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$\mathrm{PolyLog}_2 of Inverse Elliptic Nome Exponential Generating Function$

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1 Main

Let

$$G(q) = \mathrm{Li}_2(m(q)) \tag{1}$$

be an exponential generating function, where Li_2 is the polylogarithm of order 2,

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \tag{2}$$

and $m(q)$ is the inverse elliptic nome which can be expressed through the Dedakind eta function as

$$m(q) = \frac{\eta(\frac{\tau}{2})^8 \eta(2\tau)^{16}}{\eta(\tau)^{24}} \tag{3}$$

where $q = e^{i\pi\tau}$ or by Jacobi theta functions

$$m(q) = \left(\frac{\theta_2(0,q)}{\theta_3(0,q)}\right)^4 \tag{4}$$

where

$$\theta_2(0,q) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2} \theta_3(0,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} \tag{5}$$

giving explicitly

$$G(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \left(\frac{2\sum_{n=0}^{\infty} x^{(n+1/2)^2}}{1+2\sum_{n=1}^{\infty} x^{n^2}}\right)^{4k} = \sum_{k=0}^{\infty} \frac{a_k x^k}{k!} \tag{6}$$

if we consider the sequence of coefficients a_k associated with $G(x)$, modulo 1, or the fractional part of the coefficients, $\mathrm{frac}(a_k)$ we gain the following sequence

$$0,0,0,\frac{2}{3},0,\frac{4}{5},0,\frac{5}{7},0,0,0,\frac{6}{11},0,\frac{10}{13},0,0,0,\frac{1}{17},0,\frac{3}{19},0,0,0,\frac{7}{23},0,0,0,0,0,\frac{13}{19},0,\frac{15}{31},0,0,0,0,0,\frac{21}{37},0,0,0,\frac{25}{41},\cdots \tag{7}$$

we see the primes in the denominator in positions where the power of x is a prime. We also note that so far, the numerators are always less than the denominator (obviously), but count, succesively upwards, producing monotonically increasing subsequences. The prime only parts continue

$$\frac{2}{3},\frac{4}{5},\frac{5}{7},\frac{6}{11},\frac{10}{13},\frac{1}{17},\frac{3}{19},\frac{7}{23},\frac{13}{29},\frac{15}{31},\frac{21}{37},\frac{25}{41},\frac{27}{43},\frac{31}{47},\frac{37}{53},\frac{43}{59},\frac{45}{61},\frac{51}{67},\frac{55}{71},\frac{57}{73},\frac{63}{79},\frac{67}{83},\frac{73}{89},\frac{81}{97}, \tag{8}$$

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After closer inspection, we see the numerators from the point $1, 3, 7, 13, 15, 21, 25, 27, 31, 37, 43, 45, 51, 55, 57, ...$ take the form $\text{prime}(k)-16$, the numerators before this take the form $2\cdot\text{prime}(k)-16$, for $6, 10$, $3\cdot\text{prime}(k)-16$ for 5, $4\cdot\text{prime}(k)-16$ for 4 and $6\cdot\text{prime}(k)-16$ for the first numerator 2. It is likely then that for the rest of the numbers this pattern continues. This then gives for the coefficient a_k of $G(x)$, with $k > 6$,

$$\text{frac}(a_k) = \frac{k-16}{k},\ k \in \mathbb{P} \tag{9}$$

We find that if we take the original coefficients a_k, and subtract this fractional part in general

$$\delta_k = a_k - \frac{k-16}{k} \tag{10}$$

for numbers m which cannot be written as a sum of at least three consecutive positive integers, δ_m is an integer (empirical). A111774 "Numbers that can be written as a sum of at least three consecutive positive integers." apart from odd primes, numbers which cannot are powers of two.

2 Other

We find a similar relationship with

$$G_2(x) = \text{Li}_2\left(\frac{4x}{(1-x)^2\left(1-\frac{2x}{x-1}\right)^2}\right) = \sum_{k=0}^{\infty}\frac{b_k x^k}{k!} \tag{11}$$

where b_k seem to follow for $k > 2$

$$\text{frac}(b_k) = \frac{k-4}{k},\ k \in \mathbb{P} \tag{12}$$

3 Generating Function for Fractional Part

We see the Generating function for $n/2$ is

$$\frac{x}{2(x-1)^2} \tag{13}$$

but the generating function for the fractional part of $n/2$, which is $(n \mod 2)/2$, is given by

$$\frac{-x}{2(x^2-1)} \tag{14}$$

the property described is associated with the polylog, and we seen that the fractional part of

$$\text{Li}_2(2x) = \sum_{k=0}^{\infty}\frac{c_k x^k}{k!} \tag{15}$$

gives

$$\text{frac}(c_k) = \frac{k-2}{k},\ k \in \mathbb{P}0,\ \text{otherwise} \tag{16}$$

this means

$$\text{frac}\left(\frac{2^k k!}{k^2}\right) = \frac{k-2}{k},\ k \in \mathbb{P}0,\ \text{otherwise} \tag{17}$$

or

$$\text{frac}\left(\frac{2^k (k-1)!}{k}\right) = \frac{k-2}{k},\ k \in \mathbb{P}0,\ \text{otherwise} \tag{18}$$

we also see that

$$\text{frac}\left(\frac{(k-1)!}{k}\right) = \frac{k-1}{k},\ k \in \mathbb{P}\frac{1}{2}, 40,\ \text{otherwise} \tag{19}$$