# Analytic continuation of the spectral functions for the Sturm-Liouville problems with weight functions

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**Abstract**

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*Keywords*: Sturm-Liouvillve problems; Weight functions; Spectral zeta functions; Analytic continuation; WKB method.

## 1 Introduction

In this paper, we will consider the Sturm-Liouvillve eigenvalue problems with the general separated boundary conditions

$$-(py')' + qy = \lambda wy,\ y = y(x),\ x \in (a, b); \tag{1}$$

$$\cos\alpha y(a) - \sin\alpha (py')(a) = 0 = \cos\beta y(b) - \sin\beta (py')(b),\ \alpha,\ \beta \in [0, \pi), \tag{2}$$

where $1/p,\ q,\ w \in L^1[a, b],\ p(x) > 0,\ w(x) \geq 0$ a.e. $x \in [a, b]$ and $\int_a^b w > 0$. Here $q$ is called a potential function, $w$ is called a weight function and $\lambda$ is called an eigenvalue if the problem (1) with the boundary condition (2) has a nontrivial solution (cf., [11]).

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By the spectral theory (cf., [1, 2, 13]), it is known that eigenvalue problems (1) and (2) have only discrete real eigenvalues and the eigenvalues are lower bounded. Denote $\lambda_n$ the $n$-th eigenvalue of the problem (1). In the paper, without loss of generality, by translating the potential function $q$, we can suppose that all the eigenvalues of (1) and (2) are greater than zero. We have

$$0 < \lambda_1 < \cdots < \lambda_n < \cdots, \ \lambda_n \to \infty, \quad n \to \infty. \tag{3}$$

And Weyl's law tells us (cf., [1, (1.5)]) that

$$\lambda_n = \frac{n^2\pi^2}{\left(\int_a^b \sqrt{w/p}\right)^2}(1 + o(1)) \to \infty, \ n \to \infty. \tag{4}$$

Then for Sturm-Liouville problem (1) with the boundary condition (2), we can define its spectral zeta function as (cf., [12])

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}. \tag{5}$$

Thanks to Weyl's law (4), we know that $\zeta(s)$ converges at the half-plane $\Re(s) > 1/2$. Let us consider a special case of the problem (1) with Dirichlet boundary condition, see Example 3.2,

$$-y'' = \lambda y, \text{ on } (0, \pi), \ y(0) = y(\pi) = 0. \tag{6}$$

Then the $n-$th eigenvalue of (6) is $\lambda_n = n^2$, for any $n \geq 1$ and its spectral zeta function is exactly Riemann zeta function $\sum_{n=1}^{\infty} 1/n^{2s}$, that arises out of the number theory (cf., [8], [5, §16]).

Recently, many papers have studied the properties and applications of spectral zeta functions of specific operators. G. Fucci, C. Graham and K. Kirsten [3] provided an analysis of the analytic continuation of spectral zeta functions without weight functions and the coefficients of the asymptotic expansion of the associated heat kernel are given by using this analytical continuation. F. Gesztesy and K. Klaus [4] studied the trace of the resolvent, Fredholm determinant by the properties of spectral zeta functions. Using the connection with the mentioned spectral functions, K. Kirsten [6] provide a method for the calculation of heat-kernel coefficients of Laplace-like operators on Riemannian manifolds. In this paper, the spectral zeta function $\zeta(s)$ of (1) will be analytically continued to the whole complex plane, and the values of the simple poles will be determined.

The rest of this paper is organized as follows. In Section 2, by using the properties of the differential equations and WKB method, the asymptotic estimates of the solution of the Cauchy problem of (1) with respect to the complex parameter $\lambda$ tending to 0 and $\infty$ are given. Moreover, in Subsection 2.2, in order to extend the zeta function $\zeta(s)$, a simple integral representation of $\zeta(s)$ is given. In Section 3, we give the analytic continuation of $\zeta(s)$ and determine the positions of the simple poles of $\zeta(s)$.

## 2 Preliminary

For convenience of the following calculation, we can assume that $p \equiv 1$ in (1). In fact, by Sturm transform, set (cf., [9, §2], [10, §1])

$$\widetilde{x} = \int_a^x \frac{1}{p} =: f(x),\ x \in [a,b] \ \text{ and } \ \widetilde{q}(\widetilde{x}) := p(x)q(x),\ \widetilde{w}(\widetilde{x}) := p(x)w(x),$$

then $\widetilde{x} \in [0,B]$ with $B := \int_a^b 1/p$ and $x = f^{-1}(\widetilde{x})$. It follows from $\int_0^B |\widetilde{w}(\widetilde{x})| \mathrm{d}\widetilde{x} = \int_a^b w$, $\int_0^B |\widetilde{q}(\widetilde{x})| \mathrm{d}\widetilde{x} = \int_a^b q$ and $w,\ q \in L^1[a,b]$ that $\widetilde{w},\ \widetilde{q} \in L^1[0,B]$. Hence differential equations (1) can be rewritten as

$$-\phi''(r,x) + q(x)\phi(r,x) = \lambda w(x)\phi(r,x) =: r^2 w(x)\phi(r,x),\ \text{on } (a,b), \tag{1'}$$

where $r^2 := \lambda$ and $\phi'(r,x) := \partial\phi(r,x)/\partial x$.

For any $r \in \mathbb{C}$, we choose the solutions of (1'), $\phi_r$, such that they satisfy the following initial conditions:

$$\phi(r,a) = \sin\alpha,\ \phi'(r,a) = \cos\alpha. \tag{7}$$

Set

$$\omega(r) := \cos\beta\phi(r,b) - \sin\beta\phi'(r,b), \tag{8}$$

where $\alpha,\ \beta \in [0,\pi)$ are from (2). Then $\omega(\cdot)$ is an analytic function on $\mathbb{C}$ (see [10, §1.7]) and $\lambda = r^2$ is an eigenvalue of (1') and (2) if and only if $\Omega(r) = 0$. Then all zeros of function $\omega(\cdot)$ are

$$0 < r_1 = \sqrt{\lambda_1} < r_2 = \sqrt{\lambda_2} < \cdots . \tag{9}$$

Note that for any $r \in \mathbb{C}$, both $\phi(r,\cdot)$ and $\phi(-r,\cdot)$ satisfy the differential equation (1') and Cauchy condition (7). Hence $\phi(r,\cdot) = \phi(-r,\cdot)$ and

$$\omega(r) = \omega(-r),\ r \in \mathbb{C}. \tag{10}$$

The equation (10) means that $\omega(\cdot)$ is an even function. Moreover, since the conditions (2) are separated boundary conditions, $r$ is an algebraic simple zero of function $\omega(\cdot)$ (see [7, Theorem 5.4]), as $\lambda = r^2$ is any eigenvalue of (1) and (2), i.e.,

$$\omega(r) = 0 \text{ and } \omega'(r) \neq 0. \tag{11}$$

Hence by using Cauchy's residue theorem, we can get that for any eigenvalue $\lambda_n = r_n^2 > 0$, $n \geq 1$, and for the appropriate $s \in \mathbb{C}$, we have

$$\frac{1}{\lambda_n^s} = \frac{1}{r_n^{2s}} = \frac{1}{2\pi i}\int_{\mathcal{C}_n} \frac{1}{r^{2s}} \frac{\mathrm{d}}{\mathrm{d}r} \ln \omega(r) \mathrm{d}r,$$

where $\mathcal{C}_n$ is a contour on the complex plane that only encircles $r_n$ in the counterclockwise direction. Moreover, we have

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{r_n^{2s}} = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{1}{r^{2s}} \frac{\mathrm{d}}{\mathrm{d}r} \ln \omega(r) \mathrm{d}r, \tag{12}$$

where $\mathcal{C}$ is a contour on the complex plane that encircles in the counterclockwise direction all the roots of $\omega(\cdot)$, cf., [6, Figure 1].

Since we assume that all eigenvalues of problems (1), (2) are are greater than 0, see (9), we can choose the contour $\mathcal{C}$ in the right half complex plane. Hence for convenience, we can take appropriate connected branches so that the right half complex plane belongs to the same connected branch. In the next, we will select the appropriate contour to calculate (12), see Figure 1.

## 2.1 Some estimates of function $\omega$

Select contour $\mathcal{C}$ as shown in Figure 1. Set

$$\mathcal{C} := \mathcal{C}_R + I(\varepsilon, R) + \mathcal{C}_\varepsilon := \overrightarrow{-iRRiR} + \left(\overrightarrow{iRi\varepsilon} + \overrightarrow{-i\varepsilon - iR}\right) + \overrightarrow{i\varepsilon\varepsilon - i\varepsilon}.$$

To take the limit of integral (12), as $\varepsilon \to 0$ and $R \to +\infty$, we must first make some estimates of the function $\omega(\cdot)$.

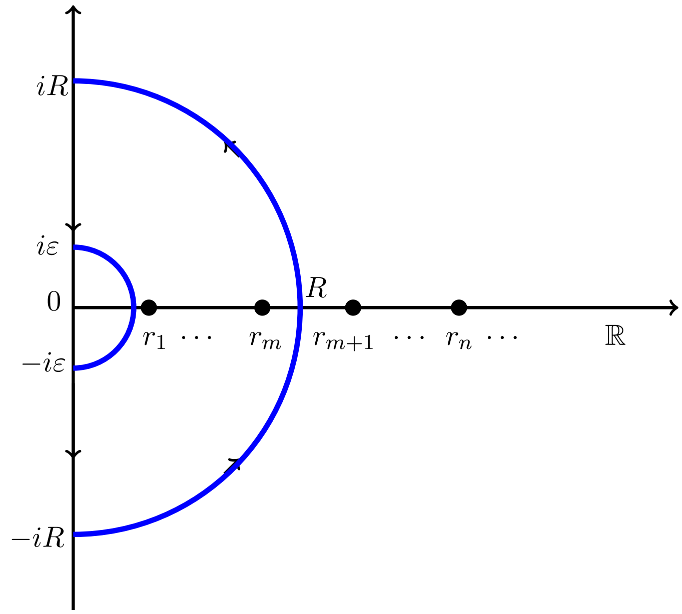


Figure 1: The contour $\mathcal{C}$.

Since $\omega(r) = \omega(-r),\ r \in \mathbb{C}$ by (10), we have that

$$\omega(r) - \omega(0) \sim cr^2,\ \text{as}\ r \to 0,$$

where $c$ is a complex constant, i.e., the linear terms in $r$ are not present in a neighborhood of $r = 0$. Moreover, $\omega(0) \neq 0$ can lead that

$$\frac{\mathrm{d}}{\mathrm{d}r} \ln \omega(r) = \frac{\omega'(r)}{\omega(r)} = \frac{2cr}{\omega(0)},\ \text{as}\ r \to 0. \tag{13}$$

This implies that for any $s \in \mathbb{C}$, $\Re(s) < 1$, the integration in (12) on the contour $\mathcal{C}_\varepsilon$ converges to 0,

$$\int_{\mathcal{C}_\varepsilon} \frac{1}{r^{2s}} \frac{\mathrm{d}}{\mathrm{d}r} \ln \omega(r) \mathrm{d}r = 0,\ \text{as}\ \varepsilon \to 0. \tag{14}$$

For judging that the convergence of the integration in (12) on the contour $\mathcal{C}_R$ either converges to 0, as $R \to +\infty$, we will use WKB analysis (cf., [3]) to estimate

$$\omega(r) = \omega(z_1 + iz_2),\ r = z_1 + iz_2,\ z_1, z_2 \in \mathbb{R},\ \text{as}\ z_2 \to +\infty.$$

Firstly, the second-order linear equation (1) is transformed into a first-order nonlinear equation, and then we can obtain the behavior of the solutions as $z_2 \to +\infty$, by using WKB expansion. In the following, power series expansion is needed, hence we assume that $q,\ w \in C^\infty[a,b]$ and $w(x) > 0$ a.e. on $x \in [a,b]$,

For any $r = z_1 + iz_2$, $z_2 \neq 0$, set $\phi(r,\cdot)$ as the solutions of (1'), i.e.,

$$-\phi''(r,x) + q(x)\phi(r,x) = z^2 w(x)\phi(r,x) \text{ on } (a,b), \tag{15}$$

and set the changing of variables,

$$\psi(r,x) := \phi'(r,x)/\phi(r,x),\ x \in (a,b). \tag{16}$$

The rationality of the changing variables of (16) needs to be explained.

In fact, if there exists $b_0 \in (a,b]$ such that $\phi(r,b_0) = 0$, then $r^2$ and $\phi(r,\cdot)$ are a pair of eigenvalue and eigenfunction of the self-adjoint operator (1) on $[a,b_0]$ with the boundary condition

$$\cos\alpha\phi(a) - \sin\alpha\phi'(a) = 0 = \phi(b_0).$$

If $z_1 \neq 0$, then we obtain a non-real eigenvalue of the self-adjoint operator, $r^2 = z_1^2 - z_2^2 + 2iz_1z_2 \in \mathbb{C}\backslash\mathbb{R}$. It is a contradiction.

If $z_1 = 0$, then $r^2 = -z_2^2$. For any $\phi \in C^\infty[a,b]$, we have

$$\int_a^{b_0} |\phi'|^2 + \int_a^{b_0} q|\phi|^2 \geq \frac{\min\{q(x):\ x \in [a,b]\}}{\max\{w(x):\ x \in [a,b]\}} \int_a^{b_0} w|\phi|^2 =: -M \int_a^{b_0} w|\phi|^2.$$

Hence the eigenvalue $r = -z_2^2$ must satisfy $-z_2^2 \geq -M$. This fact can lead that for any $z^2 > M$, $\phi(r,x) \neq 0$, $x \in (a,b)$, and hence (16) is reasonable.

Using the changing variables (16), we obtain a first order nonlinear differential equation for $\psi(r,\cdot)$,

$$\psi'(r,x) = q(x) - r^2 w - \psi^2(r,x). \tag{17}$$

Since differential equation (15) has two linearly independent solutions, and a first order nonlinear differential equation (17) also has two corresponding solutions.

By the properties of the coefficients of (17), for $|r| \to \infty$, we can assume that $\psi(r,x)$

has an asymptotic expansion

$$\psi(r,x)=\psi_{-1}(x)r+\sum_{j=0}^{\infty}\frac{\psi_j(x)}{r^j}. \tag{18}$$

Plugging this expansion into (17), we can obtain

$$\psi_{-1}^{\pm}(x)=\pm i\sqrt{w(x)},\ \psi_0^{\pm}(x)=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\ln(\psi_{-1}^{\pm}(x)),$$

$$\psi_1^{\pm}(x)=\frac{1}{2\psi_{-1}^{\pm}(x)}[q(x)-(\psi_0^{\pm})^2(x)-(\psi_0^{\pm}(x))'],$$

and for any $j\geq 1$,

$$\psi_{j+1}^{\pm}(x)=-\frac{1}{2\psi_{-1}^{\pm}(x)}\left[(\psi_j^{\pm}(x))'+\sum_{k=0}^{j}\psi_k^{\pm}\psi_{j-k}^{\pm}(x)\right].$$

The different signs correspond to the two solutions $\psi^+(r,x)$ and $\psi^-(r,x)$ to (17).

Note that

$$\psi_{-1}^{+}=-\psi_{-1}^{-},\ \psi_0^{+}=\psi_0^{-},\cdots,\ \psi_k^{+}=(-1)^k\psi_k^{-},\cdots. \tag{19}$$

Then the general solution of (15) is obtained by the linear combination of the increasing and decaying terms as follows,

$$\phi(r,x)=Ae^{\int_a^x\psi^+(r,t)\mathrm{d}t}+Be^{\int_a^x\psi^-(r,t)\mathrm{d}t}, \tag{20}$$

where $A,\ B$ are determined by the initial conditions (7). From $\psi_{-1}^{\pm}(x)=\pm i\sqrt{w(x)}$ and (18), we know for $r=z_1+iz_2\in\mathbb{C}$,

$$\psi_{-1}^{\pm}(x)r=\pm i\sqrt{w(x)}r=\mp\sqrt{w(x)}z_2\pm i\sqrt{w(x)}z_1. \tag{21}$$

And hence the asymptotic behavior of $\phi(r,x)$ for $z_2\to-\infty$ (resp., $+\infty$) is obtained by the asymptotic increase of the term $Ae^{\int_a^x\psi^+(r,t)\mathrm{d}t}$ (resp., $Be^{\int_a^x\psi^-(r,t)\mathrm{d}t}$) and the asymptotic decay of the term $Be^{\int_a^x\psi^-(r,t)\mathrm{d}t}$ (resp., $Ae^{\int_a^x\psi^+(r,t)\mathrm{d}t}$). Hence

$$\begin{aligned}e^{\int_a^b\psi^-(r,t)\mathrm{d}t}&=\varepsilon(r)e^{\int_a^b\psi^+(r,t)\mathrm{d}t},\ \text{as } z_2\to-\infty,\\ e^{\int_a^b\psi^+(r,t)\mathrm{d}t}&=\varepsilon(r)e^{\int_a^b\psi^-(r,t)\mathrm{d}t},\ \text{as } z_2\to+\infty,\end{aligned} \tag{22}$$

where $\varepsilon(r)$ is exponentially small contributions in $z_2$.

Input the cauchy condition (7) in (20) to obtain

$$A = -\frac{\sin\alpha\psi^-(r,a) - \cos\alpha}{\psi^+(r,a) - \psi^-(r,a)},\ B = \frac{\sin\alpha\psi^+(r,a) - \cos\alpha}{\psi^+(r,a) - \psi^-(r,a)}.$$

By (22) and substituting these equations into (8), we can get the next estimates,

$$\begin{aligned}
\omega(r) &= \cos\beta\phi(r,b) - \sin\beta\phi'(r,b) \\
&= [\cos\beta - \sin\beta\psi^+(r,b)]Ae^{\int_a^b \psi^+(r,t)\mathrm{d}t} + [\cos\beta - \sin\beta\psi^-(r,b)]Be^{\int_a^b \psi^-(r,t)\mathrm{d}t} \\
&= [\cos\beta - \sin\beta\psi^+(r,b)]Ae^{\int_a^b \psi^+(r,t)\mathrm{d}t}(1+\varepsilon(r)),\ \text{as } z_2 \to -\infty; \\
\text{or } &= [\cos\beta - \sin\beta\psi^-(r,b)]Ae^{\int_a^b \psi^-(r,t)\mathrm{d}t}(1+\varepsilon(r)),\ \text{as } z_2 \to +\infty,
\end{aligned}$$

and hence by (18), we know that for $r = z_1 + iz_2 \in \mathbb{C}$,

$$\ln\omega(r) = c_1 + c_2\ln r + c_3 r + \sum_{j=1}^{\infty}\frac{M_j}{r^j},\ \text{as } |z_2| \to \infty. \tag{23}$$

The estimate (23) implies that

$$\left|\frac{\mathrm{d}}{\mathrm{d}r}\ln\omega(r)\right| < c_3 + 1,\ \text{as } |z_2| \to \infty.$$

This fact can lead to that for any $s \in \mathbb{C}$, $\Re(s) > 1/2$, the integration in (12) on the contour $\mathcal{C}_R$ can converge to 0,

$$\int_{\mathcal{C}_{R_n}} \frac{1}{r^{2s}}\frac{\mathrm{d}}{\mathrm{d}r}\ln\omega(r)\mathrm{d}r = 0,\ \text{as } n \to +\infty, \tag{24}$$

where $R_n := (r_n + r_{n+1})/2 \in (r_n, r_{n+1})$, see Figure 1.

Using the two estimates (14) and (24), we can obtain an simple representation of the spectral zeta function $\zeta$.

## 2.2 An representation of the spectral zeta function

Recall the contour integral (12), we have

$$\sum_{m=1}^{n}\frac{1}{\lambda_m^s} = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{1}{r^{2s}}\frac{\mathrm{d}}{\mathrm{d}r}\ln\omega(r)\mathrm{d}r = \frac{1}{2\pi i}\int_{\mathcal{C}_{R_n}+I(\varepsilon,R_n)+\mathcal{C}_\varepsilon}\frac{1}{r^{2s}}\frac{\mathrm{d}}{\mathrm{d}r}\ln\omega(r)\mathrm{d}r.$$

Using (14), (24), and letting $\varepsilon \to 0$, $n \to +\infty$, we can get that for any $s \in \mathbb{C}$, $1 > \Re(s) > 1/2$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \frac{1}{2\pi i} \int_{+\infty}^{-\infty} \frac{1}{(iz)^{2s}} \frac{\mathrm{d}}{\mathrm{d}(iz)} \ln \omega(iz) \mathrm{d}(iz), \tag{25}$$

where the changing variable $r = iz$ is made.

First, we need to verify the convergence of the integral (25). Let us recall (13), the estimate of $\frac{\mathrm{d}}{\mathrm{d}r} \ln \omega(r)$ at $r = 0$,

$$\frac{\mathrm{d}}{\mathrm{d}r} \ln \omega(r) = \frac{2cr}{\omega(0)}, \text{ as } r \to 0. \tag{26}$$

Therefore, as $1 > \Re(s)$, the integral (25) is convergence in the neighborhood of 0. Similarly, using (23) again, we can obtain that as $\Re(s) > 1/2$, (25) is convergence at $\pm\infty$. Hence as $1 > \Re(s) > 1/2$, the integral (25) is convergence.

Furthermore, we need to simplify the integral (25).

$$\frac{1}{2\pi i} \int_{+\infty}^{0} \frac{1}{(iz)^{2s}} \frac{\mathrm{d}}{\mathrm{d}(iz)} \ln\omega(iz)\mathrm{d}(iz) = -\frac{i^{-2s-1}}{2\pi} \int_{0}^{+\infty} \frac{1}{z^{2s}} \frac{\mathrm{d}}{\mathrm{d}z} \ln\omega(iz)\mathrm{d}z.$$

Note that $\omega(\cdot)$ is an even function by (10), hence $\omega(iz) = \omega(-iz) \in \mathbb{R}$. This fact can lead to

$$\begin{aligned}
\frac{1}{2\pi i} \int_{0}^{-\infty} \frac{1}{(iz)^{2s}} \frac{\mathrm{d}}{\mathrm{d}(iz)} \ln\omega(iz)\mathrm{d}(iz) &= \frac{i^{-2s-1}}{2\pi} \int_{0}^{-\infty} \frac{1}{z^{2s}} \frac{\mathrm{d}}{\mathrm{d}z} \ln\omega(iz)\mathrm{d}z \\
&= \frac{i^{-2s-1}}{2\pi} \int_{0}^{+\infty} \frac{1}{(-z)^{2s}} \frac{\mathrm{d}}{\mathrm{d}(-z)} \ln\omega(-iz)\mathrm{d}(-z) \\
&= \frac{(-1)^{-2s} i^{-2s-1}}{2\pi} \int_{0}^{+\infty} \frac{1}{z^{2s}} \frac{\mathrm{d}}{\mathrm{d}z} \ln\omega(iz)\mathrm{d}z.
\end{aligned}$$

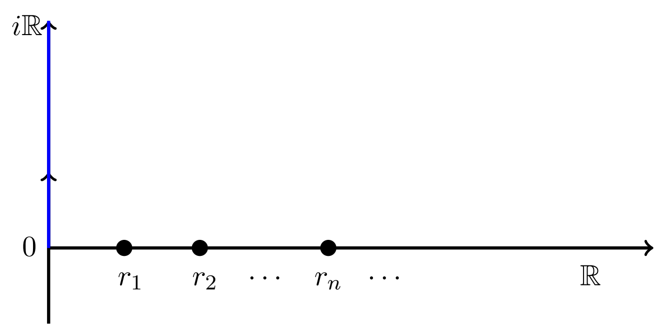


Figure 2: The simpler contour.

Using the above two equations and noting the relationship

$$\Re(i^{-2s-1}) = \Re(e^{-i\frac{\pi}{2}(2s+1)}) = -\sin(s\pi),$$

we can get a simple representation of (25) (cf., [3], [6, §2]). For any $s \in \mathbb{C}$, $1 > \Re(s) > 1/2$, we have

$$\zeta(s) = \frac{\sin \pi s}{\pi} \int_0^\infty \frac{1}{z^{2s}} \frac{\mathrm{d}}{\mathrm{d}z} \ln \omega(iz) \mathrm{d}z. \tag{27}$$

See Figure 2. Moreover, let us recall the spectral zeta function associated with (1),

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \sum_{n=1}^{\infty} \frac{1}{r_n^{2s}}, \tag{28}$$

which is convergent, due to Weyl's estimate (4), in the semi-plane $\Re(s) > 1/2$.

In the following, we will use (27) to analytically continue the spectral zeta function $\zeta(\cdot)$ to a region extending to the left of the semi-plane $1/2 < \Re(s)$. Finally, the spectral function will be analytically continued to the whole complex plane.

## 3 Analytic continuation of the spectral zeta function

Now we can give the analytic continuation of the spectral zeta function. (27) can be conveniently rewritten as

$$\zeta(s) = \frac{\sin \pi s}{\pi} \int_0^1 \frac{1}{z^{2s}} \frac{\mathrm{d}}{\mathrm{d}z} \ln \omega(iz) \mathrm{d}z + \frac{\sin \pi s}{\pi} \int_1^\infty \frac{1}{z^{2s}} \frac{\mathrm{d}}{\mathrm{d}z} \ln \omega(iz) \mathrm{d}z. \tag{29}$$

By (13), we know that the first integral is an analytic function for $\Re(s) < 1$, and the second one defines an analytic function for $\Re(s) > 1/2$, by (23). In order to analytically continue $\zeta(\cdot)$ to a region extending to the left of the strip $1/2 < \Re(s) < 1$, we need to extend the second integral. Inputting the first $L+3$ terms of (23) to the second integral (29), we can rewrite the spectral zeta function as

$$\zeta(s) = Z(s) + \frac{\sin \pi s}{\pi} \left[ \frac{c_2}{2s} + \frac{c_3}{2s-1} - \sum_{j=1}^{L} \frac{jM_j}{2s+j} \right], \tag{30}$$

where $Z(s)$ is analytic in $\Re(s) > -L/2$, $L \geq 0$. For any even number $j$, $\sin(\pi s/2) = 0$, as $s = j/2$, hence $\zeta(s)$ is a meromorphic function of $s$ with simple poles at points $s =$

$1/2,\ -1/2,\ -3/2,\ \cdots$. Hence we have obtained the next theorem,

**Theorem 3.1.** *Suppose* $p,\ q,\ w \in C^\infty[a,b],\ p(x),\ w(x) > 0$ a.e. $x \in [a,b]$. *Then the spectral zeta functions* $\zeta(s)$ *of problems* (1) *and* (2) *can be analytically continued to the whole complex plane and be with simple poles at points* $s = 1/2,\ -1/2,\ -3/2,\ \cdots$.

Riemann zeta function $\sum_{n=1}^\infty 1/n^{2s}$ is the most important special case of spectral zeta functions.

**Example 3.2.** *Let us consider a special case of* (1) *with the Dirichlet boundary condition,*

$$-y'' = \lambda y, \text{ on } (0,\pi),\ y(0) = y(\pi) = 0.$$

*Then the $n$-th eigenvalue and eigenfunction are* $\lambda_n = r_n^2 = n^2$ *and* $\sin(nx)$. *By* (18), *we can obtain that*

$$\psi^\pm(x,z) \equiv \mp z, \text{ and } \psi_{-1}^\pm = \pm i,\ \psi_j^\pm = 0,\ j \geq 0. \tag{31}$$

*By substituting* (31) *into* (23), *we obtain that*

$$\ln \omega(iz) = c_1 + c_2 \log z + c_3 z, \text{ as } z \to +\infty,$$

*and*

$$\zeta(s) = Z(s) + \frac{\sin \pi s}{\pi}\left[\frac{c_2}{2s} + \frac{c_3}{2s-1}\right].$$

*In fact,* $\omega(iz) = \frac{1}{2z}(e^z - e^{-z})$ and

$$\ln \omega(iz) = -\ln(2z) + \ln(e^z - e^{-z}) = -\ln 2 - \ln z + z, \text{ as } z \to +\infty.$$

*Hence the Riemann zeta function* $\zeta(s) = \sum_{n=1}^\infty 1/\lambda_n^s = \sum_{n=1}^\infty 1/n^{2s}$ *has been analytically continued to a meromorphic function on* $\mathbb{C}$ *and be with a simple pole only at point* $s = 1/2$.

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