Riemann Zeta Invariance Under Composed Integral Transform

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From a question I asked online [1], I had deduced that the Laplace transform could be absorbed into the inverse Mellin transform as

$$\mathcal{L}\mathcal{M}^{-1}[\phi] = -\mathcal{M}^{-1}[\phi^*] \tag{1}$$

and the Mellin transform could be absorbed into the inverse Laplace transform as

$$\mathcal{M}\mathcal{L}^{-1}[\psi] = \Gamma(q)\mathcal{L}^{-1}[\psi^*] \tag{2}$$

where

$$\phi^* = \Gamma(t)\phi(1-t) \tag{3}$$

and

$$\psi^* = \psi(-e^{-s})e^{-s} \tag{4}$$

the term of $\phi(1-t)$ reminded me of the Riemann function equation for $\zeta(s)$ which is

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{5}$$

the question I was then interested in was **what other transform when applied to the inverse Mellin transform of a function, would result in this functional equation**, or what is the transform such that $\zeta(s)$ is invariant to?

The more fundamental quantity in terms of Mellin transforms is $\Gamma(s)\zeta(s)$ which has the integral (Mellin transform) representation:

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1}\,dx = \mathcal{M}\left[\frac{1}{e^x-1}\right] \tag{6}$$

we would like to find an integral transform of a function f $\mathcal{Q}[f]$ such that

$$\mathcal{Q}[\mathcal{M}^{-1}[\phi(s)]] = \mathcal{M}^{-1}[2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\phi(1-s)] \tag{7}$$

such that

$$\mathcal{Q}[\mathcal{M}^{-1}[\Gamma(s)\zeta(s)]] = \mathcal{M}^{-1}[\Gamma(s)\zeta(s)] \tag{8}$$

by virtue of the integral equation 6 we should have something like

$$\mathcal{Q}\left[\frac{1}{e^x-1}\right](s) = \frac{1}{e^s-1} \tag{9}$$

we expect $\mathcal{Q}$ to somewhat resemble a Laplace transform because of the equation

$$\mathcal{L}\mathcal{M}^{-1}[\phi(s)] = \mathcal{M}^{-1}[\Gamma(s)\phi(1-s)] \tag{10}$$

Seems that we want something such that

$$\mathcal{Q}[x^{-s}] = \left(\frac{q}{2\pi}\right)^{s-1}\frac{1}{\pi}\sin\left(\frac{\pi}{2}(1-s)\right)\Gamma(1-s) \tag{11}$$

the trick seems to be using an inverse Mellin transform on the above to get the relationship

$$Q[x^{-s}] = \int_0^\infty x^{-s}\frac{\sin\left(\frac{q}{2\pi x}\right)}{\pi x}\,dx = q^{-s}2^s\pi^{s-1}\Gamma(s)\sin\left(\frac{\pi s}{2}\right) \tag{12}$$

this is still not quite right as we want to invert the $s \to 1-s$. It does seem (numerically) for a small region of s values (between 0 and 1?) that

$$\mathcal{Q}[x^{-s}] = \int_0^\infty x^{-s}\frac{\sin\left(\frac{qx}{2\pi}\right)}{\pi}\,dx = \left(\frac{q}{2\pi}\right)^{s-1}\frac{1}{\pi}\sin\left(\frac{\pi}{2}(1-s)\right)\Gamma(1-s) \tag{13}$$

as required. Hence our transform becomes (note the minus sign)

$$\mathcal{Q}[f] = -\int_0^\infty f(x)\frac{\sin\left(\frac{qx}{2\pi}\right)}{\pi}\,dx \tag{14}$$

which should (formally) satisfy

$$\mathcal{Q}[\mathcal{M}^{-1}[\Gamma(s)\zeta(s)] = \mathcal{M}^{-1}[\Gamma(s)\zeta(s)] \tag{15}$$

or then 'fixing' the inverse Mellin transform as given, $\Gamma(s)\zeta(s)$ is some kind of eigen-function of the transform $\mathcal{Q}$...

Checking This Follows Through

Thus

$$\mathcal{Q}[\mathcal{M}^{-1}[\phi]] = \mathcal{Q}\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\phi(s)\,ds\right]$$

$$\mathcal{Q}[\mathcal{M}^{-1}[\phi]] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{Q}\left[x^{-s}\right]\phi(s)\,ds$$

$$\mathcal{Q}[\mathcal{M}^{-1}[\phi]] = \frac{-1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \left(\frac{q}{2\pi}\right)^{s-1}\frac{1}{\pi}\sin\left(\frac{\pi}{2}(1-s)\right)\Gamma(1-s)\phi(s)\,ds$$

by letting $s-1 \to -t$ we get

$$\mathcal{Q}[\mathcal{M}^{-1}[\phi]] = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} q^{-t}2^t\pi^{t-1}\sin\left(\frac{\pi t}{2}\right)\Gamma(t)\phi(1-t)\,dt = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} q^{-t}\phi^*(t)\,dt = \boxed{\mathcal{M}^{-1}[\phi^*]}$$

where $\phi^*(t) = 2^t\pi^{t-1}\sin\left(\frac{\pi t}{2}\right)\Gamma(t)\phi(1-t)$. If we set $\phi(t) = \Gamma(t)\zeta(t)$ according to the Riemann functional equation we have

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi t}{2}\right)\Gamma(1-s)\zeta(1-s) \tag{16}$$

thus $\phi^*(t) = \phi(t) = \Gamma(t)\zeta(t)$.

Conclusion

It is formally possible to define such an integral transform. This may be possible and have better convergence for other functional relationships.

References

[1] - https://math.stackexchange.com/questions/2501698/a-pair-of-composed-integral-transforms-from-mellin-and-laplace-transforms

Appendix

If we define the forward transform as

$$\mathcal{Q}_1[f(x)](k) = \int_0^\infty \frac{f(x)}{e^{kx}-1}\,dx \tag{17}$$

we find that

$$\mathcal{Q}_1[x^{s-1}](k) = k^{-s}\Gamma(s)\zeta(s),\ s>1, t>0 \tag{18}$$

or equivalently

$$\mathcal{Q}_1[x^{s-1}](k) = k^{-s}2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(s)\Gamma(1-s)\zeta(1-s),\ s>1, t>0 \tag{19}$$

Thus

$$\mathcal{Q}_1[\mathcal{M}^{-1}[\phi]] = \mathcal{Q}_1\left[\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\phi(s)\,ds\right]$$

$$\mathcal{Q}_1[\mathcal{M}^{-1}[\phi]] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{Q}_1\left[x^{-s}\right]\phi(s)\,ds$$

$$\mathcal{Q}_1[\mathcal{M}^{-1}[\phi]] = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} q^{s-1}\Gamma(1-s)\zeta(1-s)\phi(s)\,ds$$

by letting $s-1 \to -t$ we get

$$\mathcal{Q}_1[\mathcal{M}^{-1}[\phi]] = \frac{-1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} q^{-t}\Gamma(t)\zeta(t)\phi(1-t)\,dt = \frac{-1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} q^{-t}\phi^*(t)\,dt = \boxed{-\mathcal{M}^{-1}[\phi^*]}$$

where $\phi^*(t) = \Gamma(t)\zeta(t)\phi(1-t)$. Although this is cool, it's not quite what we want.