On the Number of k-Crossing Partitions

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1 Abstract

I introduce k-crossing paths and partitions and count the number of paths for each number of desired crossings k for systems with 11 points or less. I give some conjectures into the number of possible paths for certain numbers of crossings as a function of the number of points.

2 Introduction

A order n meandric partition is a set of the integers $1 \cdots n$, such that a path from the south-west can weave through n points labeled $1 \cdots n$ without intersecting itself and finally heads east (examples are shown in Fig. 1). Counting the number of possible paths for n points is a tricky problem, and no recursion relation, generating function or explicit formula for the number of order n meandric partitions appears to have been found. This work is concerned with the number of paths that must intersect themselves exactly k times, where when k is 0, we have the meandric paths. It is possible to draw a line that deliberately crosses itself as many times as required, because of this we only consider a path to be k-crossing if k is the smallest number of crossings possible, that is a path that must cross itself k times (an example of a 3-crossing path over 9 points is given in Fig. 2).

3 Results

Define $a_k(n)$ to be the number of configurations of n points where the path through them is forced to cross itself k times. For 0-crossings on n points we have the open meandric numbers, given in the OEIS as A005316

$$a_0(n) = 1, 1, 1, 2, 3, 8, 14, 42, 81, 262, 538, 1828, 3926, \cdots, \quad n = 0, 1, \cdots \tag{1}$$

this work has counted this for $k > 0$ by calculating all $n!$ permutations of the n integers and checking to see the minimal number of crossings for each, we then have

$n =$	0	1	2	3	4	5	6	7	8
9	10	$11\cdots$							
$a_0(n) =$	1,	1,	1,	2,	3,	8,	14,	42,	81,
262,	538,	$1828,\cdots$							
$a_1(n) =$	0,	0,	1,	4,	10,	36,	85,	312,	737,
2760,	6604,	$25176,\cdots$							
$a_2(n) =$	0,	0,	0,	0,	8,	42,	168,	760,	2418,
10490,	30842,	$131676,\cdots$							
$a_3(n) =$	0,	0,	0,	0,	2,	16,	164,	944,	4386,
22240,	83066,	$398132,\cdots$							
$a_4(n) =$	0,	0,	0,	0,	1,	18,	146,	1076,	6255,
37250,	168645,	$908898,\cdots$							
$a_5(n) =$	0,	0,	0,	0,	0,	0,	96,	960,	7388,
51968,	282122,	$1711824,\cdots$							
$a_6(n) =$	0,	0,	0,	0,	0,	0,	30,	440,	6472,
55140,	384065,	$2642444,\cdots$							
$a_7(n) =$	0,	0,	0,	0,	0,	0,	14,	368,	5176,
53920,	455944,	$3575040,\cdots$							
$a_8(n) =$	0,	0,	0,	0,	0,	0,	2,	66,	3542,
45960,	484058,	$4336734,\cdots$							
$a_9(n) =$	0,	0,	0,	0,	0,	0,	1,	72,	2011,
32280,	452504,	$4661756,\cdots$							
$a_{10}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	1172,
25066,	396493,	$4709856,\cdots$							
$a_{11}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	420,
11840,	309696,	$4291440,\cdots$							
$a_{12}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	201,
8930,	225754,	$3661348,\cdots$							
$a_{13}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	40,
2240,	151849,	$2947392,\cdots$							
$a_{14}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	18,
2040,	91147,	$2103648,\cdots$							
$a_{15}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	2,
224,	55030,	$1575744,\cdots$							
$a_{16}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	1,
270,	26762,	$915924,\cdots$							
$a_{17}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	0,
0,	14627,	$665088,\cdots$							
$a_{18}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	0,
0,	5405,	$295956,\cdots$							
$a_{19}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	0,
0,	2642,	$218508,\cdots$							
$a_{20}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	0,
0,	641,	$63522,\cdots$							
$a_{21}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	0,
0,	293,	$54672,\cdots$							
$a_{22}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	0,
0,	48,	$8964,\cdots$							
$a_{23}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	0,
0,	22,	$9552,\cdots$							
$a_{24}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	0,
0,	2,	$706,\cdots$	2						
$a_{25}(n) =$	0,	0,	0,	0,	0,	0,	0,	0,	0,
0,	1,	$972,\cdots$							

(2)

where the vertical sum over columns of terms gives $n!$.

4 Conjectures

The above information has lead to a few conjectures.

$$\text{Conjecture 1: } a_{n^2}(2n) = 1 \tag{3}$$

this can be converted to words as, there is exactly one path through $2n$ points that crosses n^2 times. The partitions associated with these paths are

$$(2,1) \tag{4}$$
$$(3,1,4,2) \tag{5}$$
$$(4,1,5,2,6,3) \tag{6}$$
$$(5,1,6,2,7,3,8,4) \tag{7}$$
$$(6,1,7,2,8,3,9,4,10,5) \tag{8}$$

and a clear interlaced pattern can be seen (an example is given in Fig. 3).

$$\text{Conjecture 2: } a_{n^2-1}(2n) = 2,\ n > 1 \tag{9}$$

$$\text{Conjecture 3: } a_{n^2-2}(2n) = 4n + 2,\ n > 2 \tag{10}$$

$$\text{Conjecture 4: } a_{n^2-3}(2n) = 8n + 8,\ n > 3 \tag{11}$$

$$\text{Conjecture 5: } a_{n^2}(2n+1) = 2(n+1)3^{n-1},\ n > 1 \tag{12}$$

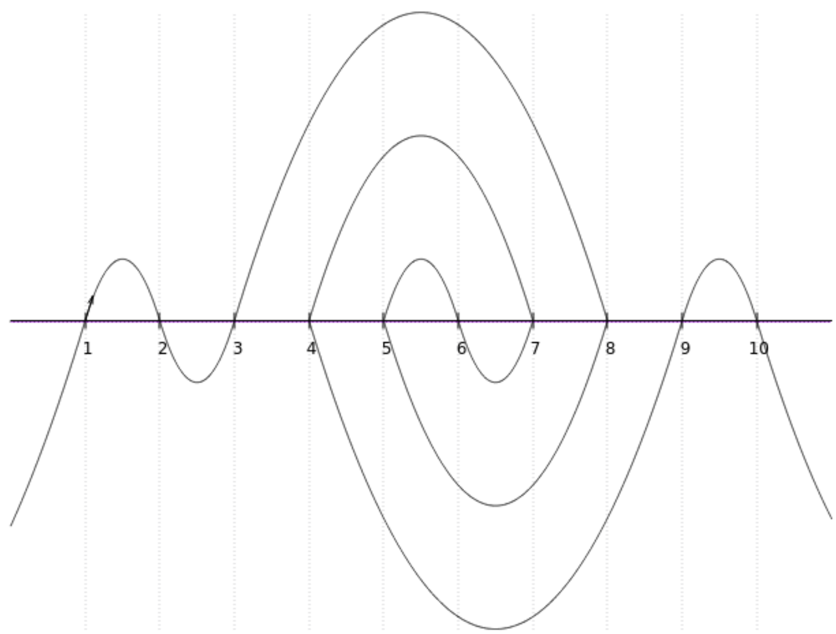

Figure 1: Fig 1. An example of a meandric path (0-crossing path) with $n = 10$. This path has meandric partition $(1, 2, 3, 8, 5, 6, 7, 4, 9, 10)$.

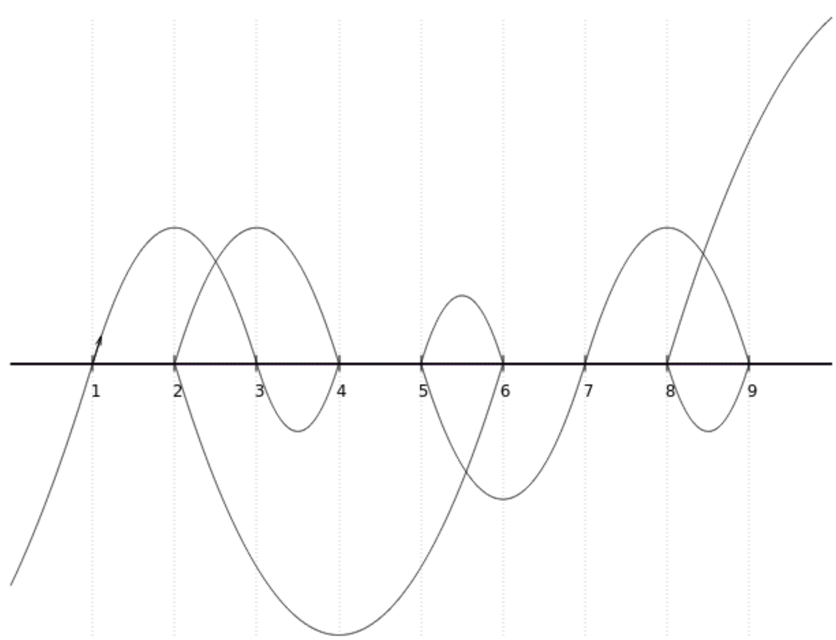

Figure 2: Fig 2. An example of a path that crosses itself 3 times on 9 points. This path has partition $(1, 3, 4, 2, 6, 5, 7, 9, 8)$.

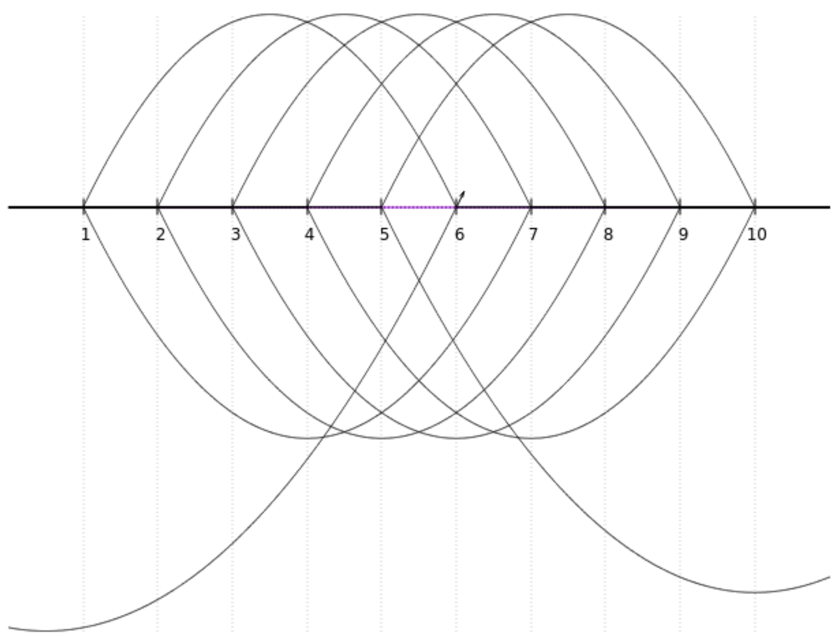

Figure 3: Fig 3. An example of a maximally crossing partition, for 10 points, there is only 1 solution, and Conjecture 1 predicts there is only 1 partition of this type for any even number of points. The number of crossings at the top of the line is a triangular number 10 and the number of crossings at the bottom is the next triangular number 15. This partition is $(6, 1, 7, 2, 8, 3, 9, 4, 10, 5)$.