[][]**References**

Theorem[section]

Resistance distance-based graph invariants and spanning trees of graphs derived from the strong prism of a graphs with given degree sequence

Leilei Zhang[1]

[1]East China Normal University

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Abstract

Let G^2_n be the graph obtained by the strong prism of a graph G_n with a given degree sequence, i.e. the strong product of K_2 and G_n. In this paper, we give the relationship between G_n^2 and G_n for Kirchhoff index and the total number of spanning tress. which generalized the main results of Z.M. Li et al. (Appl. Math. Comput., 2020, 382:125335), Y.G. Pan et al. (Int. J. Quantum Chem. 118 (2018) . E25787) and Y.G. Pan et al. (2019, ArXiv: 1906.04339). We also presented the explicit expressions for the multiplicative degree-Kirchhoff indices of G_n^2 when G_n is regular. Typically, Using this results we get the explicit expressions for the Kirchhoff index, multiplicative degree-Kirchhoff and the total number of spanning trees of the strong prism of a wheel W_n, respectively. It is surprising to find that the quotient of Kirchhoff (resp. multiplicative degree-Kirchhoff) index of W_n^2 and its Wiener (resp. Gutman) index is almost a constant. More specially, let $\mathcal{G}^2_{n,r}$ be the set of subgraphs obtained by randomly deleting r vertical edges from G^2_n, where $0\leq r \leq n$. Explicit formulas for Kirchhoff index and number of spanning trees for any graph $G^2_{n,r}\in \mathcal{G}^2_{n,r}$ are established, respectively.

Resistance distance-based graph invariants and spanning trees of graphs derived from the strong prism of a graphs with given degree sequence

Leilei Zhang]

Department of Mathematics, East China Normal University,
Shanghai 200241, P.R. China

Abstract: Let G_n^2 be the graph obtained by the strong prism of a graph G_n with a given degree sequence, i.e. the strong product of K_2 and G_n. In this paper, we give the relationship between G_n^2 and G_n for Kirchhoff index and the total number of spanning tress. which generalized the main results of Z.M. Li et al. (Appl. Math. Comput., 2020, 382:125335), Y.G. Pan et al. (Int. J. Quantum Chem. 118 (2018) . E25787) and Y.G. Pan et al. (2019, ArXiv: 1906.04339). We also presented the explicit expressions for the multiplicative degree-Kirchhoff indices of G_n^2 when G_n is regular. Typically, Using this results we get the explicit expressions for the Kirchhoff index, multiplicative degree-Kirchhoff and the total number of spanning trees of the strong

]E-mail: mathllz@qq.com(L.L. Zhang)

prism of a wheel W_n, respectively. It is surprising to find that the quotient of Kirchhoff (resp. multiplicative degree-Kirchhoff) index of W_n^2 and its Wiener (resp. Gutman) index is almost a constant. More specially, let $\mathcal{G}_{n,r}^2$ be the set of subgraphs obtained by randomly deleting r vertical edges from G_n^2, where $0 \leq r \leq n$. Explicit formulas for Kirchhoff index and number of spanning trees for any graph $G_{n,r}^2 \in \mathcal{G}_{n,r}^2$ are established, respectively.

Keywords: strong prism; Kirchhoff index; multiplicative degree-Kirchhoff index; spanning tree;

Introduction

Spectral graph theory tries to derive information about graphs from the graph spectrum (1997; 2010). There is extensive literature on works related to the spectrum on various matrices such as adjacency, Laplacian and normalized Laplacian matrices. Especially in recent years, the normalized Laplacian, which is consistent with the eigenvalues in spectral geometry and in random processes, has attracted increasing attention from researchers because many results which were only known for regular graphs can be generalized to all graphs.

In this paper, we only consider simple connected graph $G = (V_G, E_G)$ with vertex set V_G and edge set E_G. We call $n := |V_G|$ the order of G and $m := |E_G|$ the size of G and refer to Bondy and Murty (2008) for notation and terminologies used but not defined here. The adjacency matrix $A(G) := (a_{ij})_{n\times n}$ of G is a 0-1 matrix with a $a_{ij} = 1$ if $v_i \backsim v_j$ (v_i and v_j are adjacent in G) and 0 otherwise. Let d_i denote the degree of the vertex v_i in G, and thus $D(G) = diag\,(d_1, d_2, \cdots, d_n)$ is called the diagonal matrix of G. Then the matrix $L(G) := D(G) - A(G)$ is called as the Laplacian matrix of G, while the normalized Laplacian matrix of G refers to equation $\mathcal{L}(G) := D(G)^{-\frac{1}{2}}L(G)D(G)^{-\frac{1}{2}}$. Via a simple calculation, the (i,j)-entry of $\mathcal{L}(G)$ can be expressed as

$$(\mathcal{L}(G))_{ij} = \begin{cases} 1, & \text{if } i = j\ ; \\ -\frac{1}{\sqrt{d_i d_j}}, & \text{if } i \neq j \text{ and } i \sim j; \\ 0, & \text{otherwise}; \end{cases} \quad (1)$$

Let d_{ij} be the distance between vertices v_i and v_j in G, which represents the length of the shortest path connecting vertex v_i and v_j. The Wiener index of G, introduced in and widely studied in the fields of mathematics and chemistry, is defined as $W(G) = \sum_{\{v_i,v_j\}\subseteq V_G} d_{ij}$. Later, Gutman introduced the weighted version of Wiener index, namely Gutman index , which is defined as $Gut(G) = \sum_{\{v_i,v_j\}\subseteq V_G} d_i d_j d_{ij}$. Moreover, Gutman confirmed that $Gut(G) = 4W(G) - (2n-1)(n-1)$ when G is an n-vertex tree.

Based on the electronic network theory, Klein and Randić (1993) proposed a new distance-based parameter, i.e., the resistance distance, on a graph. The resistance distance between vertices v_i and v_j, written by r_{ij}, is the effective resistance between them when one puts one unit resistor on every edge of a graph G. This novel parameter is in fact intrinsic to the graph and has some nice interpretations and applications in chemistry (see (D. J. Klein, 2002; 2001) for details). As an analogue to the Wiener index, define $Kf(G) = \sum_{\{v_i,v_j\}\subseteq V_G} r_{ij}$, known as the Kirchhoff index (a structure-descriptor) of G (1993). This structure-descriptor can be expressed alternatively as

$$Kf(G) = \sum_{\{v_i,v_j\}\subseteq V_G} r_{ij} = \sum_{i=2}^{n} \frac{1}{\rho_i}.$$

where $0 = \rho_1 < \rho_2 \leq \rho_3 \cdots \leq \rho_n (2 \leq n)$ are the eigenvalues of $L(G)$.

As an analogue of Kirchhoff index of G, Chen and Zhang (2007) proposed a novel resistance distance-based graph invariant, defined by $Kf^*(G) = \sum_{\{v_i,v_j\}\subseteq V(G)} d_i d_j r_{ij}$, which is called the degree-Kirchhoff index. Just as the relationship between the Kirchhoff index and the Laplacian spectrum, the degree Kirchhoff index is just closely related to the spectrum of the normalized Laplacian matrix $\mathcal{L}(G)$, For a simple connected graph G of order n and size m, Chen and Zhang (2007) showed that

$$Kf^*(G) = \sum_{\{v_i,v_j\}\subseteq V_G} d_i d_j r_{ij} = \sum_{i=2}^{n} \frac{1}{\lambda_i}.$$

where $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \cdots \leq \lambda_n (2 \leq n)$ are the eigenvalues of $\mathcal{L}(G)$.

Note that it is difficult to carry out some algorithms to compute the Kirchhoff index in a graph from its computational complexity. Clearly, it is much more difficult to use algorithms to compute the multiplicative degree-Kirchhoff index in a graph. Hence, it is interesting to obtain the closedform formula for the multiplicative degree-Kirchhoff index of a graph G. Methods to calculate Kirchhoff index and multiplicative degree-Kirchhoff index were proposed in (2008; 2013; 2016; 2015; 2013; J. L. Palacios, 2001). In the last decades, many researchers are devoted to give closed formulas for the Kirchhoff index and the multiplicative degree-Kirchhoff index of graphs with special structures, such as cycles (1995), liner phenylenes (2019; 2019; 2019), linear polyomino chains (2016; 2008), linear pentagonal chains (2018; 2018), linear hexagonal chains (2016; 2019; 2008) , linear crossed chains (Y. G. Pan & spanning trees of the linear crossed hexagonal chains, 2018; 1080) , and composite graphs (2009). Some other studies on the Kirchhoff index and the multiplicative degree-Kirchhoff index of a graph are obtained in (2019; 1080; J. L. Palacios & the kirchhoff index, 2014; 2011; 2013; Y. J. Yang & triangulations of graphs, 2015; 2011; B. Zhou & kirchhoffindex, 2009; 2009).

Given two graphs G and H, the strong product of G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$, where two distinct vertices (u_1, v_1) and (u_2, v_2) are adjacent whenever u_1 and u_2 are equal or adjacent in G, and v_1 and v_2 are equal or adjacent in H. Specially, the strong product of G and K_2 is called the strong prism of G. Pan et al. (2020; Y. G. Pan et al., 2019; 1080) determined some resistance distance-based invariants and number of spanning trees of graphs derived from the strong prism of some special graphs, such as the path P_n, the cycle C_n and the star S_n. For the sake of convenience, we let G_n^2 be the strong prism of G_n with a given degree sequence. Obviously, $|V(G_n^2)| = 2n$ and $|E(G_n^2)| = n + 4m$. Let $E' = \{ii' : i = 1, 2, \cdots, n\}$, which is defined as the set of vertical edges of G_n^2. Let $\mathcal{G}_{n,r}^2$ be the set of subgraphs of G_n^2 obtained by randomly deleting r vertical edges from E', where $0 \leq r \leq n$. Obviously, $\mathcal{G}_{n,0}^2 = \{G_n^2\}$.

Let G_n be a graph with degree sequences $(d_1, d_2, \cdots, d_n)$. In this article, motivated by Refs. (2020; Y. G. Pan et al., 2019; 1080), using the normalized Laplacian decomposition theorem, we determine the explicit formulae for the Kirchhoff index, the multiplicative degree-Kirchhoff index, and the spanning tree numbers of G_n^2. For any graph $G_{n,r}^2 \in \mathcal{G}_{n,r}^2$, its Kirchhoff index and number of spanning trees are respectively determined. Using these results, explicit expressions for Kirchhoff index, multiplicative degree-Kirchhoff index and number of spanning trees of W_{n+1}^2 are respectively determined, where W_{n+1}^2 is the strong prism of a wheel W_{n+1} as shows in Fig 1. We also confirmed that the Kirchhoff index (resp. multiplicative degree-Kirchhoff index) of W_{n+1}^2 is almost Constant times of its Wiener index (resp. Gutman index). Obviously, our conclusions generalize the main results of Refs. (2020; Y. G. Pan et al., 2019; 1080).

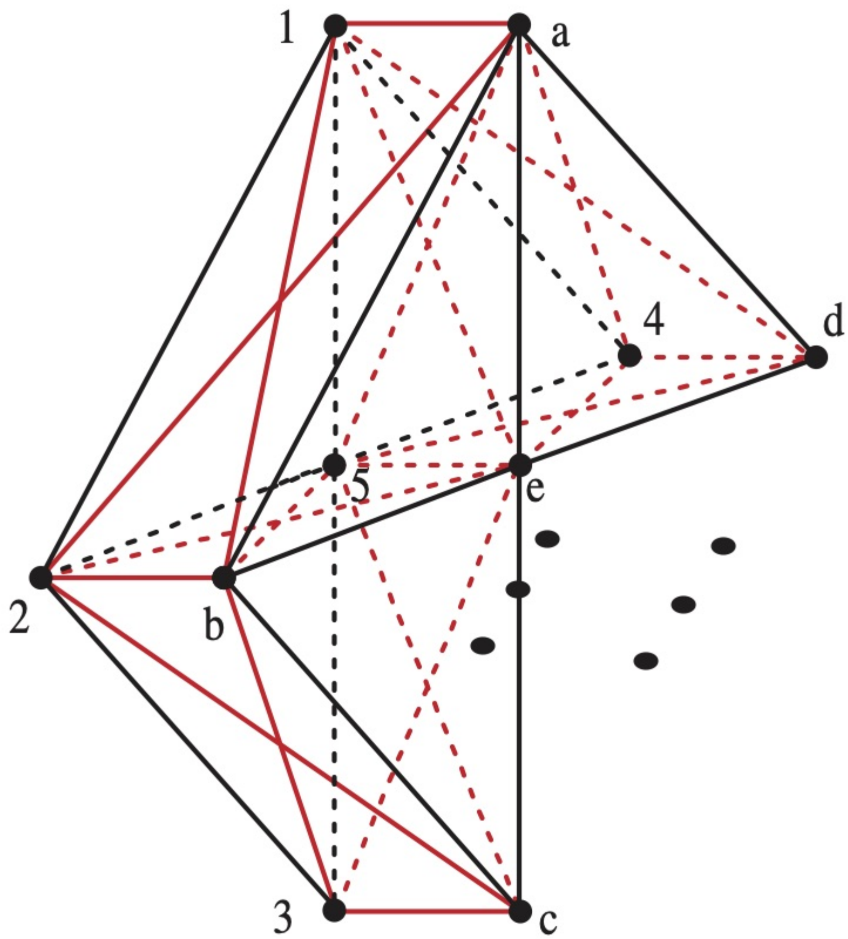

Figure 1: W_{n+1}^2 with labelled vertices.

Preliminaries

In this section, we give some preliminary results. Throughout this article, we denote by $\Phi(M,x) = det(xI - M)$ the characteristic polynomial of the square matrix M, where I is the unitary matrix with the same order as that of M.

An automorphism of G is a permutation π of $V(G)$. It has the property: uv is an edge of G if and only if $\pi(u)\pi(v)$ is an edge of G. If π is an automorphism of G, then we may write it as the product of transpositions and disjoint 1-cycles, that is,

$$\pi = (w_1)(w_2)\cdots(w_m)(u_1v_1)(u_2v_2)\cdots(u_kv_k).$$

Therefore, one has $V(G) = V_0 \cup V_1 \cup V_2$, where $V_0 = \{w_1, w_2, \cdots, w_m\}$, $V_1 = \{u_1, u_2, \cdots, u_k\}$, and $V_2 = \{v_1, v_2, \cdots, v_k\}$. Then we obtain

$$L(G) = ($$

$$\begin{matrix} L_{V_{00}} & L_{V_{01}} & L_{V_{02}} \\ L_{V_{10}} & L_{V_{11}} & L_{V_{12}} \\ L_{V_{20}} & L_{V_{21}} & L_{V_{22}} \end{matrix} \quad , \quad \mathcal{L}(G) = ($$

$$\begin{matrix} \mathcal{L}_{V_{00}} & \mathcal{L}_{V_{01}} & \mathcal{L}_{V_{02}} \\ \mathcal{L}_{V_{10}} & \mathcal{L}_{V_{11}} & \mathcal{L}_{V_{12}} \\ \mathcal{L}_{V_{20}} & \mathcal{L}_{V_{21}} & \mathcal{L}_{V_{22}} \end{matrix}$$ $.where \mathrm{L}_{V_{ij}}$ is the submatrix formed by rows corresponding to vertices in V_i and columns corresponding to vertices in V_j, $i,j = 0,1,2$. Bearing in mind that π is an automorphism of G yields $L_{V_{11}} = L_{V_{22}}$ and $\mathcal{L}_{V_{11}} = \mathcal{L}_{V_{22}}$. Let

$$T = ($$

$$\begin{matrix} I_m & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{2}}I_k & \frac{1}{\sqrt{2}}I_k \\ \mathbf{0} & \frac{1}{\sqrt{2}}I_k & -\frac{1}{\sqrt{2}}I_k \end{matrix}$$ $betheblockmatrixsuchthateachofitsblockshasthesamedimensionasthecorrespondingblockin\mathrm{L(G)}and$

$$T^T L(G) T = ($$

$$\begin{matrix} L_A(G) & \mathbf{0} \\ \mathbf{0} & L_S(G) \end{matrix} \quad , \quad T^T \mathcal{L}(G) T = ($$

$$\begin{matrix} \mathcal{L}_A(G) & \mathbf{0} \\ \mathbf{0} & \mathcal{L}_S(G) \end{matrix}$$

where

$$L_A(G) = ($$

$$\begin{matrix} L_{V_{00}} & \sqrt{2}L_{V_{01}} \\ \sqrt{2}L_{V_{10}} & L_{V_{11}} + L_{V_{12}} \end{matrix} \quad , \quad L_S(G) = L_{V_{11}} - L_{V_{12}}.$$

(1)

and

$$\mathcal{L}_A(G) = \begin{pmatrix} \mathcal{L}_{V_{00}} & \sqrt{2}\mathcal{L}_{V_{01}} \\ \sqrt{2}\mathcal{L}_{V_{10}} & \mathcal{L}_{V_{11}} + \mathcal{L}_{V_{12}} \end{pmatrix}, \quad \mathcal{L}_S(G) = \mathcal{L}_{V_{11}} - \mathcal{L}_{V_{12}}.$$
(2)

Using the above method, Yang and Yu (2008) obtained the following decomposition theorem of Laplacian polynomial, which is described in a some what different way as follows:

(2008) Let $L_A(G)$, $L_S(G)$ be defined as above. Then

$$\Phi(L(G), x) = \Phi(L_A(G), x)\Phi(L_S(G), x).$$

Huang and Li (2016) obtained the following decomposition theorem of the normalized Laplacian characteristic polynomial.

(2016) Let $\mathcal{L}_A(G)$, $\mathcal{L}_S(G)$ be matrices defined as the above forms, then

$$\Phi(\mathcal{L}(G), x) = \Phi(\mathcal{L}_A(G), x)\Phi(\mathcal{L}_S(G), x).$$

Further on we need the following lemmas.

Let G be an n-vertex connected graph and let $0 = \rho_1 < \rho_2 \leq \cdots \leq \rho_n$ be all the eigenvalues of $L(G)$. Then $Kf(G) = n\sum_{i=2}^{n} \frac{1}{\rho_i}$.

Let G be an n-vertex connected graph let $0 = \rho_1 < \rho_2 \leq \cdots \leq \rho_n$ be all the eigenvalues of $L(G)$. Then $\tau(G) = \frac{1}{n}\Pi_{i=2}^{n}\rho_i$, where $\tau(G)$ is the number of spanning trees of G.

Let G be a connected graph of order n and size m and let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be all the eigenvalues of $\mathcal{L}(G)$. Then $Kf^*(G) = 2m\sum_{i=2}^{n} \frac{1}{\lambda_i}$.

Kirchhoff index and the number of spanning trees of G_n^2.

in this section, we mainly consider two types of graph invariants of G_n^2, that is, the Kirchhoff index and the total number of spanning trees. Let $d_1, d_2, \cdots, d_n$ be the degree sequence of G. Our main aim is to give relationship between G_n^2 and G_n for these two types of graph invariants (based on the laplacian of G_n^2). Using these relationship, we get the the Kirchhoff index and the total number of spanning trees of W_{n+1}^2, where W_{n+1}^2 is the strong product of a path K_2 and a wheel W_{n+1}. We also determine the limited values for the quotients of the Kirchhoff index and the Winner index of W_{n+1}^2, respectively.

We first label the vertices of G with labels $\{1, 2, \cdots, n\}$. According to the definition of strong product, we label vertices of G_n^2 with labels $\{1, 1', 2, 2' \cdots, n, n'\}$ and i' corresponds to i in G_n^2. According to the labeled vertices of G_n^2, one may see that G_n^2 has an automorphism: $\pi = (1, 1')(2, 2') \cdots (n, n')$. That is to say, $V_0 = \emptyset$, $V_1 = \{1, 2, \cdots, n\}$ and $V_2 = \{1', 2', \cdots, n'\}$, respectively. By equation (1), we have

$$L_A(G_n^2) = L_{V_{11}}(G_n^2) + L_{V_{12}}(G_n^2), \qquad L_S(G_n^2) = L_{V_{11}}(G_n^2) - L_{V_{12}}(G_n^2).$$

Consequently, $L_{V_{11}}(G_n^2)$ and $L_{V_{12}}(G_n^2)$ are $n \times n$ matrices, which are given as follows:

$$L_{V_{11}}(G_n^2) = ($$

$$\begin{matrix} 2d_1+1 & -a_{1,2} & -a_{1,3} & -a_{1,4} & \cdots & -a_{1,n} \\ -a_{2,1} & 2d_2+1 & -a_{2,3} & -a_{2,4} & \cdots & -a_{2,n} \\ -a_{3,1} & -a_{3,2} & 2d_3+1 & -a_{3,4} & \cdots & -a_{3,n} \\ -a_{4,1} & -a_{4,2} & -a_{4,3} & 2d_4+1 & \cdots & -a_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ -a_{n,1} & -a_{n,2} & -a_{n,3} & -a_{n,4} & \cdots & 2d_n+1 \end{matrix} \quad {}_{n\times n}$$

and

$$L_{V_{12}}(G_n^2) = ($$

$$\begin{matrix} -1 & -a_{1,2} & -a_{1,3} & -a_{1,4} & \cdots & -a_{1,n} \\ -a_{2,1} & -1 & -a_{2,3} & -a_{2,4} & \cdots & -a_{2,n} \\ -a_{3,1} & -a_{3,2} & -1 & -a_{3,4} & \cdots & -a_{3,n} \\ -a_{4,1} & -a_{4,2} & -a_{4,3} & -1 & \cdots & -a_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ -a_{n,1} & -a_{n,2} & -a_{n,3} & -a_{n,4} & \cdots & -1 \end{matrix} \quad {}_{n\times n}.$$

Hence

$$L_A(G_n^2) = 2\,($$

$$\begin{matrix} d_1 & -a_{1,2} & -a_{1,3} & -a_{1,4} & \cdots & -a_{1,n} \\ -a_{2,1} & d_2 & -a_{2,3} & -a_{2,4} & \cdots & -a_{2,n} \\ -a_{3,1} & -a_{3,2} & d_3 & -a_{3,4} & \cdots & -a_{3,n} \\ -a_{4,1} & -a_{4,2} & -a_{4,3} & d_4 & \cdots & -a_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ -a_{n,1} & -a_{n,2} & -a_{n,3} & -a_{n,4} & \cdots & d_n \end{matrix} \quad {}_{n\times n}$$

and

$$L_S(G_n^2) = 2\,($$

$$\begin{matrix} d_1+1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & d_2+1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & d_3+1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & d_4+1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & d_n+1 \end{matrix} \quad {}_{n\times n}.$$

By Theorem , $\Phi(L(G), x) = \Phi(L_A(G), x)\Phi(L_S(G), x)$. Hence the Laplacian spectrum of $L(G_n^2)$ consists of eigenvalues of $L_A(G_n^2)$ and$L_S(G_n^2)$. Let $0 = \rho_1 < \rho_2 \leq \cdots \leq \rho_n$ be the L-spectrum of G, that is, the spectrum of the Laplacian matrix of G. Since $L_A(G_n^2) = 2L(G_n)$ and $L_S(G_n^2)$ is a diagonal matrix, one can easily see that $2\rho_i$, $2d_i + 2, i \in \{1, 2, \cdots, n\}$ are all the eigenvalues of $L(G_n^2)$. Then we obtain the following theorem.

For graph G_n with degree sequence $d_1, d_2, \cdots, d_n$, let $G_n^2 = K_2 \boxtimes G_n$. Then

$Kf(G_n^2) = kf(G_n) + n\sum_{i=1}^{n} \frac{1}{d_i+1}$.

$\tau(G_n^2) = \tau(G_n)4^{n-1}\Pi_{k=1}^{n}(d_i + 1)$.

Proof.

(i) Since $|V(G_n^2)| = 2n$, by lemma , we have

(ii) It follows from lemma that

□

Let W_{n+1} be a wheel with $n+1$ vertices and the degree sequence of W_{n+1} is $n, 3, 3, \cdots, 3$. It is completely determined in (1985) that the eigenvalues of $L(W_{n+1})$ are 0, $n+1$, $1+4sin^2(\frac{k\pi}{n}), k \in \{1, 2, \cdots, n-1\}$. By lemmas and , we have

$$Kf(W_{n+1}) = 1 + (n+1)\sum_{k=1}^{n-1}\frac{1}{1+4sin^2(\frac{k\pi}{n})}, \qquad \tau(W_{n+1}) = \Pi_{k=1}^{n-1}[1+4sin^2(\frac{k\pi}{n})].$$

By theorem (), we obtain the following corollary.

For $n \geq 3$, let $W_{n+1}^2 = K_2 \boxtimes W_{n+1}$. Then
$Kf(W_{n+1}^2) = \frac{n^2+n+8}{4} + (n+1)\left(\sum_{k=1}^{n-1}\frac{1}{1+4sin^2(\frac{k\pi}{n})}\right)$.
$\tau(W_{n+1}^2) = (n+1)2^{4n}\Pi_{k=1}^{n-1}[1+4sin^2(\frac{k\pi}{n})]$.

Considering the quotient of the Kirchhoff index and the Wiener index of W_{n+1}^2, we obtain the following result.

For $n \geq 3$, let $W_{n+1}^2 = K_2 \boxtimes W_{n+1}$. Then

$$\lim_{n\to+\infty}\frac{Kf(W_{n+1}^2)}{W(W_{n+1}^2)} = \frac{1}{16} + \frac{\sqrt{5}}{20}.$$

Proof. First we determine the Wiener index of W_{n+1}^2. Let $d(i, W_{n+1}^2)$ be the sum of distance from the vertex i to other vertices in W_{n+1}^2. For $i = 0$, we have

$$d(0, W_{n+1}^2) = 2n + 1.$$

For $i = \{1, 2, \cdots, n\}$, we have

$$d(i, W_{n+1}^2) = 7 + 2 \times (2n - 6) = 4n - 5.$$

Then

Let $f(x) = \frac{1}{1+4sin^2(x)}, x \in [0, \pi]$, Obviously, the function $f(x)$ is Riemann-Integrable on $[0, \pi]$. Hence

Since

$$Kf(W_{n+1}^2) = \frac{n^2+n+8}{4} + (n+1)\left(\sum_{k=1}^{n-1} \frac{1}{1+4sin^2(\frac{k\pi}{n})}\right),$$

we have

$$\lim_{n\to+\infty} \frac{Kf(W_{n+1}^2)}{W(W_{n+1}^2)} = \frac{1}{16} + \frac{\sqrt{5}}{20},$$

As desired.

□

It is completely determined in (1985) that the eigenvalues of $L(S_n)$ are 0, 1 with multiplicity $n-2$, n and the eigenvalues of $L(C_n)$ are $4sin^2(\frac{k\pi}{n}), k \in \{1, 2, \cdots, n-1, n\}$, where S_n and C_n are a star and a cycle with n vertices. By lemmas and , we have

$$Kf(S_n) = n^2 - 2n + 1, \quad \tau(S_n) = 1, \quad Kf(C_n) = \frac{n^3 - n}{12}, \quad \tau(C_n) = n.$$

(3)

The degree sequence of S_n is $\{n-1, 1, \cdots, 1\}$ and C_n is 2-regular graph. Using theorem (), we obtain the following two corollarys.

(2020) For $n \geq 3$, let $S_n^2 = K_2 \boxtimes S_n$. Then

$Kf(G_n^2) = \frac{3n^2-5n+4}{2}$,

$\tau(S_n^2) = n \cdot 2^{3n-3}$.

(Y. G. Pan et al., 2019) For $n \geq 3$, let $C_n^2 = K_2 \boxtimes C_n$. Then

$Kf(C_n^2) = \frac{n^3+4n^2-n}{12}$,

$\tau(C_n^2) = n \cdot 2^{2n-2} \cdot 3^n$.

Multiplicative degree-Kirchhoff index of $K_2 \boxtimes G_n$

G_n is a k regular graph

In this subsection, we will determine the relationship between G_n^2 and G_n for multiplicative degree-Kirchhoff index (based on the laplacian of G_n^2). Not that

$$\mathcal{L}_{V_{11}}(G_n^2) = ($$

$$\begin{array}{cccccc} 1 & \frac{-a_{1,2}}{2k+1} & \frac{-a_{1,3}}{2k+1} & \frac{-a_{1,4}}{2k+1} & \cdots & \frac{-a_{1,n}}{2k+1} \\ \frac{-a_{2,1}}{2k+1} & 1 & \frac{-a_{2,3}}{2k+1} & \frac{-a_{2,4}}{2k+1} & \cdots & \frac{-a_{2,n}}{2k+1} \\ \frac{-a_{3,1}}{2k+1} & \frac{-a_{3,2}}{2k+1} & 1 & \frac{-a_{3,4}}{2k+1} & \cdots & \frac{-a_{3,n}}{2k+1} \\ \frac{-a_{4,1}}{2k+1} & \frac{-a_{4,2}}{2k+1} & \frac{-a_{4,3}}{2k+1} & 1 & \cdots & \frac{-a_{4,n}}{2k+1} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \frac{-a_{n,1}}{2k+1} & \frac{-a_{n,2}}{2k+1} & \frac{-a_{n,3}}{2k+1} & \frac{-a_{n,4}}{2k+1} & \cdots & 1 \end{array} \quad {}_{n\times n}$$

and

$$\mathcal{L}_{V_{12}}(G_n^2) = ($$

$$\begin{array}{cccccc} \frac{-1}{2k+1} & \frac{-a_{1,2}}{2k+1} & \frac{-a_{1,3}}{2k+1} & \frac{-a_{1,4}}{2k+1} & \cdots & \frac{-a_{1,n}}{2k+1} \\ \frac{-a_{2,1}}{2k+1} & \frac{-1}{2k+1} & \frac{-a_{2,3}}{2k+1} & \frac{-a_{2,4}}{2k+1} & \cdots & \frac{-a_{2,n}}{2k+1} \\ \frac{-a_{3,1}}{2k+1} & \frac{-a_{3,2}}{2k+1} & \frac{-1}{2k+1} & \frac{-a_{3,4}}{2k+1} & \cdots & \frac{-a_{3,n}}{2k+1} \\ \frac{-a_{4,1}}{2k+1} & \frac{-a_{4,2}}{2k+1} & \frac{-a_{4,3}}{2k+1} & \frac{-1}{2k+1} & \cdots & \frac{-a_{4,n}}{2k+1} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ \frac{-a_{n,1}}{2k+1} & \frac{-a_{n,2}}{2k+1} & \frac{a_{-n,3}}{2k+1} & \frac{-a_{n,4}}{2k+1} & \cdots & \frac{-1}{2k+1} \end{array} \quad {}_{n\times n}.$$

Since $\mathcal{L}_A(G_n^2) = \mathcal{L}_{V_{11}}(G_n^2) + \mathcal{L}_{V_{12}}(G_n^2)$ and $\mathcal{L}_S(G_n^2) = \mathcal{L}_{V_{11}}(G_n^2) - \mathcal{L}_{V_{12}}(G_n^2)$, thus

$$L_A(G_n^2) = \frac{2}{2k+1}\,($$

$$\begin{array}{cccccc} k & -a_{1,2} & -a_{1,3} & -a_{1,4} & \cdots & -a_{1,n} \\ -a_{2,1} & k & -a_{2,3} & -a_{2,4} & \cdots & -a_{2,n} \\ -a_{3,1} & -a_{3,2} & k & -a_{3,4} & \cdots & -a_{3,n} \\ -a_{4,1} & -a_{4,2} & -a_{4,3} & k & \cdots & -a_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ -a_{n,1} & -a_{n,2} & -a_{n,3} & -a_{n,4} & \cdots & k \end{array} \quad {}_{n\times n}$$

and

$$L_S(G_n^2) = ($$

$$\begin{matrix} \frac{2k+2}{2k+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{2k+2}{2k+1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{2k+2}{2k+1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{2k+2}{2k+1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{2k+2}{2k+1} \end{matrix} \quad {}_{n\times n}.$$

Let $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$ be the $\mathcal{L}$-spectrum of G_n, that is, the spectrum of the normalized Laplacian matrix of G. Since $\mathcal{L}_A(G_n^2) = \frac{2k}{2k+1}\mathcal{L}(G_n)$ and $L_S(G_n^2)$ is a diagonal matrix, one can easily see that $\frac{2k}{2k+1}\lambda_i, i \in \{1, 2, \cdots, n\}$, $\frac{2k+2}{2k+1}$ with multiplicity n are all the eigenvalues of $\mathcal{L}(G_n^2)$. Then we obtain the following theorem by using Lemma .

For k regular graph G_n, let $G_n^2 = G_n \boxtimes K_2$. Then

$$Kf^*(G_n^2) = \frac{(2k+1)^2}{k^2}Kf^*(G_n) + \frac{n^2(2k+1)^2}{k+1}.$$

Since C_n is 2-regular and the eigenvalues of $\mathcal{L}(C_n)$ are $2sin^2(\frac{k\pi}{n}), k \in \{1, 2, \cdots, n-1, n\}$, we have $Kf^*(C_n) = \frac{n^3-n}{3}$. Then

(Y. G. Pan et al., 2019) For $n \geq 3$, Let $G_n = C_n \boxtimes K_2$. Then $Kf^*(G_n^2) = \frac{25n^3+100n^2-25n}{12}$.

G_n is a wheel W_n

In this subsection, we will mainly establish the explicit formulas for Multiplicative degree-Kirchhoff index of W_{n+1}^2. Obviously,

$$\mathcal{L}_{V_{11}}(W_{n+1}^2) = ($$

$$\begin{matrix} 1 & \frac{-1}{\sqrt{7(2n+1)}} & \frac{-1}{\sqrt{7(2n+1)}} & \frac{-1}{\sqrt{7(2n+1)}} & \cdots & \frac{-1}{\sqrt{7(2n+1)}} & \frac{-1}{\sqrt{7(2n+1)}} \\ \frac{-1}{\sqrt{7(2n+1)}} & 1 & -\frac{1}{7} & 0 & \cdots & 0 & -\frac{1}{7} \\ \frac{-1}{\sqrt{7(2n+1)}} & -\frac{1}{7} & 1 & -\frac{1}{7} & \cdots & 0 & 0 \\ \frac{-1}{\sqrt{7(2n+1)}} & 0 & -\frac{1}{7} & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\ \frac{-1}{\sqrt{7(2n+1)}} & 0 & 0 & 0 & \cdots & 1 & -\frac{1}{7} \\ \frac{-1}{\sqrt{7(2n+1)}} & -\frac{1}{7} & 0 & 0 & \cdots & -\frac{1}{7} & 1 \end{matrix} \quad {}_{(n+1)\times(n+1)}$$

and

$$\mathcal{L}_{V_{22}}(W_{n+1}^2) = ($$

$$\begin{array}{ccccccc}
\frac{-1}{2n+1} & \frac{-1}{\sqrt{7(2n+1)}} & \frac{-1}{\sqrt{7(2n+1)}} & \frac{-1}{\sqrt{7(2n+1)}} & \cdots & \frac{-1}{\sqrt{7(2n+1)}} & \frac{-1}{\sqrt{7(2n+1)}} \\
\frac{-1}{\sqrt{7(2n+1)}} & -\frac{1}{7} & -\frac{1}{7} & 0 & \cdots & 0 & -\frac{1}{7} \\
\frac{-1}{\sqrt{7(2n+1)}} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\
\frac{-1}{\sqrt{7(2n+1)}} & 0 & -\frac{1}{7} & -\frac{1}{7} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\
\frac{-1}{\sqrt{7(2n+1)}} & 0 & 0 & 0 & \cdots & -\frac{1}{7} & -\frac{1}{7} \\
\frac{-1}{\sqrt{7(2n+1)}} & -\frac{1}{7} & 0 & 0 & \cdots & -\frac{1}{7} & -\frac{1}{7}
\end{array}_{(n+1)\times(n+1)}.$$

Since $\mathcal{L}_A(W_{n+1}^2) = \mathcal{L}_{V_{11}}(W_{n+1}^2) + \mathcal{L}_{V_{12}}(W_{n+1}^2)$ and $\mathcal{L}_S(W_{n+1}^2) = \mathcal{L}_{V_{11}}(W_{n+1}^2) - \mathcal{L}_{V_{12}}(W_{n+1}^2)$, thus

$$\mathcal{L}_A(W_{n+1}^2) = ($$

$$\begin{array}{ccccccc}
\frac{2n}{2n+1} & \frac{-2}{\sqrt{7(2n+1)}} & \frac{-2}{\sqrt{7(2n+1)}} & \frac{-2}{\sqrt{7(2n+1)}} & \cdots & \frac{-2}{\sqrt{7(2n+1)}} & \frac{-2}{\sqrt{7(2n+1)}} \\
\frac{-2}{\sqrt{7(2n+1)}} & \frac{6}{7} & -\frac{2}{7} & 0 & \cdots & 0 & -\frac{2}{7} \\
\frac{-2}{\sqrt{7(2n+1)}} & -\frac{2}{7} & \frac{6}{7} & -\frac{2}{7} & \cdots & 0 & 0 \\
\frac{-2}{\sqrt{7(2n+1)}} & 0 & -\frac{2}{7} & \frac{6}{7} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\
\frac{-2}{\sqrt{7(2n+1)}} & 0 & 0 & 0 & \cdots & \frac{6}{7} & -\frac{2}{7} \\
\frac{-2}{\sqrt{7(2n+1)}} & -\frac{2}{7} & 0 & 0 & \cdots & -\frac{2}{7} & \frac{6}{7}
\end{array}_{(n+1)\times(n+1)}$$

and

$$\mathcal{L}_S(W_{n+1}^2) = ($$

$$\begin{array}{ccccccc}
\frac{2n+2}{2n+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{8}{7} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \frac{8}{7} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \frac{8}{7} & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \frac{8}{7} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \frac{8}{7}
\end{array}_{(n+1)\times(n+1)}.$$

By a simple calculation, we obtain that the eigenvalues of $\mathcal{L}_A(W_{n+1}^2)$ are 0, $\frac{2}{7} + \frac{2n}{2n+1}$, $\frac{2}{7}(1 + 4sin^2\frac{k\pi}{n}), k \in \{1, 2, \cdots, n-1\}$ and $\mathcal{L}_S(W_{n+1}^2)$ is a diagonal matrix. By Lemma , one can easily see that the eigenvalues of

$\mathcal{L}(W^2_{n+1})$ are 0, $\frac{2}{7}+\frac{2n}{2n+1}$, $\frac{2}{7}(1+4sin^2\frac{k\pi}{n})k\in\{1,2,\cdots,n-1\}$, $\frac{2n+2}{2n+1}$, $\frac{8}{7}$ with multiplicity n. Then we obtain the following theorem.

For $n\geq 3$, let $W^2_{n+1}=K_2\boxtimes W_{n+1}$. Then
$Kf^*(W^2_{n+1})=\frac{32+9n(1+n)(15+7n)}{4+36n}+(n+1)\left(\sum_{k=1}^{n-1}\frac{7}{1+4sin^2(\frac{k\pi}{n})}\right)$.
$\lim_{n\to+\infty}\frac{Kf^*(W^2_{n+1})}{Gut(W^2_{n+1})}=\frac{1}{16}+\frac{\sqrt{5}}{20}$.

Proof.

(i) Since $|E(W^2_{n+1})|=9n+1$, It follows directly by Lemma lemma .

(ii) First we determine the Gutman index of W^2_{n+1}. We calculate $d_id_jd_{W^2_{n+1}}(i,j)$ for all the vertices (fixed i and for all j), then add all of them together and finally the sum is divided by two.

Let $g(i,W^2_{n+1})$ be the sum of multiplicative distance from the vertex i to other vertices in W^2_{n+1}. For $i=0$

$$g(i,W^2_{n+1})=(2n+1)\times(2n+1)+(2n+1)\times 7\times 1\times 2n=(1+2n)(1+16n).$$

For $i=1,2,3,\cdots,n$

$$g(i,W^2_{n+1})=7\times[(2n+1)\times 1\times 2+7\times 1\times 5+7\times 2\times(2n-6)]=7\times(32n-47).$$

Owing to the symmetry of the G^2_n, we obtain

$$Gut(W^2_{n+1})=\frac{1}{2}\left[2g(0,W^2_{n+1})+2\sum_{i=1}^{n}g(i,W^2_{n+1})\right]=g(0,W^2_{n+1})+\sum_{i=1}^{n}g(i,W^2_{n+1})=256n^2-311n+1.$$

Note that

$$Kf^*(W^2_{n+1})=\frac{32+9n(1+n)(15+7n)}{4+36n}+(n+1)\left(\sum_{k=1}^{n-1}\frac{7}{1+4sin^2(\frac{k\pi}{n})}\right),$$

Then

$$\lim_{n\to+\infty}\frac{Kf^*(W^2_{n+1})}{Gut(W^2_{n+1})}=\frac{7}{1024}+\frac{\sqrt{5}}{1280}.$$

As desired.

□

Resistance distance-based graph invariants and the number of spanning trees of $G^2_{n,r}$

Similar to the proof of Lemma 2.1 , the spectrum of $L(G^2_{n,r})$ consists of the eigenvalues of both $L_A(G^2_{n,r})$ and $L_S(G^2_{n,r})$. Let d_i be the degree of vertex i in G_n and d'_i be the degree of vertex i in $G^2_{n,r}$. Then $d'_i = 2d_i + 1$ or $2d_i$,$(i = 1, 2, 3, \cdots, n)$ in $G^2_{n,r}$. We rearrange the vertice such that $\{11', 22', \cdots, rr'\}$ is the deleted edges of $G^2_{n,r}$. Then we have

$$L_{V_{11}}(G^2_{n,r}) = \begin{pmatrix} d'_1 & -a_{1,2} & -a_{1,3} & -a_{1,4} & \cdots & -a_{1,n} \\ -a_{2,1} & d'_2 & -a_{2,3} & -a_{2,4} & \cdots & -a_{2,n} \\ -a_{3,1} & -a_{3,2} & d'_3 & -a_{3,4} & \cdots & -a_{3,n} \\ -a_{4,1} & -a_{4,2} & -a_{4,3} & d'_4 & \cdots & -a_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ -a_{n,1} & -a_{n,2} & -a_{n,3} & -a_{n,4} & \cdots & d'_n \end{pmatrix}_{n\times n},$$

where $d'_i = 2d_i, (i = 1, 2, \cdots, r)$, $d'_i = 2d_i + 1, (i = r + 1, \cdots, n)$, and

$$L_{V_{12}}(G^2_{n,r}) = \begin{pmatrix} t_1 & -a_{1,2} & -a_{1,3} & -a_{1,4} & \cdots & -a_{1,n} \\ -a_{2,1} & t_2 & -a_{2,3} & -a_{2,4} & \cdots & -a_{2,n} \\ -a_{3,1} & -a_{3,2} & t_3 & -a_{3,4} & \cdots & -a_{3,n} \\ -a_{4,1} & -a_{4,2} & -a_{4,3} & t_4 & \cdots & -a_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ -a_{n,1} & -a_{n,2} & -a_{n,3} & -a_{n,4} & \cdots & t_n \end{pmatrix}_{n\times n},$$

where $t_i = 0, (i = 1, 2, \cdots, r)$ and $t_i = -1, (i = r+1, r+2, \cdots, n)$. Then for any graph $G^2_{n,r} \in \mathcal{G}^2_{n,r}$, $d'_i + t_i = 2d_i$ holds for all $1 \leq i \leq n$. Since $L_A(G^2_{n,r}) = L_{V_{11}}(G^2_{n,r}) + L_{V_{12}}(G^2_{n,r})$, $L_S(G^2_{n,r}) = L_{V_{11}}(G^2_{n,r}) - L_{V_{12}}(G^2_{n,r})$. we get that

$$L_A(G^2_{n,r}) = 2 \begin{pmatrix} d_1 & -a_{1,2} & -a_{1,3} & -a_{1,4} & \cdots & -a_{1,n} \\ -a_{2,1} & d_2 & -a_{2,3} & -a_{2,4} & \cdots & -a_{2,n} \\ -a_{3,1} & -a_{3,2} & d_3 & -a_{3,4} & \cdots & -a_{3,n} \\ -a_{4,1} & -a_{4,2} & -a_{4,3} & d_4 & \cdots & -a_{4,n} \\ \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ -a_{n,1} & -a_{n,2} & -a_{n,3} & -a_{n,4} & \cdots & d_n \end{pmatrix}_{n\times n}$$

and

$$L_S(G^2_{n,r}) = 2\,($$

$$\begin{matrix} 2d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 2d_2 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 2d_r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 2d_{r+1}+2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2d_n+2 \end{matrix} \quad {}_{n\times n}.$$

Since $L_A(G^2_{n,r}) = 2L(G_n)$ and $L_S(G^2_n)$ is a diagonal matrix. Thus, the eigenvalues of $L(G^2_{n,r})$ are $2\rho_i, i = \{1, 2, \cdots, n\}$, $2d_i, i = \{1, 2, \cdots, r\}$ and $2d_i + 2, i = \{r + 1, r + 2, \cdots, n\}$, where $\rho_i, i = \{1, 2, \cdots, n\}$ are the eigenvalues of $L(G_n)$. Then we obtain the following theorem.

G_n is a graph with degree sequence $\{d_1, d_2, \cdots, d_n\}$, Then for any $G^2_{n,r} \in \mathcal{G}^2_{n,r}$, we have

$Kf(G^2_{n,r}) = kf(G_n) + n\sum_{i=1}^{n} \frac{1}{s_i}$.

$\tau(G^2_{n,r}) = \tau(G_n)4^{n-1}\Pi^n_{k=1}s_i$.
Where $s_i = d_i$ if $ii' \notin E(G^2_{n,r})$ and $s_i = d_i + 1$ if $ii' \in E(G^2_{n,r})$.

Proof.

(i) Let $s_i = d_i$ if $ii' \notin E(G^2_{n,r})$ and $s_i = d_i + 1$ if $ii' \in E(G^2_{n,r})$. Since $|V(G^2_{n,r})| = 2n$, by lemma , we have

(ii) It follows from lemma that

□

Let W_{n+1} be a wheel with $n+1$ vertices and the degree sequence of W_{n+1} is $\{n, 3, 3, \cdots, 3\}$. It is completely determined in (1985) that the eigenvalues of $L(W_{n+1})$ are 0, $n+1$, $1+4sin^2(\frac{k\pi}{n}), k \in \{1, 2, \cdots, n-1\}$. Using theorem , we have the following result

For any graph $W^2_{n+1,r} \in \mathcal{W}^2_{n+1,r}$, we have

$Kf(W^2_{n+1,r}) = \{$

$\frac{3n^3+(2+r)n^2+(23+r)n+12}{12n} + (n+1)\left(\sum_{k=1}^{n-1} \frac{1}{1+4sin^2(\frac{k\pi}{n})}\right)$, if $00' \notin E(W^2_{n+1,r})$,
$\frac{3n^2+(3+r)n+24+r}{12} + (n+1)\left(\sum_{k=1}^{n-1} \frac{1}{1+4sin^2(\frac{k\pi}{n})}\right)$, if $00' \in E(W^2_{n+1,r})$.

$\tau(W^2_{n+1,r}) = \{$

$n2^{5n-2r+1}3^{r-1}\Pi_{k=1}^{n-1}[1+4sin^2(\frac{k\pi}{n})]$, if $00' \notin E(W^2_{n+1,r})$,
$(n+1)2^{4n-2r}3^{r}\Pi_{k=1}^{n-1}[1+4sin^2(\frac{k\pi}{n})]$, if $00' \in E(W^2_{n+1,r})$.

$\lim_{n\to+\infty} \frac{Kf(W^2_{n+1,r})}{W(W^2_{n+1,r})} = \frac{1}{16} + \frac{\sqrt{5}}{20}$.

Proof. (i)-(ii) Note that$|V(W^2_{n+1,r})| = 2n+2$ for any graph $W^2_{n+1,r} \in \mathcal{W}^2_{n+1,r}$. We now consider two cases.
Case 1: If the edge $00'$ is included in the deleted edges:

As the eigenvalues of $L(W^2_{n+1,r})$ are 0, $2n+2$, $2+8sin^2(\frac{k\pi}{n}), k \in \{1,2,\cdots,n-1\}$, $2n$, 6 with multiplicity $r-1$ and 8 with multiplicity $n+1-r$, by lemma , we obtain

In addition, by Lemma , we have

Case 2: If the edge $00'$ is not included in the deleted edges:

As the eigenvalues of $L(W^2_{n+1,r})$ are 0, $2n+2$ with multiplicity 2, $2+8sin^2(\frac{k\pi}{n}), k \in \{1,2,\cdots,n-1\}$, 6 with multiplicity r and 8 with multiplicity $n-r$, by lemma , we obtain

In addition, by Lemma , we have

(iii)It is obvious that $W(W^2_{n+1,r}) = W(W^2_{n+1}) + r = 4n^2 - 3n + 1 + r$, No matter which value we choose as $Kf(W^2_{n+1,r})$, we have

$$\lim_{n\to+\infty} \frac{Kf(W^2_{n+1,r})}{W(W^2_{n+1,r})} = \frac{1}{16} + \frac{\sqrt{5}}{20}.$$

□

Using theorem () and equations (3), we obtain the following two corollarys.

(2020) For any graph $S^2_{n,r} \in \mathcal{S}^2_{n,r}$, we have

$Kf(S^2_{n,r}) = \{$

$\frac{3n^3-(2r-7)n^2+(9-3r)n+r-2}{2n-2}$, if $11' \notin E(S^2_{n,r})$,
$\frac{3n^2+(r-5)n+4}{2}$, if $11' \in E(S^2_{n,r})$.

$\tau(S^2_{n,r}) = \{$

$(n-1)\cdot 2^{3n-r-2}$, if $11' \notin E(S^2_{n,r})$,
$n \cdot 2^{3n-r-3}$, if $11' \in E(S^2_{n,r})$.
where $1, 1'$ is the vertice with degree $2n-1$ in S^2_n.

(Y. G. Pan et al., 2019) For any graph $C^2_{n,r} \in \mathcal{C}^2_{n,r}$, we have

$Kf(C^2_{n,r}) = \frac{n^3+4n^2+(2r-1)n}{12}$.

$\tau(C^2_{n,r}) = n \cdot 2^{2n+r-2} \cdot 3^{n-r}$.

Concluding remarks

Let G_n be a graph with given degree sequence. By using the normalized Laplacian decomposition theorem, we first establish the relations between the Laplacian eigenvalues of $G_n \boxtimes K_2$ and G_n. Using this relations we give the explicit closed formula between G^2_n and G_n for the Kirchhoff index and the total number of spanning trees, which generalize the main results of Z.M. Li et al. (Appl. Math. Comput., 2020, 382:125335), Y.G. Pan et al. (Int. J. Quantum Chem. 118 (2018) . E25787) and Y.G. Pan et al. (2019, ArXiv: 1906.04339). Typically,using this relationship we get the explicit expressions for the Kirchhoff index and the total number of spanning trees of $W^2_{n+1} = W_{n+1} \boxtimes K_2$, where W_{n+1} is a wheel with $n+1$ vertice. We also find that the Kirchhoff index of W^2_{n+1} is almost constant times of its Wiener index. Later, we presented the explicit expressions for the multiplicative degree-Kirchhoff indices of G^2_n when G_n is regular. We also given The explicit expressions for the multiplicative degree-Kirchhoff indices of W^2_{n+1} and the limited values for the quotients of multiplicative degree-Kirchhoff index and the Gutman index of W^2_{n+1}. Finally, we construct a family of graphs $G^2_{n,r}$ obtained from G^2_n by deleting any r vertical edges of G^2_n, The Kirchhoff index and the total number of spanning tree of $G_{n,r}^2$ have been given and we show that the Kirchhoff indices of these graphs are almost constant times of their Wiener indices. It would be interesting to determine their multiplicative degree-Kirchhoff indices and Gutman indices. We will do it in the near future.

Motivated by the construction $K_2 \boxtimes P_n$ in (Y. G. Pan & spanning trees of the linear crossed hexagonal chains, 2018), $K_2 \boxtimes C_n$ in (Y. G. Pan et al., 2019), $K_2 \boxtimes S_n$ in (2020) and this paper, we propose the following two questions for further study.

For a simple connected graph G, how can we determine the Laplacian spectrum and normalized Laplacian specteum of the graph $K_n \boxtimes G$?

How to determine the Kirchhhoff index, multiplicative degree-Kirchhoff index and number of spanning trees of graphs derived from the Catersian product of K_2 and G?

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