Method to Generate Sequences

Benedict Irwin[1]

[1]University of Cambridge

January 7, 2021

Consider a general template to generate sequences (or polynomials) using the inverse Mellin transform and a kernel function $\phi(s)$

$$p_k(x) = f(x)\mathcal{M}^{-1}[\phi(s)q_k(s)](x)$$

here $p_k(x)$ and $q_k(x)$ are polynomials, and $f(x)$ is a function that cancels out with the generating form from the inverse Mellin transform. This is observed with an example setting $q_k(s) = s^k$, $\phi(s) = \Gamma(s)$ and $f(x) = e^x$, we have

$$B'_k(x) = e^x\mathcal{M}^{-1}[\Gamma(s)s^k](x)$$

where $B'_k(x)$ appear to be some form of alternating Bell polynomials, and the coefficients of these polynomials are made up of Stirling numbers of the second kind $S_2(n,k)$ as

$$B'_n(x) = \sum_{k=0}^{n}(-1)^{n-k}S_2(n,k)x^k$$

we also find that

$$\sum_{k=0}^{n}\frac{(-1)^{n-k}S_2(n,k)}{2^n}x^{k/2} = e^{\sqrt{x}}\mathcal{M}^{-1}[\Gamma(2s)s^n](x)$$

very interestingly

$$(1+x)^{n+1}\mathcal{M}^{-1}[\Gamma(s)\Gamma(1-s)s^n](x)\sum_{k=0}^{n}(-1)^{n-k-1}A[n,k]x^{k+1}, k>0$$

where $A(n,k)$ as the Eulerian numbers. The agreement is off slightly for $k=0$. There is a more general form to this

$$(1+x)^{n+t}\mathcal{M}^{-1}[\frac{\Gamma(s)\Gamma(t-s)}{\Gamma(t)}s^n](x)$$

which for $t=1$ gives the Eulerian numbers, and for $t=2$ is related to A199335. We can even insert $t=1/2$, and get a sequence which is related to A185411 (with an additional factor to $1/2^n$).

Fixing the Signs

We now consider a modification to the transform to fix the signs, define the inverse-Q transform as

$$p_n(x) = \mathcal{Q}^{-1}[\phi(s)](n,x) = \mathcal{M}^{-1}[\phi(s)(-s)^n](-x)$$

where we have chosen the inverse because of the inverse Mellin transforms, now we have

$$\mathcal{M}^{-1}[\Gamma(s)](x)\mathcal{Q}^{-1}[\Gamma(s)](n,x) = B_n(x) = \sum_{k=0}^{n}S_2(n,k)x^k$$

for Bell polynomials $B_n(x)$ and interpreting 0^0 as 1 which is common in combinatorics. It's still (perhaps) not entirely right, because for $\phi(s) = \Gamma(s)\Gamma(1-s)$ we have

$$(1-x)^{n+1}\mathcal{Q}^{-1}[\Gamma(s)\Gamma(1-s)](n,x) = xA_n(x), n > 0$$

relating to Eularian polynomials, equally one could say

$$(1-x)\mathcal{Q}^{-1}[\Gamma(s)\Gamma(1-s)](x) = \frac{xA_n(x)}{(1-x)^n}, n > 0$$

Table of Relations

we can see that the function $f(x)$ is clearly related to $\mathcal{M}^{-1}[\phi(s)]$, which is exciting because, by assuming $q_k(s) = s^k$ for all inputs it links the function $\phi(s)$ directly a special class of numbers $T(n,k)$. We can as questiosn such as , which kernel $\phi(s)$ produces the binomials?

Function Numbers	Function
$\Gamma(s)$ StirlingS2	e^x
$\Gamma(s)\Gamma(1-s)$ Eulerian Numbers	$(1+x)^{n+1}$