Periodicity of nonautonomous fuzzy neural networks with reaction-diffusion terms and distributed time delays

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Abstract

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Abstract

In this paper, the periodicity of a class of nonautonomous fuzzy neural networks with impulses, reaction-diffusion terms and distributed time delays are investigated. Some new sufficient conditions for the existence of periodic solutions and global exponential stability of the systems are obtained using time delays integral differential inequalities, Poincar mappings and fixed point theory. The validity and generality of the methods are illustrated by two numerical examples.

Keywords: nonautonomous neural networks; distributed delays; impulses; Periodic oscillatory solutions

I. Introduction

In 1996, Yang and Yang studied fuzzy cellular neural networks (FCNNs) [1]–[3] by combining fuzzy logic with traditional cellular neural networks based on the previous cellular neural networks [4]. It was shown that FCNNs play an important role in image processing problems and pattern recognition. These applications rely heavily on the dynamic behavior of FCNNs. Therefore, it is particularly important to analyze the dynamics of FCNNs. As we all know, neural networks often have delays in the process of information processing. The existence of time delays may cause the systems to oscillate, diverge or become unstable. Neural dynamics considering the delay problems are very important for the stability and balance of the neural networks. Some scholars have studied the stability of FCNNs with constant and time-varying delays [5], [6], and some have studied the stability of FCNNs with distributed time delays [7] and leaky time delays [8]. Furthermore, diffusion effects in neural networks are unavoidable when electrons move in asymmetric electromagnetic fields. Therefore, we must consider that the activation is different in time and space. A number of neural network models with reaction-diffusion terms and various delays have been developed and studied [9]- [11].

On the other hand, in neural network systems, in addition to time delays and diffusion effects, there are impulse effects, which are because of the fact that many neural networks undergo abrupt changes at a given moment due to transient disturbances. These changes occur in the fields of physics, chemistry, population dynamics, optimal control, etc. Some results about impulse effects have been obtained in time delays neural networks [12]- [20]. In particular, when we consider the long-term dynamic behaviours of systems, the parameters of the systems usually

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change over time and this nonautonomous phenomenons often occur in many practical systems. In [21], the authors studied the stability of a nonautonomous fuzzy neural network with reaction-diffusion terms without impulses. Long [22] studied the dynamic behaviors of nonautonomous cellular neural networks with time-varying delays. In [23], the authors studied the existence, uniqueness and global stability of periodic solutions of general nonautonomous impulsive cellular neural networks and obtained some criteria.

Based on what we know, there are no results on the exponential stability of FCNNs with impulses, distributed time delays and reaction-diffusion terms at the same time, which is very important in theories and applications. In terms of mathematical models, FCNNs have not only fuzzy logic but also impulse effects between its template input and/or output, except the sum of product operations. The models include reaction-diffusion terms, fuzzy logic and impulse characteristics, which have complex dynamic behaviors. It is therefore necessary to further investigate the dynamic behaviours of FCNNs. We have used the properties of M-matrices and inequality tricks to establish a new differential inequality that yields a sufficient condition for global exponentially stable periodic solutions of the systems. Finally, the validity of the results are verified by means of arithmetic examples and numerical simulations using [24].

Consider the nonautonomous FCNNs, which contains the reaction-diffusion terms, distributed time delays and impulses.

$$\begin{aligned}\frac{\partial u_p(t,x)}{\partial t} &= \sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(D_{pk}\frac{\partial u_p(t,x)}{\partial x_k}\right) - d_p(t)u_p(t,x) + \sum_{q=1}^{n} h_{pq}(t) f_q\left(u_q(t,x)\right)\\ &+ \sum_{q=1}^{n} y_{pq}(t)v_q(t) + J_p(t) + \bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,x)\right)\mathrm{d}s\\ &+ \bigwedge_{q=1}^{n}\alpha_{pq}(t)v_q(t) + \bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,x)\right)\mathrm{d}s\\ &+ \bigvee_{q=1}^{n}\beta_{pq}(t)v_q(t),\quad t\neq t_i,\quad x\in X,\end{aligned}\tag{1a}$$

$$u_p(t_i^+,x) = \varphi_{pi}(u_p(t_i^-,x)),\quad x\in X,\quad i\in N\triangleq\{0,1,2,\cdots\},\tag{1b}$$

$$u_p(t,x) = 0,\quad t\geq 0,\quad x\in\partial X,\tag{1c}$$

$$u_p(s,x) = \gamma_p(s,x),\quad s\in[-\infty,0],\tag{1d}$$

where $p = 1,2,\ldots,n$, $t\in[0,+\infty)$, $x = (x_1,x_2,\ldots,x_m)^{\mathrm{T}}\in X\subset R^m$, $X = \{x = (x_1,x_2,\ldots,x_m)^{\mathrm{T}}\,\| x_k\ | < \mathbb{L}_k, k = 1,2,\ldots,m\}$ is a bounded compact set with smooth boundary ∂X and mes $X > 0$ in space R^m ($\mathbb{L}_k > 0$); $\mathcal{K}_{pq}(\cdot)$ represent the delay kernel function which is real valued piecewise continuous. $u_p(t,x)$ denotes the pth neuron in space x and at time t; $f(\cdot)$ and $g(\cdot)$ represent the signal activation function of the qth neuron; $d_p(t) > 0$ denotes the rate of potential recovery to isolated state of the pth neuron at moment t. $h_{pq}(t)$ and $y_{pq}(t)$ denote the elements of the feedback template and the feedforward template at moment t; $\bigvee$ and $\bigwedge$ represent the fuzzy AND and fuzzy OR operations, respectively; $a_{pq}(t)$ and $b_{pq}(t)$, represent elements of fuzzy feedback MIN template and fuzzy feedback MAX template at time t,respectively; $\alpha_{pq}(t)$ and $\beta_{pq}(t)$ represent elements of fuzzy feed-forward MIN template and fuzzy feed-forward MAX template at time t, respectively; $J_p(t)$ and $v_p(t)$ represent the input and bias of the pth neuron at moment t; $D_{pk}\geq 0$ represents the transmission diffusion coefficient. In (1b), $t_i > 0$ satisfies $t_i < t_{i+1}, \lim_{i\to+\infty} t_i = +\infty; u_p\left(t_i^-,x\right)$ and $u_p\left(t_i^+,x\right)$ represent the left and right limits at t_k, respectively; φ_{pi} shows impulsive perturbation of the pth neuron at time t_i. Let $u_p\left(t_i^+, x\right) = u_p\left(t_i, x\right), i \in N$. Equations (1c) denote the Dirichlet boundary conditions and (1d) denote the initial conditions.

If impulsive operator $\varphi_{pi}\left(u_p\right) = 0, p = 1, 2, \ldots, n, i \in N$, we obtain the following systems $(2a) - (2c)$:

$$\begin{aligned}\frac{\partial u_p(t,x)}{\partial t} = &\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{pk}\frac{\partial u_p(t,x)}{\partial x_k}\right) - d_p(t)u_p(t,x) + \sum_{q=1}^{n} h_{pq}(t) f_q\left(u_q(t,x)\right)\\ &+ \sum_{q=1}^{n} y_{pq}(t)v_q(t) + J_p(t) + \bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s)g_q\left(u_q(s,x)\right)\mathrm{d}s\\ &+ \bigwedge_{q=1}^{n} \alpha_{pq}(t)v_q(t) + \bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s)g_q\left(u_q(s,x)\right)\mathrm{d}s\\ &+ \bigvee_{q=1}^{n} \beta_{pq}(t)v_q(t), \quad x \in X,\end{aligned} \tag{2a}$$

$$u_p(t,x) = 0, \quad t \geq 0, \quad x \in \partial X, \tag{2b}$$

$$u_p(s,x) = \gamma_p(s,x), \quad s \in [-\infty, 0]. \tag{2c}$$

Systems $(2a) - (2c)$ are continuous forms of systems $(1a) - (1d)$.

The main contributions of this manuscript are

(a) We have developed a new neural network model, including nonautonomous fuzzy neural networks, reaction-diffusion cellular neural networks, distributed time delays neural networks, impulsive neural networks and Dirichlet boundary conditions.

(b) We have obtained several new criteria that guarantee the exponential stability of periodic solutions for considered networks. These criteria are expressed in the forms of simple algebraic inequalities which depend only on systems $(1a) - (1d)$ parameters.

II. Preliminaries

In this section, we explain some of the necessary assumptions, associated notations and definitions.

(H1) There exist diagonal matrices F=diag$(F_1, F_2, \cdots, F_n)$ and G=diag$(G_1, G_2, \cdots, G_n)$ such that

$$F_p = \sup_{z_1 \neq z_2}\left|\frac{f_p(z_1) - f_p(z_2)}{z_1 - z_2}\right|, G_p = \sup_{z_1 \neq z_2}\left|\frac{g_p(z_1) - g_p(z_2)}{z_1 - z_2}\right|,$$

for all $z_1, z_2 \in R(z_1 \neq z_2), p = 1, 2 \cdots n$.

(H2) There exists a non-negative diagonal matrix $\Psi_i = \operatorname{diag}\left(\varphi_{1i}, \ldots, \varphi_{ni}\right)$ such that

$$\left|\varphi_{pi}(z_1) - \varphi_{pi}(z_2)\right| \leqslant \varphi_{pi}|z_1 - z_2|$$

for all $z_1, z_2 \in R(z_1 \neq z_2), p \in \{1, 2, \cdots, n\}, \quad i \in N$.

(H3) $d_p(t), h_{pq}(t), a_{pq}(t)$ and $b_{pq}(t)$ are continuous bounded function defined on $t \in [0, +\infty)$.

(H4) there exists a positive number $\sigma > 0$ such that

$$\mathbb{K}_{pq}(\lambda) = \int_0^{+\infty} e^{\lambda s}\left|\mathcal{K}_{pq}(s)\right| ds$$

is continuous for $\lambda \in [0, \sigma), p, q = 1, 2, \ldots, n$.

Let $PC(X) \triangleq \{\mu : [-\infty, 0] \times X \to R^n \mid \mu(s,x)$ is bounded on $[-\infty, 0] \times X$ and $\mu(s^+, x) = \mu(s,x)$ for $s \in [-\infty, 0]$. $\mu(s^-, x)$ exists for $s \in [-\infty, 0]$ and $\mu(s^-, x) = \mu(s,x)$ for all but a finite number of points $s \in [-\infty, 0]\}$. For $\mu(s,x) = (\mu_1(s,x), \ \mu_2(s,x), \ldots, \mu_n(s,x))^T \in PC(X)$, $\|\mu\|$ is defined as

$$\|\mu\| = \sup_{-\infty \le s \le 0} \sum_{p=1}^{n} \left\|\mu_p(s,x)\right\|_2, \tag{3}$$

where $\left\|u_p(t,x)\right\|_2 = \left[\int_X \left|u_p(t,x)\right|^2 dx\right]^{\frac{1}{2}}$, $p = 1, 2, \ldots, n$.

Let $PC \triangleq \{\mu : [-\infty, 0] \to R^n \mid \mu(s)$ is bounded on $[-\infty, 0]$ and $\mu(s^+) = \mu(s)$ for $s \in [-\infty, 0]$. $\mu(s^-)$ exists for $s \in [-\infty, 0]$ and $\mu(s^-) = \mu(s)$ for all but a finite number of points $s \in [-\infty, 0]\}$.

Let $C = (c_{pq})_{m\times n}$, and $B = (b_{pq})_{m\times n}$, then the Schur product of C and B is defined by $C \otimes B = \left(c_{pq} b_{pq}\right)_{m\times n}$.

$\mathbf{e} = (1, 1, \ldots, 1)^T \in R^n$ and E denotes a $n \times n$ identity matrix.

Definition 1: If $u(t,x)(u : R \times X \to R^n)$ satisfies

(i) $u(t,x)$ is piecewise continuous and right-continuous at every discontinuity point t_i $i \in N$ which are the first kind of discontinuity points;

(ii) $u(s,x) = \gamma(s,x)(s \in [-\infty, 0])$ satisfies systems $(1a) - (1d)$ for all $t \geqslant 0$.

Hence, $u(t,\gamma,x)$ represents the special solution of systems $(1a) - (1d)$ under initial condition $\gamma \in PC(X)$.

Definition 2: In the initial condition of $\upsilon \in PC(X)$, $u(t,\upsilon,x)$ is any solution to the systems $(1a) - (1d)$. If there exist two positive numbers $\lambda > 0$ and $M \geqslant 1$ such that

$$\|u(t,\gamma,x) - u(t,\upsilon,x)\| \leqslant M \|\gamma - \upsilon\| e^{-\lambda t} \quad \text{for all } t \geqslant 0, \tag{4}$$

then systems $(1a) - (1d)$ are globally exponentially stable.

Definition 3: [25] If $B = \left(b_{pq}\right)_{n\times n}$ is a real matrix, suppose that

(i) $b_{pq} \leqslant 0$, for all $p, q = 1, 2, \ldots, n, p \neq q$;

(ii) all successive principal minors of B are positive.

Then, B is a non-singular M-matrix.

Lemma 1: [25]Setting $B = \left(b_{pq}\right)_{n\times n}$ with $b_{pq} \leqslant 0 (p \neq q)$ for all $p, q = 1, 2, \cdots, n$. Then the necessary and sufficient conditions for B to be a non-singular M-matrix are that there exists a vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n) > 0$ such that $B\xi > 0$ or $B^T\xi > 0$.

Lemma 2: [26] Let $f(x)$ is a real-valued function and $X = \{x = (x_1, x_2, \ldots, x_m)^{\mathrm{T}} \| x_k | < \mathbb{L}_k, k = 1, 2, \ldots, m\}$ is a cube. If $f(x)$ meets $f(x)|_{\partial X} = 0$, that is $f(x)$ is equal to zero at the boundary of X. Then

$$\int_X f^2(x) dx \leqslant \mathbb{L}_k^2 \int_X \left|\frac{\partial f}{\partial x_k}\right|^2 dx.$$

Lemma 3: [25] Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ be two states of neural networks $(1a) - (1d)$, and $f(\cdot)$ be a real-valued function. Then following inequalities hold:

$$\left|\bigwedge_{q=1}^{n} \alpha_{pq}(t) f_q\left(\mu_q\right) - \bigwedge_{q=1}^{n} \alpha_{pq}(t) f_q\left(\nu_q\right)\right| \le \sum_{q=1}^{n} \left|\alpha_{pq}(t)\right| \left|f_q\left(\mu_q\right) - f_q\left(\nu_q\right)\right|$$

and

$$\left|\bigvee_{q=1}^{n} \beta_{pq}(t) f_q\left(\mu_q\right) - \bigvee_{q=1}^{n} \beta_{pq}(t) f_q\left(\nu_q\right)\right| \le \sum_{q=1}^{n} \left|\beta_{pq}(t)\right| \left|f_q\left(\mu_q\right) - f_q\left(\nu_q\right)\right|.$$

Lemma 4: Let $a < b \leqslant +\infty$, If $V(t) = (V_1(t), V_2(t), \ldots, V_n(t))^{\mathrm{T}} \in C[[a,b), R^n]$ makes the following differential inequality hold:

$$\begin{cases} D^+V(t) \leqslant P(t)V(t) + \int_0^{+\infty}(R(t) \otimes |\mathcal{K}(s)|)V(t-s)ds, & a \leqslant t < b, \\ V(a+s) \in PC, & -\infty < s \leqslant 0, \end{cases}$$

where $P(t) = \left(p_{pq}(t)\right)_{n\times n}$ with $p_{pq}(t) \geqslant 0$ $(p \neq q)$, $\mathcal{K}(s) = (\mathcal{K}_{pq}(s))_{n\times n}$, $R(t) = (r_{pq}(t))_{n\times n}$ with $r_{pq}(t) \geqslant 0$. If the initial condition meets

$$V(t) \leqslant \kappa\xi \mathrm{e}^{-\lambda(t-a)}, \kappa \geqslant 0, t \in (-\infty, a], \tag{5}$$

where $\lambda > 0$ and $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} > 0$ satisfy the following inequality:

$$[\lambda E + P(t) + R(t) \otimes K(\lambda)]\xi < 0, \quad K(\lambda) = (\mathbb{K}_{pq}(\lambda))_{n\times n}, \tag{6}$$

then $V(t) \leqslant \kappa\xi e^{-\lambda(t-a)}, \quad t \in [a,b)$.

Proof: For $p \in \{1, 2, \ldots, n\}$, $\forall \varepsilon > 0$, let $\omega_p(t) \triangleq (\kappa+\varepsilon)\xi_p \mathrm{e}^{-\lambda(t-a)}$. Then

$$V_p(t) \leqslant \omega_p(t) = (\kappa+\varepsilon)\xi_p \mathrm{e}^{-\lambda(t-a)}, \quad t \in [a,b), \quad p = 1, 2, \ldots, n. \tag{7}$$

If the above is false, that is there exist a number $t^* \in [a,b)$ and several integer l such that

$$V_l(t^*) = \omega_l(t^*), \quad D^+V_l(t^*) \geqslant \dot{\omega}_l(t^*), \quad V_p(t) \leqslant \omega_p(t), \quad t \in [a, t^*], \quad p = 1, 2, \ldots, n. \tag{8}$$

According to Lemma 4 and (7), we obtain

$$\begin{aligned} D^+V_l(t^*) &\leqslant \sum_{q=1}^{n}\left[p_{lq}(t^*)V_q(t^*) + \int_0^{+\infty} r_{lq}(t)|\mathcal{K}_{lq}(s)|V_q(t^*-s)ds\right] \\ &\leqslant \sum_{q=1}^{n}\left[p_{lq}(t^*)(\kappa+\varepsilon)\xi_q \mathrm{e}^{-\lambda(t^*-a)} + r_{lq}(t^*)\int_0^{+\infty}\left|\mathcal{K}_{lq}(s)\right|(\kappa+\varepsilon)\xi_q e^{-\lambda(t^*-s-a)}ds\right] \\ &= \sum_{q=1}^{n}\left[p_{lq}(t^*)(\kappa+\varepsilon)\xi_q \mathrm{e}^{-\lambda(t^*-a)} + r_{lq}(t^*)(\kappa+\varepsilon)\xi_q e^{-\lambda(t^*-a)}\int_0^{+\infty} e^{\lambda s}\left|\mathcal{K}_{lq}(s)\right|ds\right] \\ &\leqslant \sum_{q=1}^{n}\left[p_{lq}(t^*) + r_{lq}(t^*)K_{lq}(\lambda)\right](\kappa+\varepsilon)\xi_q \mathrm{e}^{-\lambda(t^*-a)}. \end{aligned} \tag{9}$$

From $[\lambda E + P(t) + R \otimes K(\lambda)]\xi < 0$, $p_{pq}(t^*) \geqslant 0 (p \neq q)$, $r_{pq}(t^*) \geqslant 0$ and $\mathbb{K}_{pq}(\lambda) \geqslant 0$, it follows that

$$\sum_{q=1}^{n}\left[p_{lq}(t^*) + r_{lq}(t)\mathbb{K}_{lq}(\lambda)\right]\xi_q < -\lambda\xi_l < 0,$$

From (9), we obtain

$$D^+V_l(t^*) < -\lambda\xi_l(\kappa+\varepsilon)\mathrm{e}^{-\lambda(t^*-l)} = \dot{\omega}_l(t^*). \tag{10}$$

That is

$$D^+V_l(t^*) < \dot{\omega}_l(t^*),$$

This is contradictory to (8). Therefore, the inequality (7) holds for all $t \in [a,b)$.

Now, letting $\varepsilon \to 0$ in (7), we have that

$$V_p(t) \leqslant \kappa\xi_p \mathrm{e}^{-\lambda(t-a)}, \quad t \in [a,b), \quad p = 1, 2, \ldots, n.$$

That is,

$$V(t) \leqslant \kappa\xi e^{-\lambda(t-a)} \quad \text{for all} \quad t \in [a,b).$$

■

III. Main Results

In this section, we introduce the main results of systems $(1a)-(1d)$ and their proof process.

Theorem 1: If assumptions (H1)–(H4) are satisfied, suppose that

(C1) There exist $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} > 0$ and $\lambda > 0$ such that

$$[\lambda E - W(t) + H(t)F + R(t) \otimes K(\lambda)]\xi < 0, \quad t \geqslant 0,$$

where $W(t) = \mathrm{diag}\,(w_1(t), w_2(t), \ldots, w_n(t))$ with $w_p(t) = d_p(t) + \sum_{k=1}^{m} \frac{D_{pk}}{\mathbb{L}_k^2}$, $R(t) = [A(t) + B(t)]G$, $A(t) = \left(\left|a_{pq}(t)\right|\right)_{n\times n}$, $B(t) = \left(\left|b_{pq}(t)\right|\right)_{n\times n}$, $H(t) = \left(\left|h_{pq}(t)\right|\right)_{n\times n}$, $R(t) = (r_{pq}(t))_{n\times n}$, $K(\lambda) = (\mathbb{K}_{pq}(\lambda))_{n\times n}$, $F = \mathrm{diag}(F_1, F_2, \cdots, F_n)$, $G = \mathrm{diag}(G_1, G_2, \cdots, G_n)$;

(C2) There exists a constant $\phi > 0$ such that

$$\sup_{i\in N}\left\{\frac{\ln\phi_i}{t_i - t_{i-1}}\right\} \leqslant \phi < \lambda,$$

where $\phi_i = \max_{1\leqslant p\leqslant n}\left\{1, \varphi_{pi}\right\}, i \in N$;

then systems $(1a)-(1d)$ are globally exponentially stable.

Proof: For $\theta, \vartheta \in PC(X)$, let $u(t,\theta,x) = (u_1(t,\theta,x), u_2(t,\theta,x), \ldots, u_n(t,\theta,x))^T$ and $u(t,\vartheta,x) = (u_1(t,\vartheta,x), u_2(t,\vartheta,x), \ldots, u_n(t,\vartheta,x))^T$ be solutions of systems $(1a)-(1d)$ through $(0,\theta)$ and $(0,\vartheta)$, respectively. Define $u_t(\theta,x) = u(t+s,\theta,x), u_t(\vartheta,x) = u(t+s,\vartheta,x), -\infty < s \leqslant 0, t \geqslant 0$, that is $u_t(\theta,x)$, $u_t(\vartheta,x) \in PC(X)$ for all $t \geqslant 0$.

Let $U_p(t,\gamma,x) = u_p(t,\theta,x) - u_p(t,\vartheta,x), p = 1,2,\ldots,n, \gamma = \theta - \vartheta$, then

$$\begin{aligned}
\frac{\partial U_p(t,\gamma,x)}{\partial t} &= \sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{pk}\frac{\partial U_p(t,\gamma,x)}{\partial x_k}\right) - d_p(t)U_p(t,\gamma,x) \\
&\quad - \sum_{q=1}^{n} h_{pq}(t)[f_q(u_q(t,\theta,x)) - f_q(u_q(t,\vartheta,x))] \\
&\quad + \bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\theta,x)\right)\mathrm{d}s \\
&\quad - \bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\vartheta,x)\right)\mathrm{d}s \\
&\quad + \bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\theta,x)\right)\mathrm{d}s \\
&\quad - \bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\vartheta,x)\right)\mathrm{d}s
\end{aligned} \tag{11}$$

for $t \neq t_i,\ x \in X,\ p = 1,2,\ldots,n$.

Multiply both sides of (11) by $U_p(t,\gamma,x)$ and integrate it, one can obtain

$$\begin{aligned}
&\frac{d}{dt}\int_X \left(U_p(t,\gamma,x)\right)^2 dx \\
= \quad & 2\int_X U_p(t,\gamma,x)\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(D_{pk}\frac{\partial U_p(t,\gamma,x)}{\partial x_k}\right)dx - 2\int_X d_p(t)U_p^2(t,\gamma,x)dx \\
&+2\sum_{q=1}^{n} h_{pq}(t)\int_X U_p(t,\gamma,x)\left[f_q\left(u_q(t,\theta,x)\right)-f_q\left(u_q(t,\vartheta,x)\right)\right]dx \\
&+\int_X U_p(t,\gamma,x)\Bigg[\bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\theta,x)\right)\mathrm{d}s \\
&-\bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\vartheta,x)\right)\mathrm{d}s\Bigg]dx. \\
&+\int_X U_p(t,\gamma,x)\Bigg[\bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\theta,x)\right)\mathrm{d}s \\
&-\bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\vartheta,x)\right)\mathrm{d}s\Bigg]dx. \qquad (12)
\end{aligned}$$

Due to Greens formula and Dirichlet boundary conditions, one has

$$\int_X U_p(t,\gamma,x)\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(D_{pk}\frac{\partial U_p(t,\gamma,x)}{\partial x_k}\right)dx = -\sum_{k=1}^{m}\int_X D_{pk}\left(\frac{\partial\left(U_p(t,\gamma,x)\right)}{\partial x_k}\right)^2 dx.$$

From Lemma 2, we can obtain

$$\begin{aligned}
&\int_X U_p(t,\gamma,x)\sum_{k=1}^{m}\frac{\partial}{\partial x_k}\left(D_{pk}\frac{\partial U_p(t,\gamma,x)}{\partial x_k}\right)dx \\
&\leqslant -\sum_{k=1}^{m}\frac{D_{pk}}{\mathbb{L}_k^2}\int_\Xi \left(U_p(t,\gamma,x)\right)^2 dx \\
&= -\sum_{k=1}^{m}\frac{D_{pk}}{\mathbb{L}_k^2}\left\|U_p(t,\gamma,x)\right\|_2^2. \qquad (13)
\end{aligned}$$

According to (H1) and Hoder inequality, we have

$$\begin{aligned}
&\sum_{q=1}^{n} h_{pq}(t)\int_X U_p(t,\gamma,x)\left[f_q\left(u_q(t,\theta,x)\right)-f_q\left(u_q(t,\vartheta,x)\right)\right]dx \\
\leqslant \quad & \sum_{q=1}^{n}\left|h_{pq}(t)\right|\int_X\left|U_p(t,\gamma,x)\right|\left|f_q\left(u_q(t,\theta,x)\right)-f_q\left(u_q(t,\vartheta,x)\right)\right|dx \\
\leqslant \quad & \sum_{q=1}^{n}\left|h_{pq}(t)\right|\int_X\left|U_p(t,\gamma,x)\right|\left|U_q(t,\gamma,x)\right|F_q dx \\
\leqslant \quad & \sum_{q=1}^{n}\left|h_{pq}(t)\right|\left\|U_p(t,\gamma,x)\right\|_2 F_q\left\|U_q(t,\gamma,x)\right\|_2. \qquad (14)
\end{aligned}$$

By assumption (H1), Lemma 2 and Hoder inequality, one obtains

$$\begin{aligned}
&\int_X U_p(t,\gamma,x)\Bigg[\bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\theta,x)\right)\mathrm{d}s-\bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\vartheta,x)\right)\mathrm{d}s\Bigg]dx \\
&\leqslant\sum_{q=1}^{n}\left|a_{pq}(t)\right|G_q\int_{-\infty}^{t}\left|\mathcal{K}_{pq}(t-s)\right|\left\|U_p(t,\gamma,x)\right\|_2\left\|U_q(s,\gamma,x)\right\|_2 ds. \qquad (15)
\end{aligned}$$

By the same way, we can obtain

$$\int_X U_p(t,\gamma,x)\left[\bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\theta,x)\right)\mathrm{d}s-\bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t}\mathcal{K}_{pq}(t-s)g_q\left(u_q(s,\vartheta,x)\right)\mathrm{d}s\right]dx$$
$$\leqslant \sum_{q=1}^{n}\left|b_{pq}(t)\right|G_q\int_{-\infty}^{t}\left|\mathcal{K}_{pq}(t-s)\right|\left\|U_p(t,\gamma,x)\right\|_2\left\|U_q(s,\gamma,x)\right\|_2 ds. \tag{16}$$

Applying (12)–(16) to (11), we can obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|U_p(t,\gamma,x)\right\|_2^2 \leqslant &-\left(d_p(t)+\sum_{k=1}^{m}\frac{D_{pk}}{\mathbb{L}_k^2}\right)\left\|U_p(t,\gamma,x)\right\|_2^2\\ &+\sum_{q=1}^{n}\left|h_{pq}(t)\right|\left\|U_p(t,\gamma,x)\right\|_2 F_q\left\|U_q(t,\gamma,x)\right\|_2\\ &+\sum_{q=1}^{n}\left|a_{pq}(t)\right|G_q\int_{-\infty}^{t}\left|\mathcal{K}_{pq}(t-s)\right|\left\|U_p(t,\gamma,x)\right\|_2\left\|U_q(s,\gamma,x)\right\|_2 ds\\ &+\sum_{q=1}^{n}\left|b_{pq}(t)\right|G_q\int_{-\infty}^{t}\left|\mathcal{K}_{pq}(t-s)\right|\left\|U_p(t,\gamma,x)\right\|_2\left\|U_q(s,\gamma,x)\right\|_2 ds.\end{aligned}$$

i.e.

$$\begin{aligned}D^+\left\|U_p(t,\gamma,x)\right\|_2 \leq &-\left(d_p(t)+\sum_{k=1}^{m}\frac{D_{pk}}{\mathbb{L}_k^2}\right)\left\|U_p(t,\gamma,x)\right\|_2+\sum_{q=1}^{n}\left|h_{pq}(t)\right|F_q\left\|U_q(t,\gamma,x)\right\|_2\\ &+\sum_{q=1}^{n}\left[\left|a_{pq}(t)\right|+\left|b_{pq}(t)\right|\right]G_q\int_0^{+\infty}\left|\mathcal{K}_{pq}(s)\right|\left\|U_q(t-s,\gamma,x)\right\|_2 ds.\end{aligned} \tag{17}$$

Let $V_p(t)=\left\|U_p(t,\gamma,x)\right\|_2$, $V(t)=(V_1(t),V_2(t),\ldots,V_n(t))^T$, $w_p(t)=d_p(t)+\sum_{k=1}^{m}\left(D_{pk}/\mathbb{L}_k^2\right)$, $p=1,2,\ldots,n$, $W(t)=\operatorname{diag}(w_1(t)\ w_2(t),\ldots,w_n(t))$, $P(t)=-W(t)+H(t)F, R(t)=[A(t)+B(t)]G, \mathcal{K}(s)=(\mathcal{K}_{pq}(s))_{n\times n}$. Then, (17) can be simplified into the following form:

$$D^+V(t)\leqslant P(t)V(t)+\int_0^{+\infty}(R(t)\otimes|\mathcal{K}(s)|)V(t-s)ds \tag{18}$$

From condition (C1), There exist $\xi=(\xi_1,\xi_2,\ldots,\xi_n)^{\mathrm{T}}>0$ and $\lambda>0$, then

$$[\lambda E-W(t)+H(t)F+R(t)\otimes K(\lambda)]\xi<0, \tag{19}$$

Here, taking $\kappa=\|\gamma\|/\min_{1\leqslant p\leqslant n}\left\{\xi_p\right\}$, we have

$$V(t)\leqslant\kappa\xi e^{-\lambda t},\quad t\in[-\infty,t_0],\quad t_0=0. \tag{20}$$

From Lemma 4, one can obtain

$$V(t)\leqslant\kappa\xi e^{-\lambda t},\quad t\in[t_0,t_1).$$

If the following inequality is true for $m\leqslant i$

$$V(t)\leqslant\kappa\phi_0\cdots\phi_{m-1}\xi\mathrm{e}^{-\lambda t},\quad t\in[t_{m-1},t_m),\ \phi_0=1. \tag{21}$$

When $m = i + 1$, we can obtain

$$\begin{aligned} V(t_i) &= \|u(t_i, \theta, x) - u(t_i, \vartheta, x)\|_2 \\ &= \|\varphi_i\left(u(t_i^-, \theta, x)\right) - \varphi_i\left(u(t_i^-, \vartheta, x)\right)\|_2 \\ &\leqslant \Psi_i \|V(t_i^-, \gamma, x)\|_2 \\ &= \Psi_i V(t_i^-) \\ &\leqslant \kappa\phi_0 \cdots \phi_{i-1}\phi_i \xi \lim_{t \to t_i^-} \mathrm{e}^{-\lambda t_i} \\ &\leqslant \kappa\phi_0 \cdots \phi_{i-1}\phi_i \xi \mathrm{e}^{-\lambda t_i}. \end{aligned} \tag{22}$$

By (21), (22) and $\phi_i \geqslant 1$, we have

$$V(t) \leqslant \kappa\phi_0 \cdots \phi_{i-1}\phi_i \xi \mathrm{e}^{-\lambda t}, \quad -\infty \leqslant t \leqslant t_i. \tag{23}$$

Combining (19),(20),(23) and Lemma 3, one has

$$V(t) \leqslant \kappa\phi_0 \cdots \phi_{i-1}\phi_i \xi \mathrm{e}^{-\lambda t}, \quad t_i \leqslant t < t_{i+1}. \tag{24}$$

According to mathematical induction, then we have

$$V(t) \leqslant \kappa\phi_0 \cdots \phi_{i-1} \xi \mathrm{e}^{-\lambda t}, \quad t_{i-1} \leqslant t < t_i, \quad i \in N. \tag{25}$$

applying the condition (C2) and (25), one obtain

$$V(t) \leqslant \kappa \mathrm{e}^{\phi t_1} \mathrm{e}^{\phi(t_2 - t_1)} \cdots \mathrm{e}^{\phi(t_{i-1} - t_{i-2})} \xi \mathrm{e}^{-\lambda t} \leqslant \kappa \xi \mathrm{e}^{\phi t} \mathrm{e}^{-\lambda t} = \kappa \xi \mathrm{e}^{-(\lambda - \phi)t} \tag{26}$$

for all $i \in N, \quad t_{i-1} \leqslant t < t_i$.

This implies that

$$\begin{aligned} \|u(t, \theta, x) - u(t, \vartheta, x)\| &= \sum_{p=1}^{n} \left\|u_p(t, \theta, x) - u_p(t, \vartheta, x)\right\|_2 \\ &= \sum_{p=1}^{n} V_p(t) \\ &\leqslant \sum_{p=1}^{n} \kappa \xi_p e^{-(\lambda - \phi)t} \\ &= \frac{\sum\limits_{p=1}^{n} \xi_p}{\min\limits_{1 \leqslant p \leqslant n} \left\{\xi_p\right\}} \|\theta - \vartheta\| e^{-(\lambda - \phi)t}. \end{aligned}$$

That is,

$$\|u_t(\theta, x) - u_t(\vartheta, x)\| \leqslant M \|\theta - \vartheta\| e^{-(\lambda - \phi)t}, \quad t \geqslant 0, \tag{27}$$

where $M = \left(\sum\limits_{p=1}^{n} \xi_p / \min\limits_{1 \leqslant p \leqslant n} \left\{\xi_p\right\}\right)$. ■

Remark 1: Condition (C1) is equivalent to that $\Upsilon(t) = W(t) - H(t)F - R(t) \otimes K(0)$ is a nonsingular M-matrix for all $t \geqslant 0$. As a matter of fact, if $\Upsilon(t)$ is a nonsingular M-matrix for any $t \geqslant 0$, by using Lemma 1, there exist $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} > 0$ such that

$$[W(t) - H(t)F - R(t) \otimes K(0)]\xi > 0. \tag{28}$$

By the uniform continuity, there exists $\lambda > 0$ satisfies:

$$[\lambda E - W(t) + H(t)F + R(t) \otimes K(\lambda)]\xi < 0. \tag{29}$$

It tells us that (C1) is true. Reversely, setting $\lambda = 0$ in (C1), we can easily get that $\Upsilon(t) = W(t) - H(t)F - R(t) \otimes K(0)$ is a nonsingular M-matrix for all $t \geqslant 0$.

Corollary 1: If assumptions $(H1), (H3)$ and $(H4)$ are satisfied, suppose that condition $(C1)$ holds. Then systems $(2a) - (2c)$ are globally exponentially stable.

Corollary 2: When the coefficient of system $(1a) - (1d)$ are constants, they degenerate into the following autonomous FCNNs with reaction-diffusion terms and distribution delays

$$\begin{aligned}\frac{\partial u_p(t,x)}{\partial t} = &\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(D_{pk}\frac{\partial u_p(t,x)}{\partial x_k}\right) - d_p u_p(t,x) + \sum_{q=1}^{n} h_{pq} f_q\left(u_q(t,x)\right) \\ &+ \sum_{q=1}^{n} y_{pq} v_q + J_p(t) + \bigwedge_{q=1}^{n} a_{pq}\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s) g_q\left(u_q(s,x)\right) \mathrm{d}s \\ &+ \bigwedge_{q=1}^{n} \alpha_{pq} v_q + \bigvee_{q=1}^{n} b_{pq}\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s) g_q\left(u_q(s,x)\right) \mathrm{d}s \\ &+ \bigvee_{q=1}^{n} \beta_{pq} v_q, \quad t \neq t_i, \quad x \in X, \end{aligned} \tag{30a}$$

$$u_p(t_i^+, x) = \phi_{pi}(u_p(t_i^-, x)), \quad x \in X, \quad i \in N \triangleq \{0, 1, 2, \cdots\}, \tag{30b}$$

$$u_p(t,x) = 0, x \in \partial X, \tag{30c}$$

$$u_p(s,x) = \gamma_p(s,x), \quad s \in [-\infty, 0], \tag{30d}$$

For assumption $(H1), (H2)$ and $(H4)$, Theorem 1 can be expressed in the following form:

$(C'1)$ There exist $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} > 0$ and $\lambda > 0$ such that

$$[\lambda E - W + HF + R \otimes K(\lambda)]\xi < 0,$$

where $W = \operatorname{diag}(w_1, w_2, \ldots, w_n)$ with $w_p = d_p + \sum_{k=1}^{m} \frac{D_{pk}}{\mathbb{L}_k^2}$, $R = (A+B)G$, $A = \left(\left|a_{pq}\right|\right)_{n\times n}$, $B = \left(\left|b_{pq}\right|\right)_{n\times n}$, $H = \left(\left|h_{pq}\right|\right)_{n\times n}$, $R = (r_{pq})_{n\times n}$, $K(\lambda) = (\mathbb{K}_{pq}(\lambda))_{n\times n}$, $F = \operatorname{diag}(F_1, F_2, \cdots, F_n)$, $G = \operatorname{diag}(G_1, G_2, \cdots, G_n)$;

$(C2)$ There exists a constant $\phi > 0$ such that

$$\sup_{i\in N}\left\{\frac{\ln \phi_i}{t_i - t_{i-1}}\right\} \leqslant \phi < \lambda,$$

where $\phi_i = \max_{1\leqslant p\leqslant n}\left\{1, \varphi_{pi}\right\}, i \in N$;

then systems $(30a) - (30d)$ are globally exponentially stable.

Remark 2: Some existing neural network models (see [21,26]) are special cases such as systems $(2a) - (2c)$ and systems $(30a) - (30d)$. Compared with the methods of constructing Lyapunov functional in [21], our results is more concise, and it is not difficult to find that some of the standards have been improved. Moreover, in [26], the authors gave sufficient conditions for the existence of uniqueness and global exponential stability of the equilibrium point of impulsive FCNNs with distributed time delays and reaction-diffusion terms, but we have to say that the method we using is similar.

Next, in order to consider the periodic solution of the systems $(1a) - (1d)$, we add the following two assumptions.

(H5) $d_p(t), h_{pq}(t), a_{pq}(t), b_{pq}(t), \alpha_{pq}(t), \beta_{pq}(t), y_{pq}(t), v_q(t)$ and $J_p(t)$ are periodic continuous functions with a common period $\varpi > 0$ for all $t \geqslant 0$.

(H6) For $\Psi_i = \mathrm{diag}\,(\varphi_{1i}, \ldots, \varphi_{ni})$ and the impulsive time $\{t_i\}_{i\in N}$, there exists a positive integer l such that

$$\varphi_{p(i+l)} = \varphi_{pi}, \quad t_{i+l} = t_i + \varpi.$$

Combined with assumptions (H5) and (H6), we have the following results for periodic systems $(1a)-(1d)$, based on the discussion of global exponential stability of the systems in Theorem 1.

Theorem 2: If assumptions (H1)–(H6) are satisfied, suppose that

(C1) There exist $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^{\mathrm{T}} > 0$ and $\lambda > 0$ such that

$$[\lambda E - W(t) + H(t)F + R(t) \otimes K(\lambda)]\xi < 0, \quad t \geqslant 0,$$

where $W(t) = \mathrm{diag}\,(w_1(t), w_2(t), \ldots, w_n(t))$ with $w_p(t) = d_p(t) + \sum_{k=1}^{m} \frac{D_{pk}}{\mathbb{L}_k^2}, \quad R(t) = [A(t) + B(t)]G$,
$A(t) = \left(\left|a_{pq}(t)\right|\right)_{n\times n}, \quad B(t) = \left(\left|b_{pq}(t)\right|\right)_{n\times n}, \quad H(t) = \left(\left|h_{pq}(t)\right|\right)_{n\times n}, \quad R(t) = (r_{pq}(t))_{n\times n}, \quad K(\lambda) = (\mathbb{K}_{pq}(\lambda))_{n\times n}$,
$F = \mathrm{diag}(F_1, F_2, \cdots, F_n), \quad G = \mathrm{diag}(G_1, G_2, \cdots, G_n)$;

(C2) There exists a constant $\phi > 0$ such that

$$\sup_{i\in N}\left\{\frac{\ln \phi_i}{t_i - t_{i-1}}\right\} \leqslant \phi < \lambda,$$

where $\phi_i = \max_{1\leqslant p\leqslant n}\left\{1, \varphi_{pi}\right\}, i \in N$;

then systems $(1a)-(1d)$ have exactly one globally exponentially stable ϖ-periodic solution.

Proof: To choose a positive integer $\eta > 0$ such that $Me^{-(\lambda-\phi)\eta\varpi} \leqslant \frac{1}{2}$ and we define a Poincare mapping

$$\Gamma : PC(X) \longrightarrow PC(X) \qquad \text{by} \qquad \Gamma(\theta) = u_{\varpi}(\theta, x),$$

it follows that $\Gamma^{\eta}(\theta) = u_{\eta\varpi}(\theta, x)$. Setting $t = \eta\varpi$, we get

$$\|\Gamma^{\eta}(\theta) - \Gamma^{\eta}(\vartheta)\| \leq \frac{1}{2}\|\theta - \vartheta\|.$$

Obviously, Γ^{η} is a contraction mapping, therefore there exists one unique fixed point $\theta^* \in PC(X)$ such that

$$\Gamma^{\eta}(\theta^*) = \theta^*.$$

Hence, we obtain

$$\Gamma^{\eta}(\Gamma(\theta^*)) = \Gamma(\Gamma^{\eta}(\theta^*)) = \Gamma(\theta^*),$$

this implies that $\Gamma(\theta^*) \in PC(X)$ is also a fixed point of Γ^{η}. Then

$$\Gamma(\theta^*) = \theta^*, \quad \text{i.e.} \quad u_{\theta}(\theta^*, x) = \theta^*.$$

Hence, if $u(t, \theta^*, x)$ is a solution of system (1) through $(0, \theta^*)$, then $u(t + \varpi, \theta^*, x)$ is also a solution to systems $(1a) - (1d)$. Distinctly,

$$u_{t+\varpi}(\theta^*, x) = u_t(u_{\varpi}(\theta^*, x)) = u_t(\theta^*, x), \quad t \geqslant 0,$$

i.e.

$$u(t + \varpi, \theta^*, x) = u(t, \theta^*, x).$$

This shows that $u(t,\theta^{*},x)$ has one solution for systems $(1a)-(1d)$ with ϖ-period and all other solution of systems $(1a)-(1d)$ converge exponentially to it as $t \to +\infty$. ■

Remark 3: In Theorem 1 and Theorem 2, the condition (C2) ($\phi = \sup_{i\in N}\{\ln\phi_i/t_i - t_{i-1}\}$) describes the influence of the impulsive intensity and the impulsive interval on the global exponential stability of systems $(1a)-(1d)$. In the absence of impulses, the following optimization problem can be solved in order to estimate the exponential convergence rate of the systems $(1a)-(1d)$.

$$\text{(OP)} \begin{cases} \max \lambda \\ \text{s.t. } (C1) \text{ holds.} \end{cases}$$

Obviously, λ is related to delay kernel function, diffusion coefficients, Dirichlet boundary conditions, and system parameters. Theorem 1 shows that when $\phi \in [0,\lambda)$, systems $(1a)-(1d)$ is globally exponentially stable and its exponential convergence rate equals $\lambda-\phi$.

Corollary 3: Under assumptions $(H1),(H2),(H4)$ and $(H5)$,suppose that

(C1) There exist $\xi = (\xi_1,\xi_2,\ldots,\xi_n)^{\mathrm{T}} > 0$ and $\lambda > 0$ such that

$$[\lambda E - W(t) + H(t)F + R(t)\otimes K(\lambda)]\xi < 0, \quad t \geqslant 0,$$

where $W(t) = \operatorname{diag}(w_1(t),w_2(t),\ldots,w_n(t))$ with $w_p(t) = d_p(t) + \sum_{k=1}^{m}\frac{D_{pk}}{\mathbb{L}_k^2}$, $R(t) = [A(t)+B(t)]G$, $A(t) = \left(\left|a_{pq}(t)\right|\right)_{n\times n}$, $B(t) = \left(\left|b_{pq}(t)\right|\right)_{n\times n}$, $H(t) = \left(\left|h_{pq}(t)\right|\right)_{n\times n}$, $R(t) = (r_{pq}(t))_{n\times n}$, $K(\lambda) = (\mathbb{K}_{pq}(\lambda))_{n\times n}$, $F = \operatorname{diag}(F_1,F_2,\cdots,F_n)$, $G = \operatorname{diag}(G_1,G_2,\cdots,G_n)$;

then systems $(2a)-(2c)$ have exactly one globally exponentially stable ϖ-periodic solution.

Remark 4: If $D_{pk} = 0$ or $D_{pk} \ll \mathbb{L}_k$, that is, when the diffusion effect is negligible, systems $(1a)-(1d)$ will degenerate into a non-autonomous FCNN with distributed delays. By Theorem 2, We can obtain a corollary for the global exponentially stable periodic solution of the systems $(1a)-(1d)$.

Corollary 4: If assumptions (H1)–(H6) are satisfied, suppose that

$(C''1)$ There exist $\xi = (\xi_1,\xi_2,\ldots,\xi_n)^{\mathrm{T}} > 0$ and $\lambda > 0$ such that

$$[\lambda E - D(t) + H(t)F + R(t)\otimes K(\lambda)]\xi < 0, \quad t \geqslant 0,$$

where $D(t) = \operatorname{diag}(d_1(t),d_2(t),\ldots,d_n(t))$, $R(t) = [A(t)+B(t)]G$, $A(t) = \left(\left|a_{pq}(t)\right|\right)_{n\times n}$, $B(t) = \left(\left|b_{pq}(t)\right|\right)_{n\times n}$, $H(t) = \left(\left|h_{pq}(t)\right|\right)_{n\times n}$, $R(t) = (r_{pq}(t))_{n\times n}$, $K(\lambda) = (\mathbb{K}_{pq}(\lambda))_{n\times n}$, $F = \operatorname{diag}(F_1,F_2,\cdots,F_n)$, $G = \operatorname{diag}(G_1,G_2,\cdots,G_n)$;

$(C2)$ There exists a constant $\phi > 0$ such that

$$\sup_{i\in N}\left\{\frac{\ln\phi_i}{t_i - t_{i-1}}\right\} \leqslant \phi < \lambda,$$

where $\phi_i = \max_{1\leqslant p\leqslant n}\left\{1,\varphi_{pi}\right\}, i \in N$;

then systems $(1a)-(1d)$ have exactly one globally exponentially stable ϖ-periodic solution.

IV. Illustrative examples

Finally, two examples are given to verify the validity and universality of our results.

Example 1: Consider the following two-neuron impulsive system:

$$\frac{\partial u_p(t,x)}{\partial t} = \sum_{k=1}^{1} \frac{\partial}{\partial x_k}\left(D_{pk}\frac{\partial u_p(t,x)}{\partial x_k}\right) - d_p(t)u_p(t,x) + \sum_{q=1}^{n} h_{pq}(t) f_q\left(u_q(t,x)\right)$$
$$+ \sum_{q=1}^{n} y_{pq}(t)v_q(t) + J_p(t) + \bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s) g_q\left(u_q(s,x)\right)\mathrm{d}s$$
$$+ \bigwedge_{q=1}^{n} \alpha_{pq}(t)v_q(t) + \bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s) g_q\left(u_q(s,x)\right)\mathrm{d}s$$
$$+ \bigvee_{q=1}^{n} \beta_{pq}(t)v_q(t), \quad t \neq t_i, \quad x \in X \tag{1a}$$

$$u_p(t_i^+, x) = 2.5u_p(t_i^-, x), \quad x \in X, \quad i \in N \triangleq \{0, 1, 2, \cdots\}, \tag{1b}$$

$$u_p(t,x) = 0, \quad t \geq 0, \quad x \in \partial X, \tag{1c}$$

$$u_p(s,x) = 1, \quad s \in [-\infty, 0], \tag{1d}$$

for $p = 1, 2$, where $X = [0, 1], t_i = 1.5\pi i$,$i = 1, 2, \cdots$.

In system(1a)-(1d), choosing $D_{11} = D_{21} = 1$, $d_1(t) = 5.8+|sint|$, $d_2(t) = 6.9+|sint|$, $h_{11}(t) = y_{11}(t) = y_{12}(t) = y_{21}(t) = y_{22}(t) = 0.25sint$, $h_{12}(t) = -0.2 - 0.3cost$, $h_{21}(t) = 0.5 + 0.1sint$, $h_{22}(t) = 0.6 + 0.2cost$, $a_{11}(t) = b_{11}(t) = 0.1 + 0.5cost$, $a_{12}(t) = b_{12}(t) = 0.6cost$, $a_{22}(t) = b_{22}(t) = 0.7sint$, $a_{21}(t) = b_{21}(t) = 0.8sint$, $\mathcal{K}_{11}(s) = \mathcal{K}_{12}(s) = \mathcal{K}_{21}(s) = \mathcal{K}_{22}(s) = e^{-s}$, $v_1(t) = v_2(t) = 1 + 0.25sint$, $\alpha_{11}(t) = \alpha_{21}(t) = cost$, $\alpha_{21}(t) = \alpha_{22}(t) = sint$, $\beta_{11}(t) = \beta_{12}(t) = sint + 0.8$, $\beta_{21}(t) = \beta_{22}(t) = 2cost - 0.5$, $J_1(t) = 0.3cost$, $J_2(t) = 0.6sint$, $f_1(u) = f_2(u) = g_1(u) = g_2(u) = tanhu$.

Distinctly, assumptions (H1)-(H6) are satisfied, and then we have

$$W(t) = \begin{pmatrix} 6.8 + |sint| & 0 \\ 0 & 7.9 + |sint| \end{pmatrix}, \quad F = G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$R(t) = \begin{pmatrix} |0.2 + cost| & |1.2cost| \\ |1.6sint| & |1.4sint| \end{pmatrix},$$

$$H(t) = \begin{pmatrix} |0.25sint| & |-0.2 - 0.3cost| \\ |0.5 + 0.1sint| & |0.6 + 0.2cost| \end{pmatrix}, \quad K(\lambda) = \begin{pmatrix} \frac{1}{1-\lambda} & \frac{1}{1-\lambda} \\ \frac{1}{1-\lambda} & \frac{1}{1-\lambda} \end{pmatrix}.$$

$\lambda \in [0, 1)$, $\phi_i = \max\{1, 2.5\} = 2.5$, $\gamma = \sup_{i\in N} \frac{\ln \gamma_i}{t_i - t_{i-1}} = \frac{\ln 2.5}{1.5\pi} \approx 0.1944$.

By solving the optimization problem:

$$\text{(OP)} \begin{cases} \max \lambda \\ \text{s.t. } (C1) \text{ holds.} \end{cases}$$

ones can obtain that $\lambda \approx 0.4931$ and $\xi = (2745063, 3715964) > 0$. Thus, we know that the systems $(1a)-(1d)$ have exactly one globally exponentially stable 2π-periodic solution and the estimation of its exponential convergence rate is $\lambda - \phi \approx 0.4931 - 0.1944 \approx 0.2987$.

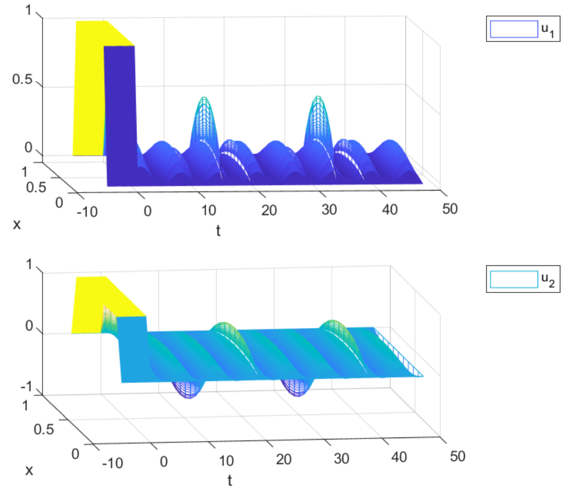

Fig. 1: 2π-periodic solutions of impulsive systems $(1a)-(1d)$ in $x \in [0,1]$ and $t \in [-5, 50]$.

Remark 5: However, the results in [21] do not contain the impulsive perturbation, and if the parameters in [21] are taken and the impulsive condition of Example 1 is added to solve the above optimization problem, we get $\lambda \approx 0.4669$ and $\xi = (9218617, 9668710) > 0$. From Theorem 1, systems $(1a)-(1d)$ are globally exponentially stable and the exponential convergence rate is estimated as $\lambda - \phi \approx 0.4669 - 0.1944 \approx 0.2725$.

Example 2: Consider the reaction-diffusion two-neuron system without impulses:

$$\begin{aligned}\frac{\partial u_p(t,x)}{\partial t} &= \sum_{k=1}^{1} \frac{\partial}{\partial x_k}\left(D_{pk}\frac{\partial u_p(t,x)}{\partial x_k}\right) - d_p(t)u_p(t,x) + \sum_{q=1}^{n} h_{pq}(t)f_q\left(u_q(t,x)\right) \\ &+ \sum_{q=1}^{n} y_{pq}(t)v_q(t) + J_p(t) + \bigwedge_{q=1}^{n} a_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s)g_q\left(u_q(s,x)\right)\mathrm{d}s \\ &+ \bigwedge_{q=1}^{n} \alpha_{pq}(t)v_q(t) + \bigvee_{q=1}^{n} b_{pq}(t)\int_{-\infty}^{t} \mathcal{K}_{pq}(t-s)g_q\left(u_q(s,x)\right)\mathrm{d}s \\ &+ \bigvee_{q=1}^{n} \beta_{pq}(t)v_q(t), \quad t \neq t_i, \quad x \in X \end{aligned} \tag{2a}$$

$$u_p(t,x) = 0, \quad t \geq 0, \quad x \in \partial X, \tag{2b}$$

$$u_p(s,x) = e^s sin(\pi x), \quad s \in [-\infty, 0], \tag{2c}$$

for $p = 1, 2$, $X = [0, 1]$.

In system $(2a)-(2c)$, choosing $D_{11} = D_{21} = 1$, $d_1(t) = 3 + |sint|$, $d_2(t) = 2 + |cost|$, $h_{11}(t) = y_{11}(t) = y_{12}(t) = y_{21}(t) = y_{22}(t) = sint$, $h_{21}(t) = h_{12}(t) = 0$ $h_{22}(t) = cost$, $a_{11}(t) = b_{11}(t) = 0.15cost$, $a_{12}(t) = b_{12}(t) = 0.1cost$, $a_{22}(t) = b_{22}(t) = 0.1sint$, $a_{21}(t) = b_{21}(t) = 0.1sint$, $\mathcal{K}_{11}(s) = \mathcal{K}_{12}(s) = \mathcal{K}_{21}(s) = \mathcal{K}_{22}(s) = e^{-s}$, $v_1(t) = v_2(t) = sint$, $\alpha_{11}(t) = \alpha_{21}(t) = cost$, $\alpha_{21}(t) = \alpha_{22}(t) = sint$, $\beta_{11}(t) = \beta_{12}(t) = sint$, $\beta_{21}(t) = \beta_{22}(t) = cost$, $J_1(t) = cost$, $J_2(t) = sint$, $f_1(u) = f_2(u) = g_1(u) = g_2(u) = \dfrac{|u+1| - |u-1|}{2}$.

Distinctly, assumptions (H1) and (H3)-(H5) are satisfied, and then we have

$$W(t) = \begin{pmatrix} 4 + |sint| & 0 \\ 0 & 3 + |sint| \end{pmatrix}, \quad F = G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R(t) = \begin{pmatrix} |0.3cost| & |0.2cost| \\ |0.2sint| & |0.2sint| \end{pmatrix},$$

$$H(t) = \begin{pmatrix} |sint| & 0 \\ 0 & |cost| \end{pmatrix}, \quad K(\lambda) = \begin{pmatrix} \frac{1}{1-\lambda} & \frac{1}{1-\lambda} \\ \frac{1}{1-\lambda} & \frac{1}{1-\lambda} \end{pmatrix}.$$

ones can obtain that $\lambda \approx 0.6659$ and $\xi = (15354274, 18850909) > 0$. Thus, the systems $(2a) - (2c)$ have exactly one globally exponentially stable 2π-periodic solution.

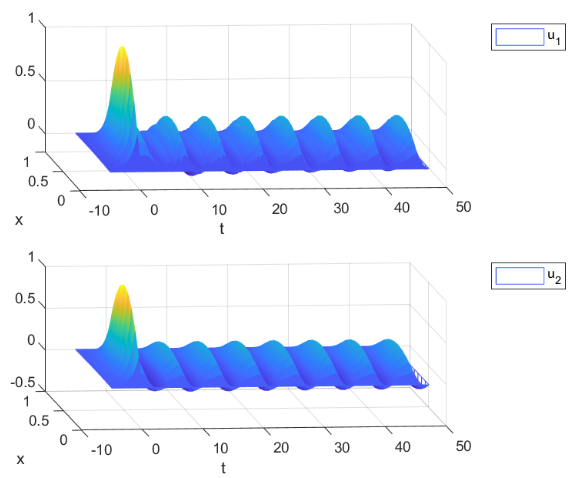

Fig. 2: 2π-periodic solutions of system $(2a) - (2c)$ without impulses in $x \in [0, 1]$ and $t \in [-5, 50]$.

V. Conclusion

We have developed and studied a new class of neural network models that bring together nonautonomous neural networks, fuzzy neural networks, reaction-diffusion terms, distributed time delays, impulses, and Dirichlet boundary conditions. In the form of a simple algebraic inequality, several new sufficient conditions are obtained to guarantee the global exponential stability and periodicity of the systems $(1a) - (1d)$. In particular, in order to estimate the exponential convergence rate of the systems $(1a)-(1d)$, an optimisation method is proposed which relies on diffusion coefficients, Dirichlet boundary conditions, distributed time delays, system parameters and impulses. The method is simpler and more effective than the Lyapunov generalized method used in much of the previous literature for the stability and periodicity analysis of complex systems. Two examples show that our results improve and generalize previously known criteria. In the near future, we will continue to investigate the global exponential stability and periodicity of nonautonomous impulsive neural networks with leakage time delays.

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