Generalised Inverse Mellin Invariant Transform

Benedict Irwin[1]

[1]University of Cambridge

January 7, 2021

In a previous article [1], we found a specific integral transform $\mathcal{Q}$, such that

$$\mathcal{Q}[\mathcal{M}^{-1}[f(s)](x)](t) = \mathcal{M}^{-1}[f(s)](t) \tag{1}$$

for the case that $f(s) = \Gamma(s)\zeta(s)$, as this had a functional equation which could be used to define an invariance. In this work we create a generalised version of this.

If we have a general functional relationship for $f(x)$

$$f(s) = h(s)f(g(s))$$

where $g(s)$ has a nice inverse $g^{-1}(s)$.

Then the prescription for getting a $\mathcal{Q}$ that meets the requirement of equation 1 is finding a kernel to define the transform $\mathcal{Q}$ for an inverse power as

$$\mathcal{Q}[x^{-s}](q, s) = \int_0^\infty x^{-s} k(x, q) \, dx = q^{-g^{-1}(s)} \frac{h(g^{-1}(s))}{g'(g^{-1}(s))} \tag{2}$$

Thus for a test function $\phi(s)$,

$$\mathcal{Q}[\mathcal{M}^{-1}[\phi]] = \mathcal{Q}\left[\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \phi(s) \, ds \right]$$

$$\mathcal{Q}[\mathcal{M}^{-1}[\phi]] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \mathcal{Q}\left[x^{-s} \right] \phi(s) \, ds$$

$$\mathcal{Q}[\mathcal{M}^{-1}[\phi]] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} q^{-g^{-1}(s)} \frac{h(g^{-1}(s))}{g'(g^{-1}(s))} \phi(s) \, ds$$

by letting $s \to g(t)$ we get

$$\mathcal{Q}[\mathcal{M}^{-1}[\phi]] = \frac{1}{2\pi i} \int_{g^{-1}(c-i\infty)}^{g^{-1}(c+i\infty)} q^{-t} h(t) \phi(g(t)) \, dt = \mathcal{M}^{-1}[h(t)\phi(g(t))]$$

the condition for this to remain an inverse Mellin transform (up to sign) is that $g^{-1}(c + i\infty) \to d \pm i\infty$ and $g^{-1}(c - i\infty) \to d \mp i\infty$. When $\phi(s) = f(s)$ which satisfies the functional equation, then equation 1 is satisfied.

Getting the Kernel Function

Because we have chosen our transform to be the inverse Mellin transform, we can extract our kernel function by taking the inverse Mellin transform of the desired result of $\mathcal{Q}[x^{-s}]$.

Then the kernel function should be given by

$$k(q,x) = \mathcal{M}^{-1}\left[q^{-s}\frac{h(s)}{g'(s)}\right](x)$$

Example

Take the Riemann zeta function again, but this time without the additional gamma function

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$$

Here we have

$$h(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)g(s) = 1 - sg^{-1}(s) = 1 - sg'(x) = -1g^{-1}(c\pm i\infty) = d\mp i\infty$$

we may need to include an additional negative factor due to the integral limits swapping sign.

We assemble

$$k_f(q,x) = \mathcal{M}^{-1}\left[-q^{-s}2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\right](x) = \frac{2\cos\left(\frac{2\pi}{qx}\right)}{qx}$$

If we take the modular form

$$y(z) = y\left(\frac{2z+3}{z+2}\right)$$

then

$$h(s) = 1g(s) = \frac{2s+3}{s+2}g^{-1}(s) = \frac{3-2s}{s-2}g'(s) = \frac{1}{(s+2)^2}$$

there is a sign swap

$$k_y(q,x) = \mathcal{M}^{-1}\left[q^{-s}(s+2)^2\right](x)$$

this doesn't work so well.

Starting from a simpler example

$$f(x) = \Gamma(x)f(1-x)$$

gives $k(x,q) = -e^{-qx}$, then

$$\mathcal{Q}[x^{-s}] = -q^{s-1}\Gamma(1-s)$$

which apparently is connected to the Laplace Transform. Likewise

$$f(s) = (-i)^s\Gamma(s)f(1-s)$$

gives the kernel $-e^{-iqx}$ which will relate to the Fourier transform.

For the functional equation

$$f(s) = \Gamma(s)f\left(\frac{s^2}{2}\right)$$

we get

$$k(q,x) = \mathcal{M}^{-1}[q^{-s}s^{-1}\Gamma(s)] = \Gamma(0,qx)$$

2

This defines a transform such that

$$\mathcal{Q}[x^n] = \frac{\Gamma(n+1)}{(n+1)q^{n+1}}, q > 0$$

which seems quite fundamental in terms of differentiation. Another interesting one satisfies

$$f(s) = \Gamma(s)f(s + s^2)$$

which gives

$$k(q, x) = e^{-qx} - \sqrt{\pi q x} + \sqrt{\pi q x}\,\mathrm{erf}(\sqrt{qx})$$

and

$$f(s) = \Gamma(s)f(s + \frac{s^2}{2})$$

$$k(q, x) = e^{-qx} - qx\Gamma(0, qx)$$

for which

$$\mathcal{Q}[x^n] = \frac{\Gamma(n+1)}{(n+2)q^{n+1}}, q > 0$$

References

[1]   -   Riemann   Zeta   Invariance   Under   Composed   Integral   Transform, https://www.authorea.com/users/5445/articles/436204-riemann-zeta-invariance-under-composed-integral-transform