Multiplicative Iteration On Infinite Products

Benedict Irwin[1]

[1]University of Cambridge

January 7, 2021

We have the product

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\frac{2}{3} \tag{1}$$

if we take the prodegrand and apply a multiplicative derivative defined by

$$f^*(x)=\exp\left(\frac{f'(x)}{f(x)}\right) \tag{2}$$

then we get

$$\prod_{n=2}^{\infty}\exp\left(\frac{6n^2}{n^6-1}\right)\sim 1.64872 \tag{3}$$

This converges to

$$\lim_{m\to\infty}\exp\left(\frac{m^4+2m^3+2m^2-3m-2}{2m(m+1)(m^2+m+1)}\right)=\sqrt{e} \tag{4}$$

likewise for

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\pi\mathrm{csch}\pi \tag{5}$$

we get

$$\prod_{n=2}^{\infty}\exp\left(\frac{4n}{n^4-1}\right)=\exp(-3/2+2\gamma+\psi_0(2-i)+\psi_0(2+i)) \tag{6}$$

we can take the product

$$\prod_{k=1}^{\infty}\frac{(1+k^{-1})^2}{(1+2k^{-1})}=2 \tag{7}$$

and find

$$\prod_{k=1}^{\infty}\exp\left(\frac{-2}{k(k+1)(k+2)}\right)=\frac{1}{\sqrt{e}} \tag{8}$$

in the same line of thought we have

$$\prod_{n=2}^{\infty}\exp\left(\frac{8n^3}{n^8-1}\right)=A \tag{9}$$

$$A=\exp\left(\sum_{\pm}-\frac{3}{2}+2\gamma-\psi_0(2\pm i)+\psi_0(2\pm(-1)^{1/4})+\psi_0(2\pm(-1)^{3/4})\right) \tag{10}$$

it seems that

$$Q_k = \prod_{n=2}^{\infty} \exp\left(\frac{(2k)n^{k-1}}{n^{2k}-1}\right) = \exp\left(\frac{3}{2} + \sum_{\pm}\sum_{l=1}^{k-1} \psi_0(2 \pm (-1)^{l/k})\right) \tag{11}$$

List of Products			
$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}$	$\frac{2}{3}$	$\prod_{n=2}^{\infty} \exp\left(\frac{6n^2}{n^6-1}\right)$	$\sqrt{e}$
$\prod_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n e^{1/(2n)-1}$	$\frac{e^{1+\gamma/2}}{\sqrt{2\pi}}$	$\prod_{n=1}^{\infty}$	$e^{1-\gamma-\pi^2/12}$