Quick Notes: The Levi-Civita Symbol, some of its properties and examples of use

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Abstract

This note is part of the collection of quick notes, corresponding to short writings on diverse topics of everyday use in the Atmospheric Sciences and Geophysical Fluid Dynamics.

This note shows some properties of the Levi-Civita Symbol, which is extremely useful in tensor manipulations, greatly simplifying the vectorial calculations related to the vector product between vectors. These notes are based on chapter 1 of the excellent book by (Sepúlveda, 2009).

"Half of mathematics is the art of saving space."
Mokokoma Mokhonoana

The Levi-Civita Symbol

Let $\{\hat{e_i}\}$ be the set of the unitary elements of a orthogonal base for the tridimensional space, where $i = 1, 2, 3$. Because the elements are orthonormal we can write:

$$\hat{e_i} \cdot \hat{e_j} = \delta_{ij}, \tag{1}$$

where δ_{ij} is the Kronecker delta defined as:

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \tag{2}$$

One definition of the Levi-Civita Symbol comes from the cross product between the elements of the orthogonal unit vectors that forms a base for the tridimensional space:

$$\hat{e_i} \times \hat{e_j} = \sum_{k=1}^{3} \epsilon_{ijk} \hat{e_k}, \tag{3}$$

where ϵ_{ijk} is the Levi-Civita symbol and is defined as:

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i.j,k) \text{ is an even permutation of } (1,2,3) \\ -1, & \text{if } (i.j,k) \text{ is an odd permutation of } (1,2,3) \\ 0, & \text{if there is a repeated index} \end{cases} \tag{4}$$

An even permutation of the index is defined in the direction of the flow of the arrows in Fig. 1. They correspond to the permutations 123, 231, and 312. An odd permutation corresponds to flow against the arrows, i.e., 132, 321, and 213.

One simple algebraic form to express these properties of the the Levi-Civita Symbol is given by:

$$\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i). \tag{5}$$

From the definition of ϵ_{ijk} given by Eq. (4), is clear that any change of contiguous index will introduce a minus sign, i.e, $\epsilon_{ijk} = -\epsilon_{ikj}$ or $\epsilon_{ijk} = -\epsilon_{jik}$.

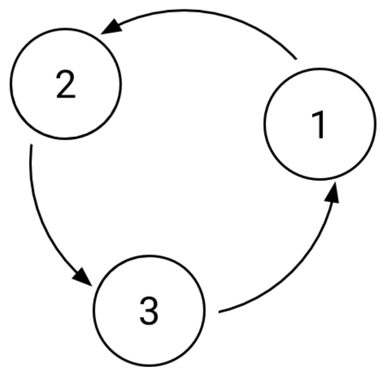

Figure 1: Even permutation of the index

Some properties of the Levi-Civita Symbol

One of the main applications of the Levi-Civita symbol is in the simplification of vector operations by using the summation notation and the benefits and simplifications that this notation brings for vector calculation. A very useful relationship is obtained from the $\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{lmk}$. Explicitly writing the terms of the summation gives:

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{lmk} = \epsilon_{ij1}\epsilon_{lm1} + \epsilon_{ij2}\epsilon_{lm2} + \epsilon_{ij3}\epsilon_{lm3} \tag{6}$$

It is clear that for each one the three elements of the right hand side, the values for the index i, j, l and m are restricted by the corresponding value of k. For example, for the first term $\epsilon_{ij1}\epsilon_{lm1}$: i, j, l and m can only be 2 or 3, otherwise this term will be zero. This leads us to consider only the next possibilities:

$$\epsilon_{ij1}\epsilon_{lm1} = \begin{cases} \epsilon_{231}\epsilon_{231} = 1 \\ \epsilon_{321}\epsilon_{321} = 1 \\ \epsilon_{231}\epsilon_{321} = -1 \\ \epsilon_{321}\epsilon_{231} = -1 \end{cases} \tag{7}$$

From Ec. (7), it is evident that $\epsilon_{ij1}\epsilon_{lm1} = 1$ if $i = l$ and $j = m$; $\epsilon_{ij1}\epsilon_{lm1} = -1$ if $i = m$ and $j = l$, and that $\epsilon_{ij1}\epsilon_{lm1} = 0$ if $i = j$ and $m = l$. It is straightforward to show that the same applies for the case $k = 2, 3$. From Eq. (6) and using Eq. (7), it is clear that when i, j, l, and m take fixed values, at most one of the three terms in Eq. (6) will be different from zero. We can write then:

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}, \tag{8}$$

Which can also be written in determinant form as:

$$\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{lmk} = \begin{vmatrix} \delta_{il} & \delta_{im} \\ \delta_{jl} & \delta_{jm} \end{vmatrix} \tag{9}$$

From Eq. (9), it seems that there is a relationship between the multiplication of two Levi-Civita symbols and the determinant. To find a relationship, let us study the multiplication of $\epsilon_{ijk}\epsilon_{lmn}$. We need to explore all of the possible non-zero elements of this product. These elements are tabulated in tables 1, 2, 3, 4 and 5.

From Table 1 for $i = l$, we get 1 if $j = m$ and $k = n$, and we get -1 if $j = n$ and $k = m$, i.e, we will have a term of the form $\delta_{il}\delta_{jm}\delta_{kn} - \delta_{il}\delta_{jn}\delta_{kn}$.

Table 1: Non-zero elements of $\epsilon_{ijk}\epsilon_{lmn}$ with $i = l$

$\epsilon_{123}\epsilon_{123} = 1$	$\epsilon_{231}\epsilon_{231} = 1$	$\epsilon_{312}\epsilon_{312} = 1$
$\epsilon_{132}\epsilon_{132} = 1$	$\epsilon_{213}\epsilon_{213} = 1$	$\epsilon_{321}\epsilon_{321} = 1$
$\epsilon_{132}\epsilon_{123} = -1$	$\epsilon_{231}\epsilon_{213} = -1$	$\epsilon_{312}\epsilon_{321} = -1$
$\epsilon_{123}\epsilon_{132} = -1$	$\epsilon_{213}\epsilon_{231} = -1$	$\epsilon_{321}\epsilon_{312} = -1$

From Table 2 for $j = m$, we get 1 if $i = l$ and $k = n$, and we get -1 if $i = n$ and $k = l$, i.e, we will have a term of the form $\delta_{il}\delta_{jm}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl}$.

Table 2: Non-zero elements of $\epsilon_{ijk}\epsilon_{lmn}$ with $j = m$

$\epsilon_{312}\epsilon_{312} = 1$	$\epsilon_{123}\epsilon_{123} = 1$	$\epsilon_{231}\epsilon_{231} = 1$
$\epsilon_{213}\epsilon_{213} = 1$	$\epsilon_{321}\epsilon_{321} = 1$	$\epsilon_{132}\epsilon_{132} = 1$
$\epsilon_{312}\epsilon_{213} = -1$	$\epsilon_{123}\epsilon_{321} = -1$	$\epsilon_{231}\epsilon_{132} = -1$
$\epsilon_{213}\epsilon_{312} = -1$	$\epsilon_{321}\epsilon_{123} = -1$	$\epsilon_{132}\epsilon_{231} = -1$

From Table 3 for $k = n$, we get 1 if $i = l$ and $j = m$, and we get -1 if $i = m$ and $j = l$, i.e, we will have a term of the form $\delta_{il}\delta_{jm}\delta_{kn} - \delta_{im}\delta_{jl}\delta_{kn}$.

Table 3: Non-zero elements of $\epsilon_{ijk}\epsilon_{lmn}$ with $k = n$

$\epsilon_{231}\epsilon_{231} = 1$	$\epsilon_{132}\epsilon_{132} = 1$	$\epsilon_{123}\epsilon_{123} = 1$
$\epsilon_{321}\epsilon_{321} = 1$	$\epsilon_{321}\epsilon_{312} = 1$	$\epsilon_{213}\epsilon_{213} = 1$
$\epsilon_{231}\epsilon_{321} = -1$	$\epsilon_{132}\epsilon_{312} = -1$	$\epsilon_{123}\epsilon_{213} = -1$
$\epsilon_{321}\epsilon_{231} = -1$	$\epsilon_{312}\epsilon_{132} = -1$	$\epsilon_{213}\epsilon_{123} = -1$

From Table 4 for $i = m$, we get 1 if $j = n$ and $k = l$, and we get -1 if $j = l$ and $k = n$, i.e, we will have a term of the form $\delta_{im}\delta_{jn}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn}$.

Table 4: Non-zero elements of $\epsilon_{ijk}\epsilon_{lmn}$ with $i = m$

$\epsilon_{123}\epsilon_{312} = 1$	$\epsilon_{231}\epsilon_{123} = 1$	$\epsilon_{312}\epsilon_{231} = 1$
$\epsilon_{132}\epsilon_{213} = 1$	$\epsilon_{213}\epsilon_{321} = 1$	$\epsilon_{321}\epsilon_{132} = 1$
$\epsilon_{123}\epsilon_{213} = -1$	$\epsilon_{231}\epsilon_{321} = -1$	$\epsilon_{312}\epsilon_{132} = -1$
$\epsilon_{132}\epsilon_{312} = -1$	$\epsilon_{213}\epsilon_{123} = -1$	$\epsilon_{321}\epsilon_{231} = -1$

Finally, From Table 5 for $i = n$, we get 1 if $j = l$ and $k = m$, and we get -1 if $j = m$ and $k = l$, i.e, we will have a term of the form $\delta_{in}\delta_{jl}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl}$.

Table 5: Non-zero elements of $\epsilon_{ijk}\epsilon_{lmn}$ with $i = n$

$\epsilon_{123}\epsilon_{231} = 1$	$\epsilon_{231}\epsilon_{312} = 1$	$\epsilon_{312}\epsilon_{123} = 1$
$\epsilon_{132}\epsilon_{321} = 1$	$\epsilon_{213}\epsilon_{132} = 1$	$\epsilon_{321}\epsilon_{213} = 1$
$\epsilon_{123}\epsilon_{321} = -1$	$\epsilon_{231}\epsilon_{132} = -1$	$\epsilon_{312}\epsilon_{213} = -1$
$\epsilon_{132}\epsilon_{231} = -1$	$\epsilon_{213}\epsilon_{312} = -1$	$\epsilon_{321}\epsilon_{123} = -1$

Note that we can summarize the results obtained from Tables 1 to 5, by writing the multiplication of $\epsilon_{ijk}\epsilon_{lmn}$ as:

$$\begin{aligned}\epsilon_{ijk}\epsilon_{lmn} &= \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{kn} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn} \\ &= \delta_{il}\left(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{kn}\right) - \delta_{im}\left(\delta_{jl}\delta_{kn} - \delta_{jn}\delta_{kl}\right) + \delta_{in}\left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right),\end{aligned} \tag{10}$$

Which can be written using the determinant as:

$$\epsilon_{ijk}\epsilon_{lmn} = \begin{vmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{vmatrix} \tag{11}$$

From Eq. (11), we can also use the Levi-Civita symbol to define the determinant of a matrix in summation form. As we know, the determinant of a matrix is given by:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{12}(a_{23}a_{31} - a_{21}a_{33}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \tag{12}$$

From Eq. (12), we can write the determinant of a matrix as the triple scalar product $\vec{a} \cdot (\vec{b} \times \vec{c})$, where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are defined as:

$$\begin{aligned}\vec{a} &= \sum_{i=1}^{3} a_{1i}\hat{e_i}, \\ \vec{b} &= \sum_{j=1}^{3} a_{2j}\hat{e_j}, \\ \vec{c} &= \sum_{k=1}^{3} a_{3k}\hat{e_k}.\end{aligned} \tag{13}$$

From these definitions and using the triple scalar product we get:

$$
\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \vec{a} \cdot \vec{b} \times \vec{c} \begin{aligned} &= \sum_{i=1}^{3} a_{1i}\hat{e_i} \cdot \sum_{jk=1}^{3} a_{2j}a_{3k}\hat{e_j} \times \hat{e_k} \\ &= \sum_{ijkl=1}^{3} a_{1i}a_{2j}a_{3k}\epsilon_{ljk}\hat{e_i} \cdot \hat{e_l} \\ &= \sum_{ijkl=1}^{3} a_{1i}a_{2j}a_{3k}\epsilon_{ljk}\delta_{il} \\ &= \sum_{ijk=1}^{3} \epsilon_{ijk}a_{1i}a_{2j}a_{3k} \end{aligned} \tag{14}
$$

Examples

Here we solve solve problems related to the use of the Levi-Civita Symbol.

Example 1.

Show that $\sum_{jk=1}^{3} \epsilon_{ijk}\epsilon_{ljk} = 2\delta_{ij}$.

Solution

Using Eq.(8) we can write:

$$
\begin{aligned} \sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{ijk}\epsilon_{ljk} &= \sum_{j=1}^{3} (\delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl}) \\ &= \sum_{j=1}^{3} (\delta_{il} - \delta_{ij}\delta_{jl}) \\ &= \sum_{j=1}^{3} (\delta_{il}) - \sum_{j=1}^{3} (\delta_{ij}\delta_{jl}) \\ &= 3\delta_{il} - \delta_{il} \\ &= 2\delta_{il} \end{aligned}
$$

Example 2

Show that $\sum_{ijk=1}^{3} \epsilon_{ijk}\epsilon_{ijk} = 6$.

Solution

Using Eq. (8) we can write:

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{ijk}\epsilon_{ijk} &= \sum_{i=1}^{3}\sum_{j=1}^{3}(\delta_{ii}\delta_{jj}-\delta_{ij}\delta_{ji}) \\
&= \sum_{i=1}^{3}\sum_{j=1}^{3}(\delta_{ii}\delta_{jj}) - \sum_{i=1}^{3}\sum_{j=1}^{3}(\delta_{ij}\delta_{ji}) \\
&= \sum_{i=1}^{3}(3\delta_{ii}) - \sum_{i=1}^{3}(\delta_{ii}) \\
&= 9-3 \\
&= 6
\end{aligned}$$

Example 3

Show that $\sum_{j=1}^{3}\delta_{ij}\epsilon_{ijk}=0$.

Solution

Using the properties of the Kronecker delta we have:

$$\begin{aligned}
\sum_{j=1}^{3}\delta_{ij}\epsilon_{ijk} &= \epsilon_{iik} \\
&= 0
\end{aligned}$$

Example 4

Show that $\vec{A}\times\vec{B}=\sum_{ijk}\hat{e_i}\epsilon_{ijk}A_jB_k$.

Solution

We can write $\vec{A}$ and $\vec{B}$ in term of their components $\{A_i\}$ and $\{B_i\}$ in the base $\{\hat{e_i}\}$. Therefore, we have:

$$\vec{A}=\sum_{j=1}^{3}A_j\hat{e_j} \text{ and } \vec{B}=\sum_{k=1}^{3}B_k\hat{e_k}$$

From this we can write:

$$\begin{aligned}
\vec{A}\times\vec{B} &= \sum_{j=1}^{3}\sum_{k=1}^{3}A_jB_k\hat{e_j}\times\hat{e_k} \\
&= \sum_{j=1}^{3}\sum_{k=1}^{3}A_jB_k\sum_{i=1}^{3}\epsilon_{jki}\hat{e_i} \text{ (Using Eq. (3))} \\
&= \sum_{ijk=1}^{3}\hat{e_i}\epsilon_{ijk}A_jB_k \text{ (Reordering the index, two changes)}
\end{aligned}$$

Example 5

Show that $\vec{A} \cdot \vec{B} \times \vec{C} = \sum_{ijk} \epsilon_{ijk} A_i B_j C_k$.

Solution

$$\begin{aligned}
\vec{A} \cdot \vec{B} \times \vec{C} &= \sum_{i=1}^{3} A_i \hat{e_i} \cdot \sum_{ljk=1}^{3} \hat{e_l} \epsilon_{ljk} B_j C_k \text{ (Using Eq.(3))} \\
&= \sum_{iljk=1}^{3} \delta_{il} \epsilon_{ljk} A_i B_j C_k \\
&= \sum_{ijk=1}^{3} \epsilon_{ijk} A_i B_j C_k
\end{aligned}$$

Example 6

Show that $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B}$.

Solution

Using the results from Example 5 we have:

$$\begin{aligned}
\vec{A} \cdot \vec{B} \times \vec{C} &= \sum_{ijk=1}^{3} \epsilon_{ijk} A_i B_j C_k \\
&= \sum_{ijkl=1}^{3} \delta_{jl} \epsilon_{ilk} A_i B_j C_k \\
&= \sum_{j=1}^{3} B_j \hat{e_j} \cdot \sum_{jkl=1}^{3} \hat{e_l} \epsilon_{lki} C_k A_i \\
&= \vec{B} \cdot \vec{C} \times \vec{A} \\
&= \sum_{ijkl=1}^{3} \delta_{kl} \epsilon_{ijl} A_i B_j C_k \\
&= \sum_{k=1}^{3} C_k \hat{e_k} \cdot \sum_{ijl=1}^{3} \hat{e_l} \epsilon_{lij} A_i B_j \\
&= \vec{C} \cdot \vec{A} \times \vec{B}
\end{aligned}$$

Example 7

Show that $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$. This is also known as the BAC-CAB rule.

Solution

$$
\begin{aligned}
\vec{A} \cdot \vec{B} \times \vec{C} &= \sum_{i=1}^{3} A_i \hat{e_i} \times \sum_{ljk=1}^{3} \hat{e_l} \epsilon_{ljk} B_j C_k \\
&= \sum_{iljk=1}^{3} A_i B_j C_k \epsilon_{ljk} \hat{e_i} \times \hat{e_l} \\
&= \sum_{iljkm=1}^{3} A_i B_j C_k \epsilon_{ljk} \epsilon_{ilm} \hat{e_m} \\
&= \sum_{ijkm=1}^{3} (-1) A_i B_j C_k \hat{e_m} \sum_{l=1}^{3} \epsilon_{jkl} \epsilon_{iml} \\
&= \sum_{ijkm=1}^{3} (-1) A_i B_j C_k \hat{e_m} \left(\delta_{ji} \delta_{km} - \delta_{jm} \delta_{ki} \right) \\
&= \sum_{ijkm=1}^{3} A_i B_j C_k \hat{e_m} \delta_{jm} \delta_{ki} - \sum_{ijkm=1}^{3} A_i B_j C_k \hat{e_m} \delta_{ji} \delta_{km} \\
&= \sum_{ik=1}^{3} A_i C_k \hat{e_i} \cdot \hat{e_k} \sum_{j=1}^{3} B_j \hat{e_j} - \sum_{ij=1}^{3} A_i B_j \hat{e_i} \cdot \hat{e_j} \sum_{k=1}^{3} C_k \hat{e_k} \\
&= (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}
\end{aligned}
$$

Example 8

Show that $\vec{A} \cdot (\vec{B} \times \vec{A}) = 0$.

Solution

Using Example 6 we have:

$$
\begin{aligned}
\vec{A} \cdot \vec{B} \times \vec{A} &= \vec{B} \cdot \vec{A} \times \vec{A} \\
&= 0
\end{aligned}
$$

Example 9

Show that $(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$.

Solution

$$
\begin{aligned}
(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) &= \left(\sum_{ijk}\hat{e}_i\epsilon_{ijk}A_jB_k\right)\cdot\left(\sum_{lmn}\hat{e}_l\epsilon_{lmn}C_mD_n\right) \\
&= \sum_{ijklmn} A_jB_kC_mD_n\epsilon_{ijk}\epsilon_{lmn}\delta_{il} \\
&= \sum_{ijkmn} A_jB_kC_mD_n\epsilon_{ijk}\epsilon_{imn} \\
&= \sum_{jkmn} A_jB_kC_mD_n\sum_{i}\epsilon_{jki}\epsilon_{mni} \\
&= \sum_{jkmn} A_jB_kC_mD_n\left(\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km}\right) \\
&= \sum_{jkmn} A_jB_kC_mD_n\delta_{jm}\delta_{kn}-\sum_{jkmn} A_jB_kC_mD_n\delta_{jn}\delta_{km} \\
&= \sum_{jm} A_jC_m\delta_{jm}\sum_{jkmn} B_kD_n\delta_{kn}-\sum_{jn} A_jD_n\delta_{jn}\sum_{km} B_kC_m\delta_{km} \\
&= (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})
\end{aligned}
$$

Example 10

Show that $(\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) = A^2B^2-(\vec{A}\cdot\vec{B})^2$.

Solution

Using Example 9 we have

$$
\begin{aligned}
(\vec{A}\times\vec{B})\cdot(\vec{A}\times\vec{B}) &= (\vec{A}\cdot\vec{A})(\vec{B}\cdot\vec{B})-(\vec{A}\cdot\vec{B})(\vec{B}\cdot\vec{A}) \\
&= A^2B^2-(\vec{A}\cdot\vec{B})^2
\end{aligned}
$$

References

Física matemática. (2009). Universidad de Antioquia.