Inverse Transform with Mellin like Feature

Benedict Irwin[1]

[1]University of Cambridge

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Create a transform somewhat in analogy to the Mellin transform which to some extent extracts sequence coefficients.

$$\mathcal{M}_s[f(x)](s) \approx \Gamma(s)\phi(-s) \tag{1}$$

where

$$f(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}\phi(s)x^s \tag{2}$$

instead consider a transform $\mathcal{I}[f(x)](s)$ such that

$$\mathcal{I}[f(x)](s) \approx \Gamma(s)\chi(-s) \tag{3}$$

where

$$\frac{f^{-1}(x)}{x} = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!}\chi(s)x^s \tag{4}$$

and example, the function

$$f(x) = x + x^2 \tag{5}$$

has inverse as series

$$f^{-1}(x) = x - x^2 + 2x^3 - 5x^4 + \cdots \tag{6}$$

$$\frac{f^{-1}(x)}{x} = 1 - x + 2x^2 - 5x^3 + \cdots = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} C_s x^s = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \frac{(2s)!}{(s+1)!} x^s \tag{7}$$

then

$$\chi(s) = \frac{\Gamma(1+2s)}{\Gamma(2+s)} \tag{8}$$

then

$$\mathcal{I}[x+x^2](s) = \frac{\Gamma(s)\Gamma(1-2s)}{\Gamma(2-s)} \tag{9}$$

$$x\mathcal{M}^{-1}[\mathcal{I}[x+x^2](s)](x) = f^{-1}(x) \tag{10}$$

Results

Then it would seem that

$$\mathcal{I}[2x^2 - \sqrt{x^2+4x^4}](s) = \frac{\Gamma(s)\Gamma(1-2s)}{\Gamma(1-s)} \tag{11}$$

$$\mathcal{I}[W(x)](s) = \Gamma(s)(-1)^s \tag{12}$$

$$\mathcal{I}[-W(-x)](s) = \Gamma(s) \tag{13}$$

$$\mathcal{I}\left[\frac{x}{1-x}\right](s) = \Gamma(s)\Gamma(1-s) \tag{14}$$

$$\mathcal{I}\left[\frac{1-\sqrt{1-4x}-2x}{2x}\right](s) = \Gamma(s)\Gamma(2-s) \tag{15}$$

$$\mathcal{I}\left[\log\left(\frac{1}{x}\right)\right](s) = \Gamma(s-1) \tag{16}$$

$$\mathcal{I}\left[W\left(\frac{1}{x}\right)\right](s) = \Gamma(s-2) \tag{17}$$

$$\mathcal{I}\left[e^x - 1\right](s) = \frac{\Gamma(s)\Gamma(1-s)^2}{\Gamma(2-s)} \tag{18}$$

$$\mathcal{I}\left[\log(x)\right](s) = (-1)^{1-s}\Gamma(s-1) \tag{19}$$

$$\mathcal{I}\left[\frac{1}{e^x-1}\right](s) = \frac{\pi\csc(\pi s)}{1-s} \tag{20}$$

$$\mathcal{I}\left[-x - W(-xe^{-x})\right](s) = \Gamma(s)\zeta(s) \tag{21}$$

$$\tag{22}$$

Some more generalised ones

$$\mathcal{I}[\log(x^k)](s) = \left(-\frac{1}{k}\right)^{1-s}\Gamma(s-1) \tag{23}$$

$$\mathcal{I}\left[W\left(\frac{1}{x^k}\right)\right](s) = k^{-1-1/k+s}\Gamma(s-1-\frac{1}{k}) \tag{24}$$

$$\mathcal{I}\left[W\left(x^k\right)\right](s) = (-k)^{-1+1/k+s}\Gamma(s-1+\frac{1}{k}) \tag{25}$$

$$\mathcal{I}\left[-\frac{x}{x+W(-e^{-x}x)}\right](s) = \frac{\Gamma(s)\Gamma(1-s)}{s} \tag{26}$$

$$\mathcal{I}\left[-2W(-\frac{\sqrt{x}}{2})\right](s) = s^2\Gamma(s) \tag{27}$$

$$\mathcal{I}\left[\frac{1}{\log(k/x)}\right](s) = k\Gamma(1-s) \tag{28}$$

$$\mathcal{I}\left[\frac{1}{1-W(ex/k)}\right](s) = ks\Gamma(1-s) \tag{29}$$

$$\mathcal{I}\left[-x^kW(\frac{x^{1-k}}{A})\right](s) = -Ax^{ks}\Gamma(s) \tag{30}$$

$$\tag{31}$$

As described in a previous article on here: It would appear that for the function

$$f(x) = x^m + x, m > 1 \tag{32}$$

we get a series

$$g(x) = \sum_{n=0}^{\infty} \binom{mn}{n} \frac{(-1)^n x^{(m-1)n+1}}{(m-1)n+1} \tag{33}$$

these then have a set of consistent, hypergeometric series explainable as

$$g(x) = {}_{(m-1)}F_{(m-2)}\left(\left\{\frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m}\right\}; \left\{\frac{2}{m-1}, \cdots, \frac{m-2}{m-1}, \frac{m}{m-1}\right\}; -\frac{m^m x^{m-1}}{(m-1)^{m-1}}\right) \cdot x \tag{34}$$

then

$$\frac{g(x)}{x} = {}_{(m-1)}F_{(m-2)}\left(\left\{\frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m}\right\}; \left\{\frac{2}{m-1}, \cdots, \frac{m-2}{m-1}, \frac{m}{m-1}\right\}; -\frac{m^m x^{m-1}}{(m-1)^{m-1}}\right) \tag{35}$$

which would give

$$\mathcal{I}[x+x^m](s) = \mathcal{M}_x[\ {}_{(m-1)}F_{(m-2)}\left(\left\{\frac{1}{m}, \frac{2}{m}, \cdots, \frac{m-1}{m}\right\}; \left\{\frac{2}{m-1}, \cdots, \frac{m-2}{m-1}, \frac{m}{m-1}\right\}; -\frac{m^m x^{m-1}}{(m-1)^{m-1}}\right)](s) \tag{36}$$

which gives

$$\mathcal{I}[x+x^2](s) = \frac{\Gamma(s/1)\Gamma(1-2s/1)}{\Gamma(2-s)} \tag{37}$$

$$\mathcal{I}[x+x^3](s) = \frac{\Gamma(1-\frac{3s}{2})\Gamma(\frac{s}{2})}{2\Gamma(2-s)} \tag{38}$$

$$\mathcal{I}[x+x^4](s) = \frac{8 \cdot 3^{s-\frac{5}{2}} \pi \Gamma(-4s/3)\Gamma(s/3)}{\Gamma(2/3-s/3)\Gamma(4/3-s/3)\Gamma(-s/3)} =?= \frac{\Gamma(1-4s/3)\Gamma(s/3)}{3\Gamma(2-s)} \tag{39}$$

it then seems like

$$\frac{\Gamma\left(1-\frac{ms}{m-1}\right)\Gamma\left(\frac{s}{m-1}\right)}{(m-1)\Gamma(2-s)} = \mathcal{I}_x[x+x^m](s) \tag{40}$$

and more generally

$$\frac{a^{\frac{s}{m-1}}\Gamma\left(1-\frac{ms}{m-1}\right)\Gamma\left(\frac{s}{m-1}\right)}{(m-1)\Gamma(2-s)} = \mathcal{I}_x[x+ax^m](s) \tag{41}$$

Further Small polynomials

There are some other small polynomials that give integer sequences upon reversion. Consider $x - x^2 - x^3$.

$$\mathcal{I}_s^{-1}[\Gamma(a+bs)](x) = (-1)^b (b+a)^b W\left(-\frac{b^{\frac{1}{b+a}} x^{\frac{1}{b+a}}}{b+a}\right)^b \tag{42}$$