Improvement master-slave robustly synchronous criteria of uncertain chaotic Lur'e systems via an augmented Lyapunov-Krasovskii functional

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July 4, 2022

Abstract

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Keywords: Chaotic Lur’e systems, Lyapunov stability, Nonlinear systems, Synchronization, Time-delayed feedback

1 Introduction

Various synchronization problems are the main research content of chaotic systems, and the chaos synchronization is a hot topic in nonlinear system science. It has important practical application and scientific research value, including chaos generator design, security communication and information science. On the other hand, Lur’e system is a typical nonlinear system, which can represent many chaotic systems such as Chua’s circuit, N-rolling attractor and hyperchaotic attractor [1]. Therefore, the stability and synchronization problems of the chaotic Lur’e system have become an important topic in control science. The study of synchronization of chaotic Lur’e systems can make an important contribution to the development of nonlinear system control theory. To analyze the synchronization problem of chaotic systems, in short, is to design an appropriate controller to make the motion behavior of the slave system and the master system in the designed chaotic system gradually tend to be the same. Therefore, the research methods mainly refer to some synchronization control methods of nonlinear systems, such as, sampling control [2], dissipative control [3], intermittent control [4], and so on.

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It is well known that time delays occur extensively in many physical processes, such as communication systems, chemical processes and biological systems. The delay problem will decrease the control performance of the system and even lead to the instability of the system. Therefore, time delay is also an important problem in the study of chaotic Lur’e system theory. Nowadays, many methods are used to study chaos Lur’e systems with time delays. The most popular method is the so-called input delay method, where the system is converted to a system with time-varying input delay caused by a zero-order hold, such as, sampled-data control [5, 6], delayed state-feedback control [7–17], etc.. The synchronization criterion is mainly derived from Lyapunov stability theory, which is a sufficient condition and inevitably conservative. Various analysis methods for time-delayed systems are applied to the chaotic Lur’e system with input delays. For example, For the constant time delay, in [5], a novel two-side sampling-interval-dependent discontinuous Lyapunov–Krasovskii functional (DLKF) was constructed, which can fully utilizes the available characteristics of actual sampling information. A novel less conservative stability criterion of the synchronization error system was derived. However, it is noted that only constant delays were considered. In fact, in practical engineering, time delays are mostly time-varying and differentiable. So, for time-varying delay, more and more researchers pay high attention to the master–slave synchronization of chaotic Lur’e systems with time-varying delays. For instance, master-slave synchronization with time-varying delay feedback control [17, 18], event-triggered synchronization of networked chaotic Lur’e systems [19], sampled-data synchronization of Markovian jump chaotic Lur’e systems [1, 20, 21], synchronization of chaotic Lur’e systems with time-delays via quantized output feedback control [22], and so on.

It is worth noting that the main analysis methods for the synchronization of chaotic Lur’e systems with time-delays are Lyapunov stability theory, which are usually sufficient, but not necessary. Thus, it is still conservative to some extent. The key to reduce the conservatism is to construct the Lyapunov–Krasovskii functional (LKF) with as much the state variable or the time-delay information as possible and to update some tight inequality techniques. The analysis methods that make full use of linear time-delayed systems have become significant and spontaneous. For example, in the LKF construction, an LKF with delay decomposition [23–25], a proposed LKF consisting of a quadratic term and integral terms for the time-varying delays and the nonlinearities [26], an LKF related to a second-order Bessel–Legendre inequality [27], an LKF related to a delay-product-type function and two delay-dependent matrices [18, 28, 29], etc.; In the application of inequality techniques, Bessel–Legendre inequality [30, 31], reciprocally convex inequality [32], second order Bessel-Legendre inequality [33], Jensen-liked inequality [34–36], a matrix-separation-based inequality [37], and so on. Thus, according to the development of stability methods for linear time-delayed systems, there is still room to further reduce the conservatism of the synchronization criterion for chaotic Lur’e systems with time-delayed feedback control, which is the contribution of this paper.

In this paper, the robust synchronization of uncertain master-slave Lur’e systems based on time-varying delay feedback control is studied. An LKF consisting of some coupling information between some system variables and the time delay intervals is constructed, which increases the coupling information between some necessary variables contained in the inequality lemmas. However, some nonlinear terms with $h^2(t)$ are introduced when augmenting the LKF. To transform the nonlinear inequalities to the linear matrix inequalities (LMIs), a negative definite inequality equivalent transformation lemma proposed in [33] is used. Thus, new synchronization stability criterion and robust synchronization stability criterion are derived based on the augmented LKF and the negative definite inequality equivalent transformation lemma. The synchronization stability criteria are less conservative than some recent published references.

This paper is organized as follows. Section 2 gives the problem statement and provides some defi-
nitions, assumptions and lemmas. Section 3 presents the synchronization stability criterion and robust synchronization stability criterion, including theorems and corollaries. Section 4 shows numerical examples. Conclusions are drawn in Section 5.

**Notation:** $P$ represents a positive definite matrix if $P > 0$, vice versa. $I$ and 0 represent appropriate dimensional unit and zero matrices. The diagonal matrix is represented by diag{⋯}. $e_i (i = 1, \ldots, m)$ are block entry matrices, for example, $e_3^T = \begin{bmatrix} 0 & 0 & I & 0 \cdots 0_{m-3} \end{bmatrix}$, where $m$ is the length of the vector $\xi(t)$ in theorems and corollaries. $*$ denotes the symmetric terms in a block matrix. $F[h(t), d(t)], G[x(t)]$ denote $F, G$ are the function of $h(t), d(t)$ and $x(t)$, respectively. $\text{Sym}\{B\} = B + B^T$.

## 2 Problem formulation and preliminary

The following master-slave synchronous uncertain Lur’e system descriptions are given:

$$
\begin{align*}
M & : \begin{cases}
\dot{x}(t) = (A + \Delta A(t))x(t) + (A_1 + \Delta A_1(t))x(t - h(t)) + (B + \Delta B(t))f(Cx(t)), \\
p(t) = Hx(t),
\end{cases} \\
S & : \begin{cases}
\dot{y}(t) = (A + \Delta A(t))y(t) + (A_1 + \Delta A_1(t))y(t - h(t)) + (B + \Delta B(t))f(Cy(t)) + u(t), \\
q(t) = Hy(t),
\end{cases} \\
C & : u(t) = K(p(t - h(t)) - q(t - h(t))),
\end{align*}
$$

where $x(t), y(t), u(t) \in \mathbb{R}^n$ are the state vectors of subsystems and the control input of the slave system. $p(t), q(t) \in \mathbb{R}^l$ are the output vectors of subsystems. $A, A_1, B, C$ and $H$ represent the constant matrices with appropriate dimension. $K \in \mathbb{R}^{n \times l}$ is the gain matrix of the time-delayed feedback controller to be designed. $h(t)$ represents the time-varying delay with some constraints:

$$
0 \leq h(t) \leq h, \quad \mu_1 \leq \dot{h}(t) \leq \mu_2, \quad \forall t \geq 0.
$$

(2)

$\Delta A(t), \Delta A_1(t)$ and $\Delta B(t)$ are the parameter uncertainty matrices satisfying the following sector condition for constant matrices $D, E_a, E_{a1}$ and $E_b$ with appropriate dimensions and a continuous time-varying nonlinear function $F(t)$ satisfying $F^T(t)F(t) \leq I$.

$$
[\Delta A(t) \hspace{10pt} \Delta A_1(t) \hspace{10pt} \Delta B(t)] = DF(t) [E_a \hspace{10pt} E_{a1} \hspace{10pt} E_b].
$$

(3)

$f(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a diagonal nonlinearity vector in the feedback path of subsystems, and satisfies the following sector condition for two arbitrary real scalars $k_i^−$ and $k_i^+$

$$
k_i^− \leq \frac{f_i(w) - f_i(v)}{w - v} \leq k_i^+, \quad \forall w \neq v, \ i = 1, 2, \cdots, m.
$$

(4)

Let the synchronization error state $e(\cdot) = x(\cdot) - y(\cdot), g(Ce, y) = f(Ce + Cy) - f(Cy)$, then, the error system can be redescribed as the following system.

$$
\dot{e}(t) = (A + \Delta A(t))e(t) + (A_1 - KH + \Delta A_1(t))e(t - h(t)) + (B + \Delta B(t))g(Ce(t)),
$$

(5)

where $g(Ce)$ satisfies the following inequality constraints according to (4)

$$
k_i^- \leq \frac{g_i(c_i^T e)}{c_i^T e} = \frac{f_i(c_i^T (e + y)) - f_i(c_i^T y)}{c_i^T e} \leq k_i^+, \quad c_i^T e \neq 0, \ i = 1, 2, \cdots, m,
$$

(6)
where $C = [c_1 c_2 \cdots c_m]^T$ with $c_i \in \mathbb{R}^n, i = 1, 2, \ldots, m$.

The synchronization stability of the master-slave system (1) is a prerequisite for other performance studies. Therefore, the main purpose in this paper is to reduce the conservatism of the synchronization stability condition via Lyapunov stability theory. The following main lemmas are necessary for this purpose.

**Lemma 1** [30]. For a positive definite matrix $R$ and differentiable function $x$ in $[a, b] \rightarrow \mathbb{R}^n$, the followings hold

$$\int_a^b \dot{x}^T(s)R\dot{x}(s)ds \geq \frac{1}{b-a}\omega^T R\omega,$$

where $\mathcal{R} = \text{diag}\{R, 3R, 5R\}$, $\omega = \text{col}\{\omega_1, \omega_2, \omega_3\}$ with $\omega_1 = x(b) - x(a), \omega_2 = x(b) + x(a) - \frac{2}{b-a} \int_a^b x(s)ds, \omega_3 = \omega_1 - \frac{6}{b-a} \int_a^b x(s)ds + \frac{12}{(b-a)^2} \int_a^b (b-s)x(s)ds$.

**Lemma 2** [32]. For positive definite matrices $R_1, R_2 \in \mathbb{R}^{n\times n}$, vectors $v_1, v_2 \in \mathbb{R}^n$ and a scalar $\alpha \in [0, 1]$, if there exist symmetric matrices $X_1, X_2 \in \mathbb{R}^{n\times n}$ and any matrices $S_1, S_2 \in \mathbb{R}^{n\times n}$ such that

$$\begin{bmatrix} R_1 - X_1 & S_1 \\ * & R_1 \end{bmatrix} \geq 0, \begin{bmatrix} R_2 - X_2 & S_2 \\ * & R_2 \end{bmatrix} \geq 0,$$

the following inequality holds

$$\frac{1}{\alpha}v_1^T R_1 v_1 + \frac{1}{1-\alpha}v_2^T R_2 v_2 \geq v_1^T [R_1 + (1-\alpha)X_1] v_1 + v_2^T [R_2 + \alpha X_2] v_2 + 2v_1^T \alpha S_1 + (1-\alpha)S_2] v_2.$$

**Lemma 3** [33]. Let symmetric matrices $A_0, A_1, A_2 \in \mathbb{R}^{m\times m}$ and a vector $\zeta \in \mathbb{R}^m$. Then, the following inequality

$$\zeta^T (h_t^2 A_2 + h_t A_1 + A_0) \zeta < 0$$

holds for all $h_t \in [0, \bar{h}]$ if and only if there exist a positive definite matrix $D \in \mathbb{R}^{m\times m}$ and a skew-symmetric matrix $G \in \mathbb{R}^{k\times k}$ such that

$$\begin{bmatrix} A_0 & \frac{1}{2} A_1 \\ * & A_2 \end{bmatrix} < \begin{bmatrix} C & 0 \\ 0 & -D \end{bmatrix}^T \begin{bmatrix} -D & G \\ * & D \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix},$$

where $C = \begin{bmatrix} \frac{\bar{h}}{2}I & 0 \end{bmatrix}$ and $J = \begin{bmatrix} \frac{\bar{h}}{2}I & -I \end{bmatrix}$.

**Lemma 4** [38]. Given matrices $\Gamma, \Xi$ and $\Omega = \Omega^T$, the following inequality

$$\Omega + \Gamma F(\sigma) \Xi + \Xi^T F^T(\sigma) \Gamma^T < 0$$

holds for any $F(\sigma)$ satisfying $F^T(\sigma)F(\sigma) \leq I$, if and only if there exists a scalar $\varepsilon > 0$ such that

$$\Omega + \varepsilon^{-1} \Gamma \Gamma^T + \varepsilon \Xi \Xi^T \Xi < 0.$$
3 Main results

In order to make the calculation process more concise, some expressions are given in advance as below

\[ h_t = h(t), \bar{h}_t = h - h(t), h_d = 1 - \dot{h}(t), \]
\[ \eta_0(t) = \text{col} \{ e(t), e(t - h_t), e(t - h) \}, \]
\[ \eta_1(t) = \text{col} \left\{ e(t), e(t - h_t), v_1(t), \int_{t-h_t}^{t} e(s) ds \right\}, \]
\[ \eta_2(t) = \text{col} \left\{ e(t - h_t), e(t - h), v_2(t), \int_{t-h}^{t-h_t} e(s) ds \right\}, \]
\[ \eta_3(t, s) = \text{col} \left\{ \dot{e}(s), e(s), \eta_0(t), g^T(Ce(s)), \int_{s}^{t-h} e(\theta) d\theta, \int_{t-h}^{t-h_t} e(\theta) d\theta, \int_{t}^{t-h} e(\theta) d\theta \right\}, \]
\[ \eta_4(t, s) = \text{col} \left\{ \dot{e}(s), e(s), \eta_0(t), g^T(Ce(s)), \int_{t-h}^{t-h_t} e(\theta) d\theta, \int_{t}^{t-h} e(\theta) d\theta \right\}, \]
\[ v_1(t) = \int_{t-h}^{t-h_t} g^T(Ce(s)) ds, \quad v_2(t) = \int_{t-h}^{t-h_t} g^T(Ce(s)) ds, \]
\[ \rho_1(t) = \int_{t-h}^{t-h_t} \frac{e(s) ds}{h_t}, \quad \rho_2(t) = \int_{t-h}^{t-h_t} \frac{(t-h_t-s)e(s) ds}{h_t^2}, \]
\[ \rho_3(t) = \int_{t-h_t}^{t} \frac{e(s) ds}{h_t}, \quad \rho_4(t) = \int_{t-h}^{t-h_t} \frac{(t-s)e(s) ds}{h_t^2}, \]
\[ \xi(t) = \text{col} \left\{ e(t), e(t - h_t), e(t - h), \dot{e}(t), \dot{e}(t - h_t), \dot{e}(t - h), \right\}, \]
\[ \rho_1(t), \rho_2(t), \rho_3(t), \rho_4(t), v_1(t), v_2(t), \]
\[ g(Ce(t)), g(Ce(t - h)), g(Ce(t - h_t)) \}, \]
\[ K_1 = \text{diag} \left\{ k_1^+, k_2^+, \cdots, k_n^+ \right\}, \quad K_2 = \text{diag} \left\{ k_1^-, k_2^-, \cdots, k_n^- \right\}. \]

3.1 Synchronously Stable Criterion

Firstly, the following theorem will give a new synchronously stable criterion for the error system (5) without uncertainties described as:

\[ \dot{e}(t) = Ae(t) + (A_1 - KH)e(t - h(t)) + Bg(Ce(t)). \]

Theorem 1. Given positive scalars \( h, \mu_1, \mu_2 \) and \( \varepsilon_j \), the error system (7) is stable, if there exist positive definite matrices \( S_1 \in \mathbb{R}^{n \times 4n}, Q_i \in \mathbb{R}^{n \times n}, R_i \in \mathbb{R}^{n \times n}, Z \in \mathbb{R}^{m \times m}, D_i \in \mathbb{R}^{(10n+5m) \times (10n+5m)} \), positive definite diagonal matrices \( H_p = \text{diag} \{ h_{p1}, \cdots, h_{pm} \} \}, \) \( Y_{ir}, \Theta_{ir} \in \mathbb{R}^{m \times m} \), symmetric matrices \( S_{i1} \in \mathbb{R}^{4n \times 4n}, X_i \in \mathbb{R}^{3n \times 3n}, M_i \in \mathbb{R}^{m \times m}, \) skew-symmetric matrices \( G_i \in \mathbb{R}^{(10n+5m) \times (10n+5m)} \), any matrices \( Y_{i} \in \mathbb{R}^{3n \times 3n}, N_{i} \in \mathbb{R}^{m \times m}, U \in \mathbb{R}^{(3n+m) \times n} \), \( i, r = 1, 2; j = 1, \cdots, 4; p = 1, \cdots, 6 \), such that the following matrix inequalities hold for \( \bar{h}_t \equiv \mu_i \in \{ \mu_1, \mu_2 \} \),

\[ hS_{i1} + S_2 > 0, \quad \Sigma = \begin{bmatrix} Z - M_i \ N_i \\ * \ Z \end{bmatrix} > 0, \quad \begin{bmatrix} R_i - X_i \ Y_i \\ * \ R_i \end{bmatrix} > 0, \quad \begin{bmatrix} C^T & J \end{bmatrix} \begin{bmatrix} -D_i & G_i \\ * & D_i \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix} < 0, \]

\[ hS_{i1} + S_2 > 0, \quad \Sigma = \begin{bmatrix} \Omega_0(\mu) & \frac{1}{2} \Omega_1(\mu) \\ * & \Omega_2(\mu) \end{bmatrix} - \begin{bmatrix} C^T & J \end{bmatrix} \begin{bmatrix} -D_i & G_i \\ * & D_i \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix} < 0, \]

\[ \Sigma = \begin{bmatrix} \Omega_0(\mu) & \frac{1}{2} \Omega_1(\mu) \\ * & \Omega_2(\mu) \end{bmatrix} - \begin{bmatrix} C^T & J \end{bmatrix} \begin{bmatrix} -D_i & G_i \\ * & D_i \end{bmatrix} \begin{bmatrix} C \\ J \end{bmatrix} < 0, \]
where the delayed feedback gain matrix is given by $K = U^+ L$. The symbol $U^+$ represents the generalized inverse of a matrix $U$ and other symbols and matrices can be found below.

$$
\Delta_0 = \text{col}\{\varepsilon_{1e_1}, \varepsilon_{2e_2}, \varepsilon_{3e_3}, \varepsilon_{4e_4}, \varepsilon_{1e_{13}}\}, \ \Pi_0 = A e_1^T + (A_1 - KH)e_2^T + Be_{13}^T - e_4^T,$$

$$
\Delta_{11} = \text{col}\{e_1, e_2, e_{11}, e_0\}, \ \Delta_{12} = \text{col}\{e_0, e_0, e_0, e_9\}, \ \Delta_{13} = \text{col}\{e_4, h_4 e_5, e_{13} - h_4 e_{14}, e_1 - h_4 e_2\},
$$

$$
\Delta_{21} = \text{col}\{e_2, e_3, e_{12}, h e_7\}, \ \Delta_{22} = \text{col}\{e_0, e_0, e_0, -e_7\}, \ \Delta_{23} = \text{col}\{h_4 e_5, e_6, h_4 e_{14} - e_{15}, h_4 e_2 - e_3\},
$$

$$
\Delta_{31} = \text{col}\{e_4, e_1, e_2, e_3, e_{13}, e_0, e_0, h e_7\}, \ \Delta_{32} = \text{col}\{e_0, e_0, e_0, e_0, e_0, e_0, e_9, -e_7\},
$$

$$
\Delta_{33} = \text{col}\{e_5, e_2, e_4, e_3, e_{14}, e_0, e_0, h e_7\}, \ \Delta_{34} = \text{col}\{e_0, e_0, e_0, e_0, e_0, e_9, -e_7\},
$$

$$
\Delta_{41} = \text{col}\{e_6, e_3, e_1, e_2, e_{15}, h e_7, e_0\}, \ \Delta_{42} = \text{col}\{e_0, e_0, e_0, e_0, e_0, -e_7, e_9, e_0\},
$$

$$
\Delta_{43} = \text{col}\{e_5, e_2, e_1, e_2, e_{13}, e_0, e_0, h e_7\}, \ \Delta_{44} = \text{col}\{e_0, e_0, e_0, e_0, e_0, e_9, -e_7\},
$$

$$
\Lambda_{10} = \text{col}\{e_1 - e_2, e_0, e_0, e_0, e_{11}, e_0, e_0, e_0\},
$$

$$
\Lambda_{11} = \text{col}\{e_0, e_9, e_1, e_2, e_3, e_0, e_0, h e_7\},
$$

$$
\Lambda_{12} = \text{col}\{e_0, e_0, e_0, e_0, e_0, e_9 - e_{10}, e_{10}, -e_7\},
$$

$$
\Lambda_{20} = \text{col}\{e_2 - e_3, h e_7, h e_1, h e_2, h e_3, e_{12}, h^2(e_7 - e_8), e_0, h^2 e_8\},
$$

$$
\Lambda_{21} = \text{col}\{e_0, -e_7, -e_1, -e_2, -e_3, e_0, -2h(e_7 - e_8), h e_9, -2h e_8\},
$$

$$
\Lambda_{22} = \text{col}\{e_0, e_0, e_0, e_0, e_0, e_8 - e_7, -e_9, e_8\},
$$

$$
\Lambda_1 = \text{col}\{e_0, e_0, e_4, h_4 e_5, e_6, e_0, e_1, -h_4 e_2, h_4 e_2 - e_3\},
$$

$$
\Lambda_2 = \text{col}\{e_0, e_0, e_4, h_4 e_5, e_6, e_0, h_4 e_2, e_1 - h_4 e_2, -e_3\},
$$

$$
\Gamma_1 = \text{col}\{e_2 - e_3, e_2 + e_3 - 2e_7, e_2 - e_3 + 6e_7 + 12e_8\},
$$

$$
\Gamma_2 = \text{col}\{e_1 - e_2, e_1 + e_2 - 2e_9, e_1 - e_2 + 6e_9 + 12e_10\},
$$

$$
\mathcal{C} = \begin{bmatrix} \frac{h}{2} & 0 \\ 0 & \frac{h}{2} \end{bmatrix}, \ J = \begin{bmatrix} \frac{h}{2} & I \\ -I & \frac{h}{2} \end{bmatrix}, \ R_i = \text{diag}\{R_i, 3R_i, 5R_i\},
$$

$$
\Omega_0(\mu_i) = \text{Sym}\left\{ \begin{align*}
\Lambda_{13}^T S_{12} \Delta_{11} + \Delta_{23}^T (h S_{21} + S_{22}) \Delta_{21} + & \Lambda_{10}^T Q_{1} A_1 + \Lambda_{20}^T Q_{2} A_2 \\
[& e_{13}^T - e_j^T C^T K_2^T] H_1 C e_4 + [e_1^T C^T K_1^T - e_{13}^T] H_2 C e_4 \\
+ & h_4 [e_{14}^T - e_2^T C^T K_2^T] H_3 C e_5 + h_4 [e_2^T C^T K_1^T - e_{14}^T] H_4 C e_5 \\
+ & [e_{15}^T - e_3^T C^T K_2^T] H_5 C e_6 + [e_3^T C^T K_1^T - e_{15}^T] H_6 C e_6 \\
+ & \hat{h}_1 \Delta_{11}^T S_{11} \Delta_{11} - \hat{h}_1 \Delta_{21}^T S_{21} \Delta_{11} + & \Delta_{10}^T Q_{1} \Delta_{31} - \Delta_{41}^T Q_{2} \Delta_{11} - h_4 \Delta_{52}^T Q_{1} \Delta_{33} + h_4 \Delta_{53}^T Q_{2} \Delta_{43} \\
+ & e_j^T (h^2 R_2) e_4 + h^2 h_4 e_j^T (R_1 - R_2) e_5 + e_{13}^T (h^2 Z) e_13 + & \Gamma_1^T \hat{R}_1 \Gamma_1 + \Gamma_2^T (\hat{R}_2 + X_2) \Gamma_2 \\
+ & \text{Sym}\{\Gamma_1^T Y_1 \Gamma_2\} + e_{11}^T Z e_{11} + e_{12}^T (Z + N_2) e_{12} + & \text{Sym}\{e_{11}^T N_1 e_{12}\} \\
+ & \text{Sym}\{\Delta_0^T U \Pi_0 - \Delta_0^T L H e_2^T\} + & \text{Sym}\left\{ \sum_{j=1}^{3} \left[ e_{10+j}^T - e_j^T C^T K_2^T \right] \Psi_{2j} [K_1 C e_j - e_{10+j}] \right\} \\
+ & \text{Sym}\left\{ \sum_{j=1}^{2} \left[ e_{10+j}^T - e_{11+j}^T\right] (e_j^T - e_{1+j}^T) C^T K_2^T \right\} \Theta_{2j} [K_1 C (e_j - e_{1+j}) - (e_{10+j} - e_{11+j})] \right\},
\end{align*}\right.$$
\[ \Omega_1(\mu_i) = \text{Sym} \left\{ \Delta_{13}^T S_{11} \Delta_{11} + \Delta_{13}^T S_{12} \Delta_{12} + \dot{h}_t \Delta_{11}^T S_{11} \Delta_{12} + \Delta_{23}^T (h S_{21} + S_{22}) \Delta_{22} - \Delta_{23}^T S_{21} \Delta_{21} - h_t \Delta_{23}^T S_{21} \Delta_{21} + \Lambda_{11}^T Q_1 \Lambda_1 + \Lambda_{21}^T Q_2 \Lambda_2 \right\} + \Delta_{32}^T Q_1 \Delta_{32} - \Delta_{42}^T Q_2 \Delta_{42} - h_t \Delta_{44}^T Q_1 \Delta_{34} + h_d \Delta_{44}^T Q_2 \Delta_{44} - h h_d e^T_t (R_1 - R_2) e_5 \\
+ \frac{1}{h} [I^T_t X_1 I_1 - I^T_t X_2 I_2] + \frac{1}{h} \text{Sym} \{ I^T_t (Y_1 - Y_2) I_2 \} \\
- \frac{1}{h} [e^T_{11} M_1 e_{11} - e^T_{12} M_2 e_{12}] + \frac{1}{h} \text{Sym} \{ e^T_{11} (N_1 - N_2) e_{12} \} \\
+ \text{Sym} \left\{ \frac{1}{h} \sum_{j=1}^{3} \left[ (e^T_{10+j} - e^T_{j} C^T K_2^T) (\Psi_{1j} - \Psi_{2j}) \right] [K_1 C e_j - e_{10+j}] \\
+ \frac{1}{h} \sum_{j=1}^{2} \left[ (e^T_{10+j} - e^T_{j} C^T K_2^T) (\Theta_{1j} - \Theta_{2j}) \right] [K_1 C (e_j - e_{1+j}) - (e_{10+j} - e_{11+j})] \right\}, \]

\[ \Omega_2(\mu_i) = \text{Sym} \left\{ \Delta_{13}^T S_{11} \Delta_{12} - \Delta_{23}^T S_{21} \Delta_{22} + \Lambda_{12}^T Q_1 \Lambda_1 + \Lambda_{22}^T Q_2 \Lambda_2 \right\} + h_t \Delta_{12}^T S_{11} \Delta_{12} - h_t \Delta_{22}^T S_{21} \Delta_{22} + \Delta_{32}^T Q_1 \Delta_{32} - \Delta_{42}^T Q_2 \Delta_{42} - h_d \Delta_{34}^T Q_1 \Delta_{34} + h_d \Delta_{44}^T Q_2 \Delta_{44}. \]

**Proof:** Construct an improved LKF described as follows:

\[ V(t) = \sum_{i=1}^{5} V_i(t) \tag{11} \]

with

\[ V_1(t) = \eta_1^T(t) S_1(t) \eta_1(t) + \eta_2^T(t) S_2(t) \eta_2(t), \]

\[ V_2(t) = \int_{t-h_t}^{t} \eta_3^T(t, s) Q_1 \eta_3(s, t) ds + \int_{t-h_t}^{t-h} \eta_4^T(t, s) Q_2 \eta_4(s, t) ds, \]

\[ V_3(t) = h \int_{t-h}^{t} (h - t + s) \dot{\epsilon}^T(s) R_1 \dot{\epsilon}(s) ds + h \int_{t-h}^{t} (h - t + s) \dot{\epsilon}^T(s) R_2 \dot{\epsilon}(s) ds, \]

\[ V_4(t) = 2 \sum_{i=1}^{m} \int_{0}^{c_i^T(t)} \left[ h_{1i}(g_i(s) - k_i^- s + h_{2i}(k_i^+ s - g_i(s))) \right] ds \]

\[ + 2 \sum_{i=1}^{m} \int_{0}^{c_i^T(t-h(t))} \left[ h_{3i}(g_i(s) - k_i^- s + h_{4i}(k_i^+ s - g_i(s))) \right] ds \]

\[ + 2 \sum_{i=1}^{m} \int_{0}^{c_i^T(t-h)} \left[ h_{5i}(g_i(s) - k_i^- s + h_{6i}(k_i^+ s - g_i(s))) \right] ds, \]

\[ V_5(t) = h \int_{-h}^{0} \int_{t+\theta}^{t} g^T(Ce(s)) Z g(Ce(s)) ds d\theta. \]

where \( S_1(t) = h_t S_{11} + S_{12} \) and \( S_2(t) = \tilde{h}_t S_{21} + S_{22} \).

Calculating the derivative of \( V(t) \), we can obtain the following formulas

\[ \dot{V}_1(t) = 2 \eta_1^T(t) S_1(t) \dot{\eta}_1(t) + \eta_1^T(t) \dot{S}_1(t) \eta_1(t) \]

\[ + 2 \eta_2^T(t) S_2(t) \dot{\eta}_2(t) + \eta_2^T(t) \dot{S}_2(t) \eta_2(t) \]

\[ = 2 \eta^T(t) \Delta_{13} (h_t S_{11} + S_{12}) (\Delta_{11} + h_t \Delta_{12}) \xi(t) \]

\[ + \xi^T(t) (\Delta_{11} + h_t \Delta_{12})^T h_t S_{11} (\Delta_{11} + h_t \Delta_{12}) \xi(t) \]

\[ + 2 \xi^T(t) \Delta_{23} \tilde{h}_t S_{21} + S_{22}) (\Delta_{21} + h_t \Delta_{22}) \xi(t) \]

\[ - \xi^T(t) (\Delta_{21} + h_t \Delta_{22})^T h_t S_{21} (\Delta_{21} + h_t \Delta_{22}) \xi(t), \tag{12} \]
\[
\begin{align*}
\dot{V}_2(t) &= \eta_3^T(t, t) Q_1 \eta_3(t, t) - \eta_2^T(t, t-h) Q_2 \eta_4(t, t-h) \\
&- h_1 d^T(t, t-h) Q_1 \eta_3(t, t-h) \\
&+ h_1 d^T_t(t, t-h) Q_2 \eta_4(t, t-h) \\
&+ 2 \int_{t-h}^{t} \eta_3^T(t, s) \frac{\partial}{\partial t} \eta_3(s, t) ds \\
&+ 2 \int_{t-h}^{t-h} \eta_1^T(t, s) Q_2 \frac{\partial}{\partial t} \eta_4(s, t) ds \\
&= \xi^T(t) \left[ (\Delta_{31} + h_t \Delta_{32}) Q_1 (\Delta_{31} + h_t \Delta_{32}) \\
&- (\Delta_{41} + h_t \Delta_{42}) Q_2 (\Delta_{41} + h_t \Delta_{42}) \right] \xi(t) \\
&- h_1 d^T(t) \left[ (\Delta_{33} + h_t \Delta_{34}) Q_1 (\Delta_{33} + h_t \Delta_{34}) \\
&- (\Delta_{43} + h_t \Delta_{44}) Q_2 (\Delta_{43} + h_t \Delta_{44}) \right] \xi(t) \\
&+ 2 \xi^T(t) \left[ (\Delta_{10} + h_l \Delta_{11} + h_{l}^2 \Delta_{12}) Q_1 \Lambda_1 \\
&+ (\Delta_{20} + h_l \Lambda_{21} + h_{l}^2 \Lambda_{22}) Q_2 \Lambda_2 \right] \xi(t), \\
\dot{V}_3(t) &= h^2 e^T(t) R_2 e(t) + h h_d \ddot{e}(t) - h_1 (R_1 - R_2) \dot{e}(t-h_t) \\
&- h \left( \int_{t-h}^{t} \ddot{e}(s) R_1 \dot{e}(s) ds + \int_{t-h}^{t} \ddot{e}(s) R_2 \dot{e}(s) ds \right), \\
\dot{V}_4(t) &= 2 g^T(Ce(t))(H_1 - H_2) C \dot{e}(t) + 2 e^T(t) C^T(K_1 H_2 - K_2 H_1) C \dot{e}(t) \\
&+ 2 h_d g^T(Ce(t-h_t))(H_3 - H_4) C \dot{e}(t-h_t) \\
&+ 2 h_d e^T(t-h(t)) C^T(K_1 H_4 - K_2 H_3) C \dot{e}(t-h(t)) \\
&+ 2 g^T(Ce(t-h_t))(H_5 - H_6) C \dot{e}(t-h_t) \\
&+ 2 e^T(t-h(t)) C^T(K_1 H_6 - K_2 H_3) C \dot{e}(t-h(t)) \\
&= \xi^T(t) \text{Sym}\left\{ [e_{13} - e_1 C^T K_2^T] H_1 C e_4 + [e_1 C^T K_1^T - e_{13}] H_2 C e_4 \right. \\
&+ h_d [e_{14} - e_2 C^T K_2^T] H_3 C e_5 + h_d [e_2 C^T K_1^T - e_{14}] H_4 C e_5 \right. \\
&+ [e_{15} - e_3 C^T K_2^T] H_5 C e_6 + [e_3 C^T K_1^T - e_{15}] H_6 C e_6 \left. \right\} \xi(t), \\
\dot{V}_5(t) &= h^2 g^T(Ce(t)) Z g(Ce(t)) - h \int_{t-h}^{t} g^T(Ce(s)) Z g(Ce(s)) ds \\
&- h \int_{t-h}^{t-h} g^T(Ce(s)) Z g(Ce(s)) ds.
\end{align*}
\]
\[-h \left( \int_{t-h_t}^{t} g^T(Ce(s))Zg(Ce(s))ds + \int_{t-h}^{t-h_t} g^T(Ce(s))Zg(Ce(s))ds \right) \]
\[ \leq -\frac{1}{\beta^2} \xi^T(t) e_{11} z e^T_{11}(t) \xi(t) - \frac{1}{1-\beta^2} \xi^T(t) e_{12} z e^T_{12}(t) \xi(t) \]
\[ \leq -\xi^T(t) \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix}^T \begin{bmatrix} Z + \beta M_1 (1-\beta)N_1 + \beta N_2 \\ Z + (1-\beta)M_2 \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{12} \end{bmatrix} \xi(t). \quad (17) \]

For any matrix \( U \in \mathbb{R}^{n \times n} \), it is true that
\[ 0 = 2\xi^T(t) [\varepsilon e_1 + \varepsilon_2 e_2 + \varepsilon_3 e_4 + \varepsilon_4 e_{13}] U [A e^T_1 + (A_1 - KH)e^T_2 + B e^T_3 - e^T_4] \xi(t). \quad (18) \]

Letting \( L = UK \), the equation (18) can be rewritten as:
\[ 0 = 2\xi^T(t) [\varepsilon e_1 + \varepsilon_2 e_2 + \varepsilon_3 e_4 + \varepsilon_4 e_{13}] U [A e^T_1 + A_1 e^T_2 + B e^T_3 - e^T_4] \xi(t) \]
\[ - 2\xi^T(t) [\varepsilon e_1 + \varepsilon_2 e_2 + \varepsilon_3 e_4 + \varepsilon_4 e_{13}] L H e^T_2 \xi(t). \quad (19) \]

If the nonlinear constraint condition (6) is partitioned according to time-varying delay, we can obtain
\[ \lambda_k(s) \triangleq 2 [g(Ce(s)) - K_2 C e(s)]^T \left( \frac{h_{1k}}{\lambda} \Theta_{1k} + \frac{h_{2k}}{\lambda} \Theta_{2k} \right) [K_1 C e(s) - g(Ce(s))] \geq 0, \quad (20) \]
\[ \delta_i(s_1, s_2) \triangleq 2 [g(Ce(s_1)) - g(Ce(s_2))] - K_2 C (e(s_1) - e(s_2))]^T \left( \frac{h_{1i}}{\lambda} \Theta_{1j} + \frac{h_{2i}}{\lambda} \Theta_{2j} \right) \]
\[ \times [K_1 C (e(s_1) - e(s_2))] - g(Ce(s_1)) + g(Ce(s_2))] \geq 0, \quad (21) \]

where \( \Theta_{rk} \) and \( \Theta_{rj} \) \((j, r = 1, 2; k = 1, 2, 3)\) are positive definite diagonal matrices.

Thus, the following inequalities hold
\[ \lambda_1(t) + \lambda_2(t - h_1) + \lambda_3(t - h) \geq 0, \quad (22) \]
\[ \delta_1(t, t - h_1) + \delta_2(t - h_1, t - h) \geq 0. \quad (23) \]

Finally, from the above derivation (12)-(23), we have
\[ \dot{V}(t) \leq \xi^T(t) \left[ \Omega_0(\hat{h}_t) + h_t \Omega_1(\hat{h}_t) + h^2_2 \Omega_2(\hat{h}_t) \right] \xi(t). \quad (24) \]

According to Lemma 3, the matrix inequalities (10) together with \( \hat{h}_t \triangleq \mu_i \in \{ \mu_1, \mu_2 \} \) imply that \( \dot{V}(t) < 0 \). Therefore, by Lyapunov stability theorem, it can guarantee that the error system (7) is asymptotically stable. And that further proves that the master-slave Lur’e system (1) without uncertainties is synchronized stabilized by the time-delayed controller \( C \), which completes the proof.

**Remark 1.** Theorem 1 mainly reduces the conservatism of the synchronization stability criterion via an augmented LKF and an equivalent transformation lemma application. The major contributions to reducing the conservatism of the synchronization stability criteria for the error system (7) are summarized as follows.

- To avoid quadratic terms \( h^2(t) \), in [18], the components of the integrated vector of \( V_2(t) \) of the LKF use \( \int_{a}^{b} e(s)ds \) instead of the integral component. Indeed, \( \int_{a}^{b} e(s)ds \) can be rewritten as \( e(b) - e(a) \), with virtually no information about the single integral component \( \int_{a}^{b} e(s)ds \) involved. However, in this paper, \( V_2(t) \) of the LKF directly introduces the integral form \( \int_{t-h_1}^{t} e(s)ds, \int_{t-h}^{t-h_1} e(s)ds \), which includes the integral components necessary in the inequality lemmas 1 and 2. This not only increases some coupling information between some system variables, but also increases the coupling.
information between some necessary variables included in the inequality lemmas 1 and 2, which can make full use of the inequality lemmas 1 and 2 and reduce the conservatism of the synchronization stability criterion.

- In [15, 17, 18, 36], \( \int_{-h}^{0} I_{t+h} e^T(s) \dot{R}e(s)dsd\theta \) of the LKF directly constructs the double integral term in the whole delay interval \([0, h]\), which inevitably brings certain conservatism. \( V_3(t) \) in this paper, the double integral term is respectively constructed in two delay subinterval \([0, h_t] \) and \([h_t, h]\), where the energy relations in different delay subintervals are coupled by different Lyapunov matrices \( R_1 \) and \( R_2 \).

- However, due to the integral form \( \int_{-h}^{t} e(s)ds \) and \( \int_{-h}^{t-h} e(s)ds \), the final form of the upper bound on the derivative of the LKF includes \( h^2(t) \Omega_2(h_t) \), which cannot be solved by MATLAB. Some researchers, such as [35], decomposed this nonlinear matrix inequality into three LMI conditions via a sufficient condition constraint lemma application. However, the sufficient condition constraint lemma is quite conservative [33]. To overcome the nonlinear matrix inequality in the stability criterion, the novel negative definite inequality equivalent transformation lemma (Lemma 3) is used to transform the inequality (24) to the LMI (10) equivalently, which can be easily solved by the MATLAB LMI-Toolbox.

**Remark 2.** Usually, for time-varying delay constraints (2), the matrix inequalities in the stability conditions are \( h(t)\)- and \( \dot{h}(t)\)-dependent. According to the convexity of the matrix inequality, \( h(t) \) and \( \dot{h}(t) \) are set as the vertices of the interval of delay and delay derivative, and then the \( h(t)\)- and \( \dot{h}(t)\)-dependent matrix inequality is converted into a strict LMI, which can be easily solved via MATLAB LMI-tool box. That is a usual set \([h_t, h]\) set as \( H_1 = [0, h] \times [\mu_1, \mu_2] \) or a new allowable delay set \([h_t, h]\) is a valid set \([0, h] \times [\mu_1, \mu_2], [h_t, h]\) proposed by A. Seuret [31]. The authors of [31] pointed that improper combination of \( [0, h] \) and \([-\mu, \mu]\) will lead to varying degrees of conservatism. However, the stability conditions obtained in this paper are only \( \dot{h}(t)\)-dependent not \( h(t)\)-dependent, which avoids introducing the combinatorial relation between \([0, h]\) and \([\mu_1, \mu_2]\). This reduces the number of LMI constraints and reduces the conservatism caused by the different combination of \([0, h]\) and \([\mu_1, \mu_2]\). The conservatism caused by selecting different allowable delay set is avoided.

### 3.2 Robustly Synchronous Stability

Secondly, the following criterion extends the above synchronous stability criterion of the error system (5) without uncertainties to the form with uncertainties constrained by (3).

**Theorem 2.** Given positive scalars \( h, \mu, \varepsilon \) and \( \delta_1 \) the system (5) is robustly stable, if there exist positive definite matrices \( S_{i2} \in \mathbb{R}^{4n \times 4n}, Q_i \in \mathbb{R}^{9n \times 9n}, R_i \in \mathbb{R}^{n \times n}, Z \in \mathbb{R}^{m \times m}, D_i \in \mathbb{R}^{(10n+5m) \times (10n+5m)} \), positive definite diagonal matrices \( H_p = \text{diag}(h_1, \cdots, h_{pn}) \), \( \Upsilon_{ir}, \Theta_{rr} \in \mathbb{R}^{m \times m} \), symmetric matrices \( S_{i3} \in \mathbb{R}^{4n \times 4n}, X_i \in \mathbb{R}^{3n \times 3n}, M_i \in \mathbb{R}^{m \times m} \), skew-symmetric matrices \( G_i \in \mathbb{R}^{(10n+5m) \times (10n+5m)} \), any matrices \( Y_i \in \mathbb{R}^{3n \times 3n}, N_i \in \mathbb{R}^{m \times m}, \ U \in \mathbb{R}^{(3n+1)m \times n}, \) such that the LMIIs (8), (9) and the following LMI hold for \( h_t \triangleq \mu_i \in \{\mu_1, \mu_2\} \):

\[
\left[ \begin{array}{cc}
\Sigma_i + \delta \Phi_1^T \Phi_1 & \Phi_2^T \\
* & -\delta I
\end{array} \right] < 0,
\]

where

\[
\Phi_1 = \begin{bmatrix}
E_a e_1^T + E_a 1 e_2^T + E_b e_3^T & 0
\end{bmatrix}, \quad \Phi_2 = \begin{bmatrix}
\Delta_0 UD & 0
\end{bmatrix}^T.
\]
**Proof:** If we replace $A$, $A_1$ and $B$ in LMIs (10) with $A + DF(t)E_a$, $A_1 + DF(t)E_{a1}$, $B + DF(t)E_b$, respectively. The following inequalities can be obtained for $i = 1, 2$.

$$
\Sigma_i + \Phi_2^T F(t)\Phi_1 + \Phi_1^T F^T(t)\Phi_2 < 0. \tag{26}
$$

According to Theorem 1, it is obvious that the uncertain system (5) is robustly stable for any nonlinear function $g(\cdot)$ satisfying (6) and all admissible uncertainties (3), if LMIs in (26) hold. It follows from Lemma 4 that LMIs in (26) hold if and only if there exist positive scalars $\delta_i$ such that the following matrix inequalities hold:

$$
\Sigma_i + \delta_1^{-1}\Phi_2^T \Phi_2 + \delta_2 \Phi_1 \Phi_1^T < 0 \tag{27}
$$

which are equivalent to LMIs in (25), respectively, by Schur complement equivalence. This completes the proof.

### Table 1: MADUBs $h$ for $\mu_1 = \mu_2 = 0$ (Example 1)

<table>
<thead>
<tr>
<th>Methods</th>
<th>$h$</th>
<th>Control gain $K$</th>
<th>NoVs</th>
</tr>
</thead>
<tbody>
<tr>
<td>[9]</td>
<td>0.1411</td>
<td>[6.0229 1.3367 − 2.1264]$^T$</td>
<td>21</td>
</tr>
<tr>
<td>[10]</td>
<td>0.1622</td>
<td>[6.0229 1.3367 − 2.1264]$^T$</td>
<td>7</td>
</tr>
<tr>
<td>[12]</td>
<td>0.1830</td>
<td>[4.1455 0.9250 − 4.2596]$^T$</td>
<td>93</td>
</tr>
<tr>
<td>[14]</td>
<td>0.1850</td>
<td>[4.0779 0.9087 − 4.3430]$^T$</td>
<td>63</td>
</tr>
<tr>
<td>[16]</td>
<td>0.2722</td>
<td>[3.4852 0.6906 − 3.2849]$^T$</td>
<td>158</td>
</tr>
<tr>
<td>[17]</td>
<td>0.3237</td>
<td>[4.9158 0.6562 − 5.8369]$^T$</td>
<td>1755</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>0.3419</td>
<td>[6.3229 1.3205 − 3.6351]$^T$</td>
<td>4539</td>
</tr>
</tbody>
</table>

### 4 Numerical examples

In this section, we give two examples to show the effectiveness of the criteria proposed in this paper. The conservatism of the criteria is checked by comparing maximal admissible delay upper bounds (MADUBs) with some recent published literatures. 'NoVs' denotes the number of decision variables involved in stability criteria, and '−' is unsolvable. As we can see from the data in the example, the MADUBs obtained by the synchronization stability criteria proposed in this paper are larger than some recent published literatures, however, the NoVs are more than those published literatures. That is, the conservatism of synchronization stability criterion is reduced at the cost of increasing solution complexity.

**Example 1.** [7] The following Chua’s circuit system can be modeled as Lur’e systems.

$$
\begin{align*}
\dot{x} &= \alpha(y - \phi(x)) \\
\dot{y} &= x - y + z \\
\dot{z} &= -\beta y
\end{align*}
$$

with $\phi(x) = m_1 x + 0.5(m_0 - m_1)(|x + 1| - |x - 1|)$, $m_0 = -1/7$, $m_1 = 2/7$, $\alpha = 9$, $\beta = 14.28$ and $c = 1.$
A master-slave synchronization scheme for the Chua’s circuit can be written as (1) with system parameters described as follows

\[
A = \begin{bmatrix}
-\alpha m_1 & \alpha & 0 \\
1 & -1 & 1 \\
0 & -\beta & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
-\alpha (m_0 - m_1) \\
0 \\
0
\end{bmatrix},
\]

\[
C = H = [1 \ 0 \ 0], \quad A_1 = 0, \quad f(\theta) = 0.5(|\theta + 1| - |\theta - 1|) \in [0, 1].
\]

Table 2: MADUBs $h$ for different $\mu$ (Example 1)

<table>
<thead>
<tr>
<th>Methods $\backslash$ $\mu$</th>
<th>0.1622</th>
<th>0.1591</th>
<th>0.1566</th>
<th>0.1527</th>
<th>NoVs</th>
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<tbody>
<tr>
<td>[10]</td>
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<td></td>
<td></td>
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<tr>
<td>[34]</td>
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<td>0.1698</td>
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<td>[24]</td>
<td>0.1747</td>
<td>0.1710</td>
<td>0.1703</td>
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<td>[25]</td>
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<td>0.1721</td>
<td>0.1715</td>
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<td>[26]</td>
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<td>0.1894</td>
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<td>0.2540</td>
<td>0.2510</td>
<td>1359</td>
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<tr>
<td>[28]</td>
<td>0.2707</td>
<td>0.2700</td>
<td>0.2545</td>
<td>0.2544</td>
<td>1548</td>
</tr>
<tr>
<td>[36]</td>
<td>0.2790</td>
<td>0.2691</td>
<td>0.2553</td>
<td>0.2463</td>
<td>197</td>
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<td>[18]</td>
<td>0.2998</td>
<td>0.2857</td>
<td>0.2792</td>
<td>0.2618</td>
<td>1755</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>0.3238</td>
<td>0.2999</td>
<td>0.2902</td>
<td>0.2711</td>
<td>4539</td>
</tr>
</tbody>
</table>

- Firstly, consider constant $h$, that is $\mu_1 = \mu_2 = 0$. The MADUBs $h$ and the corresponding synchronization controller gain $K$ can be calculated by solving the LMIs (8)-(10) in Theorem 1 via Matlab LMI-TOOL box. The corresponding results and those compared with some literatures [7–15, 17, 18, 36] are listed in Table 1, which shows that the theorem 1 proposed in this paper is less conservative than those given in some recent literatures. However, the number of decision variables in LMIs (8)-(10) is more than those given in some recent literatures, which shows the conservatism of synchronization stability criterion is reduced at the cost of increasing solution complexity.

- Secondly, $\mu_1 = \mu_2 = \mu$, the system matrix $A$ is rewritten as $\tilde{A} = A + K_a$ with $K_a = -I$. Just fix the corresponding synchronization controller gain $K = \text{col}\{6.0229, 1.3367, -2.1264\}$ and calculate the corresponding MADUBs $h$, which can fully demonstrate that the stability margin obtained by the method in this paper is larger than some literatures under the same control weight. The corresponding results and those compared with some literatures [10, 18, 23–27, 34] are listed in Table 2.

- To confirm the obtained results from Tables 1-2, the simulation result is shown in Figs. 1-4. The Figs. 1-2 are the state responses of the systems (1) and the error state responses of the systems (5) with the random initial, $h(t) = 0.3419$ and $K = \text{col}\{5.0024, 0.5785, -3.3254\}$. The Figs. 3-4 are the state responses of the systems (1) and the error state responses of the systems (5) with the random initial, $h(t) = 0.3238$ and $K = \text{col}\{6.0229, 1.3367, -2.1264\}$. It is obvious that the master-slave synchronous Lur’e system (1) is synchronously stable.
Example 2. Consider uncertain system (1) with the following parameters:

\[ A = \begin{bmatrix} -7.2 & 1.6 & 0.8 \\ 1 & -5.8 & 0 \\ 3 & -11.25 & -4 \end{bmatrix}, \quad A_1 = 0_3, \quad B = \begin{bmatrix} 3.7 \\ 5.6 \\ 4.1 \end{bmatrix}, \quad C = H = [1 \ 0 \ 0], \]

\[ E_a = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad E_{a1} = 0_3, \quad E_b = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

\[ f(\eta) = |\sin| \eta|| \in [0, 1]. \]

For this example, our results of MADUBs \( h \) are listed in Table 3 based on Theorem 2, which are
compared with some existing literature [16–18]. The detail analysis and discussions are summarized as follows.

Table 3: MADUBs $h$ for $\mu_1 = 0$ (Example 2)

<table>
<thead>
<tr>
<th>Methods</th>
<th>$\mu_2$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>NoVs</th>
</tr>
</thead>
<tbody>
<tr>
<td>[16, Th. 3.1]</td>
<td>0.412</td>
<td>0.391</td>
<td>0.349</td>
<td>0.302</td>
<td>0.246</td>
<td>0.159</td>
<td>137</td>
<td></td>
</tr>
<tr>
<td>[17, Th. 2]</td>
<td>0.5621</td>
<td>0.5620</td>
<td>0.5617</td>
<td>0.5611</td>
<td>0.5607</td>
<td>0.5606</td>
<td>129</td>
<td></td>
</tr>
<tr>
<td>[18]</td>
<td>0.7884</td>
<td>0.7875</td>
<td>0.7875</td>
<td>0.7875</td>
<td>0.7875</td>
<td>0.7875</td>
<td>1755</td>
<td></td>
</tr>
<tr>
<td>Theorem 2</td>
<td>0.8324</td>
<td>0.8324</td>
<td>0.8318</td>
<td>0.8305</td>
<td>0.8305</td>
<td>0.8305</td>
<td>4539</td>
<td></td>
</tr>
</tbody>
</table>
The states $x_i(t)$ and $y_i(t)$

Figure 5: The state responses of the system (1) for Example 2.

Figure 6: The error state responses of the system (5) for Example 2.

Table 4: MADUBs $h$ for $\mu_1$ and $\mu_2$ (Example 2)

<table>
<thead>
<tr>
<th>$\mu_1 \setminus \mu_2$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.5</td>
<td>0.8538</td>
<td>0.8519</td>
<td>0.8501</td>
<td>0.8492</td>
<td>0.8476</td>
<td>0.8438</td>
</tr>
<tr>
<td>-0.1</td>
<td>0.8417</td>
<td>0.8409</td>
<td>0.8383</td>
<td>0.8351</td>
<td>0.8331</td>
<td>0.8319</td>
</tr>
<tr>
<td>0.0</td>
<td>0.8324</td>
<td>0.8324</td>
<td>0.8318</td>
<td>0.8305</td>
<td>0.8305</td>
<td>0.8305</td>
</tr>
<tr>
<td>0.1</td>
<td>–</td>
<td>0.8287</td>
<td>0.8276</td>
<td>0.8204</td>
<td>0.8198</td>
<td>0.8182</td>
</tr>
<tr>
<td>0.3</td>
<td>–</td>
<td>–</td>
<td>0.8115</td>
<td>0.8078</td>
<td>0.8004</td>
<td>0.8004</td>
</tr>
<tr>
<td>0.5</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>0.8043</td>
<td>0.8004</td>
<td>0.8001</td>
</tr>
</tbody>
</table>

- Firstly, consider $\mu_1 = 0$. The MADUBs $h$ can be calculated by solving the LMIs (8)-(9) and (25) in
Theorem 2 via Matlab LMI-TOOL box. The corresponding results and those compared with some literatures [16–18] are listed in Table 3, which shows that the theorem 2 proposed in this paper is less conservative than those given in some recent literatures. However, the number of decision variables in LMIs (8)-(9) and (25) is more than those given in some recent literatures, which shows the conservatism of synchronization stability criterion is reduced at the cost of increasing solution complexity. The corresponding synchronization controller gain $K = U^+L$. For example, $\mu_1 = 0$, $\mu_2 = 0.1$ and $h = 0.8324$, the corresponding synchronization controller gain $K = \text{col}\{3.1243, 3.1556, -4.5269\}$.

- Secondly, consider $\mu_1 \neq 0$, which is ignored in [16–18]. The corresponding MADUBs $h$ are listed in Table 4 for different $\mu_1$ and $\mu_2$. It is found that the MADUBs $h$ are decreasing as $\mu_1$ increases. For $\mu_1 = -0.5$, $\mu_2 = 0.5$ and $h = 0.8492$, the corresponding synchronization controller gain $K = \text{col}\{1.5314, 1.2057, -7.2891\}$.

- To confirm the obtained results from Tables 3-4, the simulation result is shown in Figs. 5-8. The Figs. 5-6 are the state responses of the systems (1) and the error state responses of the systems (5) with the random initial, $h(t) = 0.8324 + 0.8324\sin\left(\frac{0.2}{0.8324}\right)$ and $K = \text{col}\{3.1243, 3.1556, -4.5269\}$. The Figs. 7-8 are the state responses of the systems (1) and the error state responses of the systems (5) with the random initial, $h(t) = 0.8492 + 0.8492\sin\left(\frac{t}{0.8492}\right)$ and $K = \text{col}\{1.5314, 1.2057, -7.2891\}$. It is obvious that the master-slave synchronous Lur’e system (1) is synchronously stable.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{The state responses of the system (1) for Example 2.}
\end{figure}

5 Conclusion

In this paper, the synchronization of master-slave Lur’e systems with time-delayed feedback control is further studied. A new delay-dependent synchronization stability criterion is proposed via an augmented LKF application. Some single integral components are introduced into the LKF, and the double integral term is respectively constructed in two delay subinterval, which can make full use of the inequality...
lemmas and reduces the conservatism of the synchronization stability criterion. Moreover, to overcome the nonlinear matrix inequality in the stability criterion, the novel negative definite inequality equivalent transformation lemma (Lemma 3) is used to transform the nonlinear matrix inequality to the LMIs equivalently, which can be easily solved by the MATLAB LMI-Toolbox, which avoids introducing the conservatism caused by selecting different allowable delay set. Finally, the effectiveness of the proposed method is illustrated by comparison and discussion in numerical examples.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

**Acknowledgments**

This work is supported partially by the Yellow Sea Rookie of Yancheng Institute of Technology, the '333’ Talent Project of Jiangsu Province.

**References**


