i) = (i+1, i+1, 4n-i+3), 1 \leq i \leq 4n+1$

$r(d_i) = (2i-1, 2i+1, 4n-2i+3), 1 \leq i \leq 2n+1$

$r(f_i) = (2i+1, 2i-1, 4n-2i+5), 1 \leq i \leq 2n+1$

$r(a_j) = (4j+i-1, 4j+i+3, 4n-i), 1 \leq i \leq 2n-2j+1$ and $1 \leq j \leq n-1$

$r(b_j) = (4j+2i-3, 4j+2i+1, 4n-2i+1), 1 \leq i \leq 4n-4j+1$ and $1 \leq j \leq n-1$

$r(g_j) = (4j+2i+1, 4j+2i-3, 4n-2i+5), 1 \leq i \leq 2n-2j+1$ and $1 \leq j \leq n-1$

$r(h_j) = (4j+i+3, 4j+i-1, 4n-i-5), 1 \leq i \leq 4n-4j+1$ and $1 \leq j \leq n-1$

Let $v_1$ and $v_2$ are two distinct vertices from $V(\text{SAZ}(n))$, then $r(v_1) = r(v_2) \Rightarrow (l-1, l+3, 4n-l+1) = (p-1, p+3, 4n-p+1)$ which is contradiction.

Let $c_1$ and $c_2$ are two distinct vertices from $V(\text{SAZ}(n))$, then $r(c_1) = r(c_2) \Rightarrow (l+1, l+1, 4n-l+3) = (p+1, p+1, 4n-p+3)$ which is contradiction.

Let $a_1$ and $a_2$ are two distinct vertices from $V(\text{SAZ}(n))$, then $r(a_1) = r(a_2) \Rightarrow (4j+l-1, 4j+l+3, 4n-l+1) = (4j+p-1, 4j+p+3, 4n-p+1)$ which is contradiction.

Now we will consider those vertices which are from opposite sides, as follows.

Let $v_1$ and $u_1$ are two distinct vertices from $V(\text{SAZ}(n))$, then $r(v_1) = r(u_1) \Rightarrow (l-1, l+3, 4n-l+1) = (p+3, p-1, 4n-p+5)$ which is contradiction.

Let $c_1$ and $a_2$ are two distinct vertices from $V(\text{SAZ}(n))$, then $r(c_1) = r(a_2) \Rightarrow (l+1, l+1, 4n-l+3) = (4j+p-1, 4j+p+3, 4n-p+1)$ which is contradiction.

Let $f_1$ and $h_2$ are two distinct vertices from $V(\text{SAZ}(n))$, then $r(f_1) = r(h_2) \Rightarrow (2l+1, 2l+1, 4n-2l+5) = (4j+p+3, 4j+p-1, 4n-p+5)$ which is contradiction.

Let $b_1$ and $g_2$ are two distinct vertices from $V(\text{SAZ}(n))$, then $r(b_1) = r(g_2) \Rightarrow (4j+2l-3, 4j+2l+1, 4n-2l+1) = (4j+2p+1, 4j+2p-3, 4n-2p+5)$ which is contradiction.

Let $a_1$ and $h_2$ are two distinct vertices from $V(\text{SAZ}(n))$, then $r(a_1) = r(h_2) \Rightarrow (4j+l-1, 4j+l+3, 4n-l+1) = (4j+p+3, 4j+p-1, 4n-p+5)$ which is contradiction.
Continue this process by choosing any other options of two vertices we will get contradiction as above. Hence, every vertex has distinct representation with respect to $D$, so $D$ is a resolving set for $SAZ N(n)$; $n \geq 2$.

The following Figure 5 represents the justification for the above theorem.

**FIGURE 5** Subdivide SAZ$n(n)$

## 4 | CONCLUSIONS

In this article, firstly, we investigated the of structures $HCN(n)$, $SHCN(n)$, $AZN(n)$ and $SAZ N(n)$. Then we established the results and showed that the metric dimensions of $SHCN(n)$, $AZN(n)$ and $SAZ N(n)$ are 2 for $n = 1$ and 3 for $n \geq 2$, respectively. We are raising the following problems for future perspective.

**Open Problem 1.** Determine the metric dimension of subdivided Honeycomb network $SHCN(n)$ for $k$ subdivision.

**Open Problem 2.** Determine the metric dimension of subdivided Aztec diamond network $SAZ N(n)$ for $k$ subdivision.
Acknowledgements

xx xx xx xx.

Conflict of interest

The authors have no conflict of interest.

References


