The Averaged Hydrostatic Boussinesq Equations in Generalized Vertical Coordinates

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Abstract

Due to their limited resolution, numerical ocean models need to be interpreted as representing filtered or averaged equations. How to interpret models in terms of formally averaged equations, however, is not always clear, particularly in the case of hybrid or generalized vertical coordinate models. We derive the averaged hydrostatic Boussinesq equations in generalized vertical coordinates for an arbitrary thickness weighted-average. We then consider various special cases and discuss the extent to which the averaged equations are consistent with existing model formulations. As previously discussed, the momentum equations in existing depth-coordinate models are best interpreted as representing Eulerian averages (i.e., averages taken at fixed depth), while the tracer equations can be interpreted as either Eulerian or thickness-weighted isopycnal averages. Instead we find that no averaging is fully consistent with existing formulations of the parameterizations in semi-Lagrangian discretizations of generalized vertical coordinate ocean models. Perhaps the most natural interpretation of generalized vertical coordinate models is to assume that the average follows the model’s coordinate surfaces. However, the existing model formulations are generally not consistent with coordinate-following averages, which would require “coordinate-aware” parameterizations that can account for the changing nature of the eddy terms as the coordinate changes. Alternatively, the model variables can be interpreted as representing either Eulerian or (thickness-weighted) isopycnal averages, independent of the model coordinate that is being used for the numerical discretization. Existing parameterizations in generalized vertical coordinate models, however, are usually not fully consistent with either of these interpretations. We discuss what changes are needed to achieve consistency.
\[ b = b^#(x', z') \]

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Key Points:

\begin{itemize}
  \item We derive and discuss the arbitrarily-averaged hydrostatic Boussinesq equations in generalized vertical coordinates.
  \item Known results for Eulerian- and isopycnal-mean equations in depth and isopycnal coordinates are recovered as special cases.
  \item No choice of averaging leads to generalized vertical coordinate equations that are fully consistent with existing semi-Lagrangian models.
\end{itemize}

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Abstract

Due to their limited resolution, numerical ocean models need to be interpreted as representing filtered or averaged equations. How to interpret models in terms of formally averaged equations, however, is not always clear, particularly in the case of hybrid or generalized vertical coordinate models. We derive the averaged hydrostatic Boussinesq equations in generalized vertical coordinates for an arbitrary thickness weighted-average. We then consider various special cases and discuss the extent to which the averaged equations are consistent with existing model formulations. As previously discussed, the momentum equations in existing depth-coordinate models are best interpreted as representing Eulerian averages (i.e., averages taken at fixed depth), while the tracer equations can be interpreted as either Eulerian or thickness-weighted isopycnal averages. Instead we find that no averaging is fully consistent with existing formulations of the parameterizations in semi-Lagrangian discretizations of generalized vertical coordinate ocean models. Perhaps the most natural interpretation of generalized vertical coordinate models is to assume that the average follows the model’s coordinate surfaces. However, the existing model formulations are generally not consistent with coordinate-following averages, which would require “coordinate-aware” parameterizations that can account for the changing nature of the eddy terms as the coordinate changes. Alternatively, the model variables can be interpreted as representing either Eulerian or (thickness-weighted) isopycnal averages, independent of the model coordinate that is being used for the numerical discretization. Existing parameterizations in generalized vertical coordinate models, however, are usually not fully consistent with either of these interpretations. We discuss what changes are needed to achieve consistency.

Plain Language Summary

Numerical ocean models represent continuous three-dimensional physical fields using discrete data points and hence cannot adequately represent variability at all scales. Instead model variables need to be interpreted as representing a filtered version of the full physical fields. We here derive the evolution equations for horizontally-filtered fields, where the horizontal filtering follows arbitrary surfaces, with the only requirement being that the surfaces do not fold over (i.e. they are iso-surfaces of a field that is monotonic in depth). The equations for the filtered variables are formulated using an arbitrary vertical coordinate system, thus making them applicable to a wide range of different numerical ocean models. We then consider different physically-motivated choices for the averaging surfaces and express the equations using different common choices for the vertical coordinate system. We conclude by discussing which equation sets are consistent with the equations solved by existing numerical models and/or which modifications are needed to achieve consistency. The results have important implications for ocean model development, because the models need to include "parameterizations" that represent the effect of motions that have been removed by the filter. The formulation of these parameterizations needs to be consistent with the filtering operation assumed in the model equations.

1 Introduction

Due to their limited resolution, numerical ocean models cannot resolve the full spectrum of motions. This limitation is widely recognized and motivates the need for so-called parameterizations (or closures), i.e., additional terms added to the model’s equations that are meant to capture the effect of dynamics that cannot be explicitly resolved by the models. To formalize the idea that the model resolves only part of the motion, a suitable averaging can be applied to the equations of motion to obtain equations for the evolution of these averaged quantities. Due to the nonlinearity of the equations of motion these
averaged equations retain terms that involve deviations from the average (i.e., contributions from the unresolved flow), which then need to be parameterized. Applying a formal averaging to the equations of motion allows us to identify (1) what the model variables are meant to represent and (2) what needs to be captured by the parameterizations. An explicitly defined averaging procedure is therefore necessary to cleanly compare the model results against observations and to devise and test parameterizations.

Types of averages

In practice it is often not clear what kind of average the model variables should represent, and many averaging procedures have been suggested. Existing averaging procedures may be broadly organized into three categories: (1) Reynolds averaging (2) (non-Reynolds) spatial filtering (3) grid-cell averaging as part of the numerical discretization. The third category applies most immediately to finite-volume methods, where the discrete model variables by construction represent grid-box or grid-face averages (e.g. Griffies et al., 2020). If this is the only averaging considered, the nature of the “sub-grid” terms that need to be parameterized depends directly on choices of the numerical discretization, which complicates the development of adequate parameterizations. Moreover, full resolution of variability on all scales that can be represented by the model grid is an unrealistic goal, given imperfect numerical methods. We will therefore here focus on the first two approaches, where the continuous equations are averaged explicitly before the discretization. The goal of the averaging is to obtain fields that are relatively smooth at the grid-scale, such that the results are less sensitive to the details of the numerical discretization, although we notice that a second grid-cell averaging (of the filtered equations) may still be performed as part of a finite-volume discretization.

Eddy parameterizations are often motivated by the eddy terms arising in the Reynolds-averaged equations. Reynolds averages are defined by the convenient property that \( \overline{ab} = \overline{a} \overline{b} \), where the overbar denotes the averaging operator, which is commonly taken to be a time-mean (e.g. Gent et al., 1995; McDougall & McIntosh, 2001), a zonal-mean in a zonally re-entrant domain (e.g. Bachman & Fox-Kemper, 2013) or an ensemble mean over different possible realizations of the turbulent flow (e.g. Young, 2012; Uchida et al., 2022).

However, it has also long been acknowledged that models are fundamentally limited in their ability to represent small spatial scales, such that a spatial filtering with a finite-size filter-stencil would be the more appropriate averaging operator (McDougall & McIntosh, 2001; Fox-Kemper & Menemenlis, 2008). Spatial filters are generally not Reynolds operators and can only be approximated as such in the presence of a clear scale separation, which does not generally exist in the ocean. Luckily, many of the results obtained using Reynolds averaging readily carry over to spatial filters, as will be made explicit for the results of this manuscript.

Averaging coordinate surfaces

Independent of whether the average is in space, time, or over an ensemble, different choices can be made for the averaging coordinate, with the most widely used approaches being “Eulerian” averages, where the average is taken at fixed height (e.g. Bachman & Fox-Kemper, 2013), and isopycnal averages, where the average is taken along a surface of constant potential or neutral density (e.g. Young, 2012). For isopycnal averages we can further employ thickness-weighted and non-weighted averages.

Although it is natural to interpret the model variables as representing averages along the model coordinate, this is not the only possible interpretation. Indeed, McDougall & McIntosh (2001) argued that the formulation of the Gent & McWilliams (1990) (hereafter GM) parameterization used in many z-coordinate ocean models is not fully consistent with the form of the eddy terms in the Eulerian averaged tracer equations. For better consistency, they argued that the model’s tracer variables should instead be interpreted
as thickness-weighted isopycnal averages (with the average taken along the isopycnal whose mean depth equals the depth of the model level), while the model velocities represent Eulerian averages. However, this interpretation has not been consistently adopted (e.g. Bachman & Fox-Kemper, 2013; Zanna & Bolton, 2020).

Similarly, Loose et al. (2023) pointed out that the common formulation of isopycnal ocean models is not fully consistent with an isopycnally-averaged interpretation, neither with nor without thickness weighting. In the non-thickness weighted isopycnal-mean interpretation, the parameterized (“GM-like”) eddy flux term in the continuity equation can be interpreted as representing the bolus transport (which is not quite the same as the interpretation of the GM parameterization in z-coordinates—see McDougall & McIntosh, 2001), but the eddy terms in the momentum and tracer equations become non-conservative with this average, which is both undesirable and inconsistent with the existing parameterization approaches. A thickness-weighted average yields no eddy term in the continuity equation but instead introduces an eddy pressure gradient contribution (related to the form stress) in the momentum equations. Loose et al. (2023) propose that this term can be parameterized in the form of a vertical viscosity, following the ideas of Rhines & Young, (1982) and Greatbatch & Lamb (1990), which in practice yields similar results as the existing GM-like parameterization. However, this approach is not presently used in any operational ocean models. In summary, the question of how to formally interpret the numerical model equations in terms of averaged equations remains not fully settled in both z-coordinate and isopycnal ocean models.

Matters are further complicated by the rise of hybrid or “generalized” vertical coordinate models, such as the Hybrid Coordinate Model (HYCOM, Bleck, 2002; Chassignet et al., 2007), the General Estuarine Transport Model (GETM, Hofmeister et al., 2010), the ocean component of the Model for Prediction Across Scales (MPAS-Ocean, Petersen et al., 2015), or the Modular Ocean Model 6 (MOM6, Adcroft et al., 2019). In MOM6 and HYCOM the time discretization is semi-Lagrangian, which allows for wide flexibility in the choice of target vertical coordinate (Griffies et al., 2020). The MOM6 configuration employed in the Geophysical Fluid Dynamics Laboratory’s current global climate and Earth System models uses a hybrid target coordinate, following potential density surfaces over most of the ocean’s interior ocean while transitioning to depth coordinates near the surface and in unstratified regions (Adcroft et al., 2019). A similar hybrid vertical coordinate is used in HYCOM, where the vertical coordinate further transitions to being terrain-following over shallow coastal seas (Chassignet et al., 2007). How to interpret these models in terms of formally averaged equations remains largely unexplored.

**Manuscript overview**

In this manuscript we derive the averaged dynamical equations in arbitrary vertical coordinates and discuss implications for the interpretation of numerical model variables and the need for parameterizations. We start in section 2 by deriving the averaged hydrostatic Boussinesq equations in arbitrary (“generalized”) vertical coordinates, with the average following the vertical coordinate. We focus on a generalized thickness-weighted average, which is the only conservative average in a non-Eulerian coordinate system (c.f. Loose et al., 2023). In the special case where the vertical coordinate is height, \( z \), the thickness-weighting has no effect, such that the traditional Eulerian averaged equations are retained as a special case of the generalized thickness weighted average equations. For completeness, we will briefly discuss the non-thickness-weighed average in Appendix A.

In section 3 we consider the case where the averaging surfaces do not line up with the vertical coordinate used to express the final averaged equations. This most general case is derived by transforming the averaged equations obtained in section 2 into a second arbitrary vertical coordinate system. Armed with this general result we consider var-
Table 1. Summary of select symbols used in the text.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
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<tbody>
<tr>
<td>$c$</td>
<td>arbitrary scalar tracer (incl. temperature and salinity)</td>
</tr>
<tr>
<td>$\mathbf{v}$</td>
<td>3D velocity vector: $\mathbf{v} = u \mathbf{i} + v \mathbf{j} + w \mathbf{k}$</td>
</tr>
<tr>
<td>$\mathbf{u}$</td>
<td>horizontal velocity vector: $\mathbf{u} = u \mathbf{i} + v \mathbf{j}$</td>
</tr>
<tr>
<td>$\nabla$</td>
<td>3D nabla operator: $\nabla = i \partial_x + j \partial_y + k \partial_z$</td>
</tr>
<tr>
<td>$\nabla_r$</td>
<td>“horizontal” nabla operator with derivatives at fixed $r$: $\nabla_r = i \partial_x</td>
</tr>
<tr>
<td>$\phi$</td>
<td>geopotential: $\phi = g z$</td>
</tr>
<tr>
<td>$z_r$</td>
<td>generalized “thickness” w.r.t. coordinate $r$: $z_r \equiv \partial_r z$</td>
</tr>
<tr>
<td>$\dot{r}$</td>
<td>Lagrangian rate of change of “$r$”: $\dot{r} \equiv \frac{D r}{D t}$</td>
</tr>
<tr>
<td>$\dot{c}$</td>
<td>tracer tendencies due to diffusion and sources or sinks</td>
</tr>
<tr>
<td>$\mathcal{F}_{x/y}$</td>
<td>zonal/meridional accelerations due to frictional and external forces</td>
</tr>
<tr>
<td>$\langle \cdot \rangle$</td>
<td>average at fixed $r$</td>
</tr>
<tr>
<td>$\langle \cdot \rangle^r$</td>
<td>generalized thickness-weighted average at fixed $r$: $\langle \cdot \rangle^r \equiv \overline{z_r(\cdot)} / \overline{z^r}$</td>
</tr>
<tr>
<td>$\nabla^r \cdot \mathbf{J}^{c/u/v}$</td>
<td>eddy tracer/momentum flux divergence (Eqs. 18, 19)</td>
</tr>
<tr>
<td>$\nabla^r \cdot \mathbf{E}^{u/v}$</td>
<td>“EP” flux divergence (Eqs. 22, 23)</td>
</tr>
<tr>
<td>$\mathbf{v}_{#}^a$</td>
<td>Lagrangian rate of change of $r$ following $a$-averaged flow (Eq. 30)</td>
</tr>
<tr>
<td>$\mathbf{w}_{#}^b$</td>
<td>“residual” vertical velocity (Eq. 61)</td>
</tr>
<tr>
<td>$b_{#}(x, y, z, t)$</td>
<td>buoyancy surface whose mean height is $z$, i.e., $z_{#}(x, y, b_{#}(x, y, z), t) = z$</td>
</tr>
<tr>
<td>$\langle \cdot \rangle_{b_{#}}$</td>
<td>thickness-weighted average along buoyancy surface $b = b_{#}$ (Eq. 62)</td>
</tr>
<tr>
<td>$\langle \cdot \rangle_{#}$</td>
<td>denotes a model variable</td>
</tr>
<tr>
<td>$\tau_{u/v}$</td>
<td>zonal/meridional component of (parameterized) stress tensor</td>
</tr>
<tr>
<td>$\mathbf{D}$</td>
<td>(parameterized) Diffusivity tensor</td>
</tr>
<tr>
<td>$\mathbf{V}_{GM}$</td>
<td>(parameterized) 3D eddy advective velocity (following Gent &amp; McWilliams, 1990)</td>
</tr>
<tr>
<td>$\mathbf{A}$</td>
<td>(parameterized) skew flux tensor (Eq. 81)</td>
</tr>
<tr>
<td>$\mathbf{u}_{GM}$</td>
<td>(parameterized) 2D eddy advective velocity</td>
</tr>
</tbody>
</table>

ious special cases in section 4, some of which have been discussed before. For example, the residual mean equations are obtained as a special case of the general equations if the averaging coordinate is chosen as potential density and the averaged equations are expressed using a vertical coordinate that represents the averaged height of isopycnals. In section 5 we compare the derived equation sets to those solved in numerical models, to determine which interpretations, if any, are most consistent with the existing model formulations, and to identify which changes to the models may improve consistency with the equations. We conclude in section 6 with a discussion of the main results and conclusions.

2 The Generalized Thickness-Weighted Averaged Equations

2.1 The Hydrostatic Boussinesq Equations in Generalized Vertical Coordinates

In this paper we focus on the hydrostatic Boussinesq equations, which are used in most global ocean models. However, we note that all results are generalizable to the compressible equations since the hydrostatic non-Boussinesq equations with pressure-based vertical coordinates are isomorphic to the hydrostatic Boussinesq equations in $z$-coordinates (Marshall et al., 2004).
The hydrostatic Boussinesq equations in generalized vertical coordinates can be written as

\[\partial_t |_r z r + \nabla_r \cdot (z_r \mathbf{u}) + \partial_r (z_r \dot{\mathbf{c}}) = 0 \]  
(1)
\[\partial_t |_r c + \mathbf{u} \cdot \nabla_r c + \dot{r} \partial_r c = \dot{c} \]  
(2)
\[\partial_t |_r u + \mathbf{u} \cdot \nabla_r u + \dot{r} \partial_r u - f v = -\rho_0^{-1} \partial_r |_z p + \mathcal{F}_x \]  
(3)
\[\partial_t |_r v + \mathbf{u} \cdot \nabla_r v + \dot{r} \partial_r v + f u = -\rho_0^{-1} \partial_r |_z p + \mathcal{F}_y \]  
(4)
\[\partial_r p = -\rho \partial_r \Phi, \]  
(5)

where \(r\) is a generalized vertical coordinate (assumed to be monotonic in \(z\)), \(z_r \equiv \partial_r |_z \) is the generalized thickness (also the Jacobian of the transformation between \(z\)-coordinates and \(r\)-coordinates), \(\mathbf{u} \equiv u \hat{i} + v \hat{j}\) is the horizontal velocity, and \(\nabla_r\) denotes the horizontal nabla operator with derivatives taken at fixed \(r\) (i.e., \(\nabla_r c \equiv \hat{r} \partial_r c + \dot{r} \partial_r |_r c\)). \(\Phi = g z\) is the geopotential, and \(\mathcal{F}_x/y\) represents accelerations by friction and external forces.

The horizontal pressure gradient at fixed \(z\) can be written as

\[\partial_x |_z p = \partial_x |_r p + \rho \partial_x |_r \Phi \]  
(6)
and

\[\partial_y |_z p = \partial_y |_r p + \rho \partial_y |_r \Phi \]  
(7)
in generalized vertical coordinates. However, we note that in numerical models the pressure gradient acceleration is often not evaluated in the specific form of the two terms on the RHS of (6) and (7), which often cancel at leading order (e.g., Lin, 1997; Bleck, 2002; Adcroft et al., 2008). Eq. (2) is a placeholder for multiple tracer equations, with temperature and salinity being the dynamically active tracers, which can be used to compute density via a suitable equation of state (i.e., \(\rho = \tilde{\rho}(S, T, \Phi)\)). We will here only provide limited discussion of how to approximate the equation of state in terms of averaged quantities (see McDougall & McIntosh (1996), Blankart (2013), and Stanley et al. (2020)).

Using equation (1), the tracer equation (2) can alternatively be expressed in “flux form” as

\[\partial_t |_r (z_r c) + \nabla_r \cdot (z_r \mathbf{u} c) + \partial_r (z_r \dot{c}) = z_r \dot{c}. \]  
(8)

Similarly, the momentum equations can be written as

\[\partial_t |_r (z_r u) + \nabla_r \cdot (z_r \mathbf{u} u) + \partial_r (z_r \dot{u}) = -z_r \rho_0^{-1} \partial_r |_z p + z_r \mathcal{F}_x \]  
(9)
\[\partial_t |_r (z_r v) + \nabla_r \cdot (z_r \mathbf{u} v) + \partial_r (z_r \dot{v}) + f z_r u = -z_r \rho_0^{-1} \partial_r |_z p + z_r \mathcal{F}_y. \]  
(10)

Before proceeding, we notice that the momentum equations can also be rewritten in the form

\[\partial_t |_r u + \dot{r} \partial_r u - (f + \zeta) v + \partial_z |_r (|\mathbf{u}|^2/2) = -\rho_0^{-1} \partial_z |_z p + \mathcal{F}_x \]  
(11)
\[\partial_t |_r v + \dot{r} \partial_r v + (f + \zeta) u + \partial_y |_r (|\mathbf{u}|^2/2) = -\rho_0^{-1} \partial_y |_z p + \mathcal{F}_y. \]  
(12)

where \(\zeta = \dot{\partial}_z |_v - \dot{\partial}_y |_u\). The form of the momentum equations in (9) and (10) is advantageous for the averaging due to the appearance of the advection terms in the form of a flux divergence (which naturally leads to eddy contributions in the form of a flux divergence) and we will therefore work with that representation in the following. However, note that if the averaged equations can be written in the form of equations (11)-(12), we can always re-write the momentum equations in the form of (11) and (12) after the averaging.
2.2 Averaged Equations

Taking an average at fixed $r$ of Eqs. (1), (8), (9), (10), and (5) yields

\begin{equation}
\partial_r \langle \tau^r \rangle + \nabla_r \cdot (\langle \tau^r \rangle \hat{u}^r) + \partial_r (\langle \tau^r \rangle \hat{c}^r) = 0 \tag{13}
\end{equation}

\begin{equation}
\partial_r \langle \tau^r \cdot \hat{c}^r \rangle + \nabla_r \cdot (\langle \tau^r \rangle \hat{u}^r \hat{c}^r) + \partial_r (\langle \tau^r \rangle \hat{v}^r \hat{c}^r) = -\frac{\rho_{\tau}}{\tau_{\tau}} \nabla_r \cdot \hat{J}^r + \frac{\kappa_c}{\tau_{\tau}} \hat{c}^r \tag{14}
\end{equation}

\begin{equation}
\partial_r \langle \tau^r \rangle \hat{u}^r + \nabla_r \cdot (\langle \tau^r \rangle \hat{u}^r \hat{u}^r) + \partial_r (\langle \tau^r \rangle \hat{v}^r \hat{u}^r) - f \langle \tau^r \rangle \hat{v}^r = -\frac{\rho_{\tau}}{\tau_{\tau}} \nabla_r \cdot \hat{J}^r - \rho_0 \frac{1}{\tau_{\tau}} \partial_x \langle \tau^r \rangle \hat{p}^r + \frac{\tau_{\tau}}{\tau_{\tau}} \hat{F}_x^r \tag{15}
\end{equation}

\begin{equation}
\partial_r \langle \tau^r \rangle \hat{v}^r + \nabla_r \cdot (\langle \tau^r \rangle \hat{u}^r \hat{v}^r) + \partial_r (\langle \tau^r \rangle \hat{v}^r \hat{v}^r) + f \langle \tau^r \rangle \hat{u}^r = -\frac{\rho_{\tau}}{\tau_{\tau}} \nabla_r \cdot \hat{J}^r - \rho_0 \frac{1}{\tau_{\tau}} \partial_y \langle \tau^r \rangle \hat{p}^r + \frac{\tau_{\tau}}{\tau_{\tau}} \hat{F}_y^r \tag{16}
\end{equation}

\begin{equation}
\partial_r \hat{p}^r = -g \tau_{\tau} \hat{c}^r \tag{17}
\end{equation}

where $\langle \cdot \rangle$ denotes an average at fixed $r$, which we here assumed to commute with the derivatives, and $\langle \cdot \rangle \equiv \tau_{\tau} / \tau_{\tau}$ denotes the corresponding thickness weighted average. For simplicity we have here assumed that $f$ is constant over the filter stencil, such that $\hat{f} \hat{u}^r = f \hat{u}^r$. If that assumption is relaxed, we obtain an additional eddy Coriolis term in the momentum equations.

In general, the advective eddy flux divergence takes the form

\begin{equation}
\nabla^r \cdot \hat{J}^r \equiv \langle \tau^r \rangle^{-1} \nabla_r \cdot (\langle \tau^r \rangle (\hat{u}^c \hat{c}^c + \hat{u}^r \hat{c}^r)) = \langle \tau^r \rangle^{-1} \nabla_r \cdot (\hat{c}^c \hat{c}^c + \hat{c}^r \hat{c}^r), \tag{18}
\end{equation}

where $\nabla^r$ denotes a divergence in $(x, y, \tau^r)$-space, which arises as the natural space for the averaged equations. (We will return to this point in section 4.2.) For a Reynolds average, the eddy flux divergence can be written as

\begin{equation}
\nabla^r \cdot \hat{J}^r \equiv \langle \tau^r \rangle^{-1} \nabla_r \cdot (\langle \tau^r \rangle \hat{u}^r \hat{c}^r) = \langle \tau^r \rangle^{-1} \nabla_r \cdot (\hat{u}^r \hat{c}^r) \tag{19}
\end{equation}

where double primes denote deviations from the thickness-weighted average.

For Eqs. (13) to (17) to form a closed set of equations, we need closures for the eddy tracer and momentum flux divergences $\nabla^r \cdot \hat{J}^r$, $\nabla^r \cdot \hat{u}^r$, $\nabla^r \cdot \hat{v}^r$, an equation of state for the mean quantities $\hat{p}^r = \hat{p}^r (\hat{S}^r, \hat{T}^r, \hat{F})$ (e.g. McDougall & McIntosh, 1996; Brankart, 2013; Stanley et al., 2020), and a closure for the eddy components of the pressure gradient term. For a Reynolds average, the pressure gradient acceleration can be expressed as

\begin{equation}
\frac{\partial}{\partial x} \langle \tau^r \rangle \hat{p}^r = \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{p}^r + \frac{1}{\tau_{\tau}} \tau_{\tau}^t \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{p}^r \tag{20a}
\end{equation}

\begin{equation}
= \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{p}^r + \hat{p}^r \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{F}^r + \frac{1}{\tau_{\tau}} \tau_{\tau}^t \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{p}^r \tag{20b}
\end{equation}

\begin{equation}
= \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{p}^r + \hat{p}^r \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{F}^r + \hat{p}^r \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{F}^r + \frac{1}{\tau_{\tau}} \tau_{\tau}^t \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{p}^r \tag{20c}
\end{equation}

\begin{equation}
= \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{p}^r + \hat{p}^r \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{F}^r + \frac{1}{\tau_{\tau}} \tau_{\tau}^t \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{p}^r \tag{20d}
\end{equation}

(and analogously for $\frac{\partial}{\partial y} \langle \tau^r \rangle \hat{p}^r$). Notice that the mean part of the pressure gradient force is given by the mean pressure gradients at fixed $\tau^r$. The term $\hat{p}^r \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{F}^r$, which for a non-Reynolds average is simply $\rho \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{F}^r = \rho \frac{\partial}{\partial x} \langle \tau^r \rangle \hat{F}^r$, is equal to the difference between the mean pressure gradient at fixed height, $z^r$, and the mean pressure gradient at fixed mean height, $\tau^r$: $\frac{\partial}{\partial z} \langle \tau^r \rangle \hat{p}^r - \frac{\partial}{\partial \tau^r} \hat{p}^r$. When $f \neq 0$, the last term on the RHS of (20d) can be directly related to the geostrophic eddy thickness flux, $\tau_{\tau}^t \frac{\partial}{\partial z} \langle \tau^r \rangle \hat{p}^r = f \rho_0 \tau_{\tau}^t \hat{p}^r$. For a spatial filter that is not a Reynolds average this term becomes $\tau_{\tau}^t \frac{\partial}{\partial z} \langle \tau^r \rangle \hat{p}^r = f \rho_0 (\tau_{\tau}^t \hat{p}^r - \tau_{\tau}^t \hat{p}^r)$.\n
-7-
We can alternatively write the thickness weighted pressure gradient acceleration as

\[
\begin{align*}
\overline{\tau^r} \partial_z \rho & = \frac{\partial_x \rho}{\partial_x^r} + \frac{\partial \rho}{\partial x^r} \\
& = \overline{\tau_x} \partial_z \rho + \overline{\tau_x} \rho \partial_x \Phi^r \\
& = \frac{\partial_x \rho}{\partial_x^r} + \partial_y \rho \partial_y \Phi^r \\
& = \partial_x \rho \partial_x \Phi^r - \partial_y \rho \partial_y \Phi^r \\
& = \partial_x \rho \partial_x \Phi^r - \partial_y \rho \partial_y \Phi^r \\
& = \overline{\tau^r} \partial_z \rho - \partial_x \rho \partial_x \Phi^r \\
& = \overline{\tau^r} \partial_z \rho - \partial_x \rho \partial_x \Phi^r
\end{align*}
\] (21a)

and similarly for the \( y \)-component. We here again assumed a Reynolds average for notational convenience, but the general form is readily obtained by substituting \( \tau^r \rho^r \rightarrow \tau^r \rho^r - \tau^r \rho^r \rightarrow \overline{\tau^r} \rho^r \rightarrow \partial_x \overline{\tau^r} \rho^r \). The formulation of the eddy terms in (21f) or (21g) clarifies that the eddy pressure gradient force only redistributes momentum, with the vertical momentum flux \( \overline{\rho \partial_x | z^r} \) readily identifiable as the form stress acting on a generalized coordinate surface.

We can absorb the eddy contributions to the pressure gradient force into the eddy momentum flux divergence by defining (c.f. Young, 2012)

\[
\begin{align*}
\nabla^r \cdot \mathbf{\overline{u}}^r & = \partial_z \rho \partial_z \Phi^r + (\partial_x \rho \partial_x \Phi^r) \\
\nabla^r \cdot \mathbf{\overline{E}}^r & = \partial_z \rho \partial_z \Phi^r + (\partial_x \rho \partial_x \Phi^r)
\end{align*}
\] (22)

where \( \overline{\tau^r} \rho^r \) and \( \nabla^r \mathbf{\overline{u}}^r \) are defined as in (19).

Using (22) and (23) we can write the generalized TWA horizontal momentum equations as

\[
\begin{align*}
\partial_z \rho \partial_z \Phi^r + \partial_x \rho \partial_x \Phi^r + \partial_x \rho \partial_x \Phi^r & = -\overline{\tau^r \overline{\nabla \cdot \mathbf{\overline{u}}}}^r - \rho \overline{\tau^r \partial_x | \tau^r \rho^r} \\
\partial_z \rho \partial_z \Phi^r + \partial_x \rho \partial_x \Phi^r + \partial_x \rho \partial_x \Phi^r & = -\overline{\tau^r \overline{\nabla \cdot \mathbf{\overline{u}}}}^r - \rho \overline{\tau^r \partial_x | \tau^r \rho^r}
\end{align*}
\] (24)

Excluding the additional eddy terms, equations (13), (14), (17), (24) and (25) have the same form as the unaveraged equations in section 2.1, with the substitutions \( z \rightarrow z^r, p \rightarrow \rho^r, u \rightarrow \mathbf{\overline{u}}^r, r \rightarrow \overline{r}^r, c \rightarrow \overline{c}^r \) and \( \rho \rightarrow \rho^r \). Except for the specific form of the pressure gradient term (and associated eddy flux contributions), the equations are also mathematically identical to the isopycnal TWA equations as discussed in Young (2012), with his \( b \) replaced by the generalized vertical coordinate, \( r \).

The equations, moreover, reduce to the Eulerian mean equations for \( r = z \), in which case \( z_r \rightarrow z_z = 1 \) and hence the generalized “thickness weighted” average simply reduces to the Eulerian average. Because the Eulerian-averaged equations are already a subset of the generalized TWA equations, and because non-thickness weighted averaging with any vertical coordinate that has non-constant thickness is non-conservative and therefore undesirable (see Loose et al. 2023), we will only consider generalized “thickness weighted” averaging in the main part of this manuscript. A brief discussion of the non-thickness-weighted averaged momentum equations in Appendix A.

### 3 Coordinate transformation of averaged equations

In general, the averaging may not need to follow the same vertical coordinate as the model. For example, McDougall & McIntosh (2001) argue that the tracers in \( z \)-coordinate
Figure 1. Illustration of an averaging along an arbitrary vertical coordinate, \( a \), and expression of the averaged field using another arbitrary vertical coordinate, \( r \). The figure on the top left shows an example field \( c(x, z) \) (shading) in \((x, z)\)-space, together with isolines of \( a \) (solid) and \( r \) (dashed). We can first express this field in \((x, a)\)-space to obtain \( c(x, a) \), which is shown on the bottom left. Taking the average (at fixed \( a \)) we obtain \( \bar{c}^a(x, a) \) (bottom right) which can finally be transformed into \((x, r)\)-space to obtain \( \bar{c}^a(x, r) \) (top right). Notice that \( \bar{c}^a(x, r) \) is not generally constant in the averaging dimension (here \( x \)), unless \( r \) is itself constant along \( a \) surfaces in the averaging dimension (as is the case if \( r \) is itself an averaged quantity). We will return to this issue below in the context of the residual mean equations (c.f. Fig. 2).
models that use the GM parameterization should be interpreted as thickness-weighted
isopycnal averages. To see how such interpretations can be justified more generally, we
here derive the arbitrarily averaged generalized vertical coordinate equations. That is,
we start from the equations in an arbitrary vertical coordinate system (we will call that
vertical coordinate \( a \)), take a (thickness-weighted) average at fixed \( a \) and then transform
the resulting equations into a different coordinate system with vertical coordinate \( r \) (as
sketched in Fig. 1).

Replacing \( r \) with \( a \) in the averaged tracer equation (14), we can write the \( a \)-averaged
tracer equation as

\[
\partial_t |_a (\bar{z} \bar{c}^a) + \nabla_a \cdot (\bar{\mathbf{u}}^a \bar{c}^a) + \partial_a (\bar{z} \bar{\alpha}^a \bar{c}^a) = -\bar{z} \bar{\nabla}^a \cdot \mathbf{J}^a + \bar{z} \bar{c}^a. \tag{26}
\]

Using the following identities

\[
\partial_a = (\partial_{\bar{r}})^{-1} \partial_{\bar{r}}, \quad \nabla_a = \nabla_r - (\nabla_r a) (\partial_{\bar{r}})^{-1} \partial_{\bar{r}}, \quad \partial_{|a} = \partial_{|r} - (\partial_{|a} a) (\partial_{\bar{r}})^{-1} \partial_{\bar{r}}, \tag{27}
\]

we can transform the LHS of equation (26) into \( r \)-coordinates as

\[
\partial_{|a} (\bar{z} \bar{c}^a) + \nabla_a \cdot (\bar{\mathbf{u}}^a \bar{c}^a) + \partial_a (\bar{z} \bar{\alpha}^a \bar{c}^a) = \partial_{|r} (\bar{z} \bar{c}^a) - (\partial_{|r} a) a \partial_{|r} (\bar{z} \bar{c}^a) + \nabla_r \cdot (\bar{\mathbf{u}} \bar{c}^a) - (\partial_r a) \bar{\nabla} \cdot \bar{\mathbf{u}} \bar{c}^a + (\partial_{|r} a) a \partial_{|r} (\bar{z} \bar{c}^a).
\tag{28}
\]

With \( \bar{z}_r^a = \partial_{|r} \bar{z}^a = \partial_{|r} \bar{z} \partial_{|a} a = \bar{z} \bar{\alpha} \partial_{|r} a \) we get

\[
\bar{z}_r^{\bar{a}} \left[ \partial_{|a} (\bar{z} \bar{c}^a) + \nabla_a \cdot (\bar{\mathbf{u}}^a \bar{c}^a) + \partial_a (\bar{z} \bar{\alpha}^a \bar{c}^a) \right] = \bar{z}_r^{\bar{a}} \partial_{|r} (\bar{z} \bar{c}^a) - (\partial_{|r} a) a \partial_{|r} (\bar{z} \bar{c}^a) + \nabla_r \cdot (\bar{\mathbf{u}} \bar{c}^a) - (\partial_r a) \bar{\nabla} \cdot \bar{\mathbf{u}} \bar{c}^a + (\partial_{|r} a) a \partial_{|r} (\bar{z} \bar{c}^a) \tag{29a}
\]

\[
\bar{z}_r^{\bar{a}} \partial_{|r} (\bar{z} \bar{c}^a) + \nabla_r \cdot (\bar{\mathbf{u}}^a \bar{c}^a) + \partial_r (\bar{z} \bar{\alpha}^a \bar{c}^a) - \bar{z}_r^{\bar{a}} \partial_{|r} a - \bar{z}_r^{\bar{a}} \bar{\nabla} \cdot \bar{\mathbf{u}} \bar{c}^a + (\partial_{|r} a) a \partial_{|r} (\bar{z} \bar{c}^a) \tag{29b}
\]

\[
\bar{z}_r^{\bar{a}} \partial_{|r} (\bar{z} \bar{c}^a) + \nabla_r \cdot (\bar{\mathbf{u}}^a \bar{c}^a) + \partial_r (\bar{z} \bar{\alpha}^a \bar{c}^a) - (\partial_r a) \bar{\nabla} \cdot \bar{\mathbf{u}} \bar{c}^a + (\partial_{|r} a) a \partial_{|r} (\bar{z} \bar{c}^a) \tag{29c}
\]

\[
\bar{z}_r^{\bar{a}} \partial_{|r} (\bar{z} \bar{c}^a) + \nabla_r \cdot (\bar{\mathbf{u}}^a \bar{c}^a) + \partial_r (\bar{z} \bar{\alpha}^a \bar{c}^a) - (\partial_r a) \bar{\nabla} \cdot \bar{\mathbf{u}} \bar{c}^a + (\partial_{|r} a) a \partial_{|r} (\bar{z} \bar{c}^a) \tag{29d}
\]

where

\[
\bar{r}^{\#a} \equiv \frac{\bar{D}}{\bar{D}t} \bar{r} \tag{30a}
\]

\[
\equiv \partial_{|a} a r + \bar{\mathbf{u}} \cdot \nabla_a \nabla a + \bar{\alpha} \partial_a r \tag{30b}
\]

\[
= - (\partial_{|r} a)^{-1} \partial_{|r} a - (\partial_r a)^{-1} \bar{\mathbf{u}} \cdot \nabla_a + \bar{\alpha} (\partial_r a)^{-1} \tag{30c}
\]

is the Lagrangian rate of change of \( r \) following the \( a \)-averaged flow. Using equation (29d),
it follows that the \( a \)-averaged tracer equation can be expressed using an arbitrary ver-
tical coordinate, \( r \), as

\[
\partial_{|r} (\bar{z} \bar{c}^a) + \nabla_r \cdot (\bar{\mathbf{u}}^a \bar{c}^a) + \partial_r (\bar{z} \bar{\alpha}^a \bar{c}^a) = -\bar{z} \bar{\nabla} \cdot \mathbf{J}^a + \bar{z} \bar{c}^a. \tag{31}
\]

The continuity equation follows immediately from equation (31) with \( c = 1 \) as

\[
\partial_{|r} (\bar{z} \bar{u}^a) + \nabla_r \cdot (\bar{\mathbf{u}}^a \bar{u}^a) + \partial_r (\bar{z} \bar{\alpha}^a \bar{u}^a) - f \bar{z} \bar{v}^a = -\bar{z} \bar{\nabla} \cdot \bar{\mathbf{E}}^a + \rho_0^{-1} \bar{z} \bar{\partial}_x \bar{p} + \bar{z} \bar{\bar{c}}^a. \tag{32}
\]
and
\[
\partial_t |_r (\tau^a_r \nabla^a) + \nabla_r \cdot (\tau^a_r \mathbf{u}^a \nabla^a) + \partial_r (\tau^a_r \dot{\mathbf{v}}^a) + f \tau^a_r \dot{v}^a = -\tau^a_r \nabla^a \cdot \mathbf{E}^a - \rho^a_0 \dot{v}^a \partial_y |_r \mathbf{p}^a + \tau^a_r \mathbf{F}^a_y .
\]  
(34)

The mean pressure gradient term can be expressed in \( r \)-coordinates as
\[
\partial_x |_r \mathbf{p}^a = \partial_x |_r \mathbf{p}^a + \rho^a \partial_x |_r \mathbf{F}^a ,
\]  
(35)

where we used that \( \partial_x |_r = \partial_x |_r - (\partial_x |_r |_a) \mathbf{p}^a = \partial_x |_r \mathbf{p}^a \) and employed hydrostatic balance as
\[
\partial_r \mathbf{p}^a = \partial_r \mathbf{p}^a - (36a)
\]
\[
= -g \tau^a_\alpha \rho^a \partial_r \mathbf{a} \quad \text{(36b)}
\]
\[
= -g \partial_r \tau^a_\alpha \rho^a \quad \text{(36c)}
\]
\[
= -\partial_r \tau^a \mathbf{p}^a \quad \text{(36d)}
\]

where in the second step we used the averaged hydrostatic balance in Eq. (17) with \( r \rightarrow a \).

To summarize, the \( a \)-averaged primitive equations can be expressed in \( r \)-coordinates as:
\[
\partial_t |_r (\tau^a_\alpha) + \nabla_r \cdot (\tau^a_\alpha \mathbf{u}^a_\alpha) + \partial_r (\tau^a_\alpha \dot{\mathbf{v}}^\alpha) = 0
\]  
(37)
\[
\partial_t |_r (\tau^a_\alpha \mathbf{c}^a_\alpha) + \nabla_r \cdot (\tau^a_\alpha \mathbf{c}^a_\alpha \mathbf{c}^a_\alpha) + \partial_r (\tau^a_\alpha \dot{\mathbf{v}}^\alpha) = -\tau^a_\alpha \nabla^a \cdot \mathbf{E}^a + \tau^a_\alpha \mathbf{c}^a_\alpha
\]  
(38)
\[
\partial_t |_r (\tau^a_\alpha \mathbf{u}^a) + \nabla_r \cdot (\tau^a_\alpha \mathbf{u}^a \mathbf{u}^a) + \partial_r (\tau^a_\alpha \dot{\mathbf{v}}^\alpha) - f \tau^a_\alpha \dot{v}^a = -\tau^a_\alpha \nabla^a \cdot \mathbf{E}^a - \rho^a_0 \dot{v}^a \partial_y |_r \mathbf{p}^a + \tau^a_\alpha \mathbf{F}^a_y
\]  
(39)
\[
\partial_t |_r (\tau^a_\alpha \mathbf{c}^a_\alpha) + \nabla_r \cdot (\tau^a_\alpha \mathbf{c}^a_\alpha \mathbf{c}^a_\alpha) + \partial_r (\tau^a_\alpha \dot{\mathbf{v}}^\alpha) + f \tau^a_\alpha \dot{v}^a = -\tau^a_\alpha \nabla^a \cdot \mathbf{E}^a - \rho^a_0 \dot{v}^a \partial_y |_r \mathbf{p}^a + \tau^a_\alpha \mathbf{F}^a_y
\]  
(40)
\[
\partial_r \mathbf{p}^a = -\rho^a \partial_r \mathbf{F}^a
\]  
(41)

with \( \dot{\mathbf{v}}^\alpha \) defined in equation (30). Notice that the eddy fluxes in the generalized average equations (37-41) depend on the averaging choice but not on the coordinate system ultimately chosen to express the equations.

Except for the additional eddy contributions, Eqs. (37) to (41) are again formally identical to the unaveraged equations in section 2.1 with the substitutions \( \mathbf{u} \rightarrow \mathbf{u}^a \), \( c \rightarrow \mathbf{c}^a \), \( \dot{c} \rightarrow \dot{\mathbf{c}}^a \), \( \rho \rightarrow \rho^a \), \( \mathbf{F}_{x/y} \rightarrow \mathbf{F}_{x/y}^a \), \( p \rightarrow \mathbf{p}^a \), \( \Phi \rightarrow \mathbf{E}^a \), \( \mathbf{z}_r \rightarrow \mathbf{z}^a_r \) and \( \dot{r} \rightarrow \dot{r}^a \). The last two substitutions warrant some further discussion. Notice that \( \mathbf{z}^a_r \equiv \partial_y \mathbf{z}^a \neq \mathbf{z}^a_\alpha \), i.e. \( \mathbf{z}^a \) is not necessarily equal to the \( a \)-averaged generalized thickness but is instead defined directly as the \( r \)-derivative of the \( a \)-averaged height. Moreover, \( \dot{r}^a \neq \dot{r}^a \), i.e. \( \dot{r}^a \) is not simply the generalized TWA of \( \dot{r} \). Instead, \( \dot{r}^a \) is the rate of change of \( r \) following the generalized TWA flow. This point was also emphasized in Section 3 of Young (2012) for the special case of the buoyancy-averaged equations expressed in depth coordinates.
4 Special cases

4.1 The Eulerian mean equations in generalized vertical coordinates: \( a \to z \)

With \( a = z \) (in which case \( \widehat{a}^a \to \widehat{a}^z = (\widehat{a}^z) \)) we obtain the Eulerian-averaged equations in generalized vertical coordinates as

\[
\begin{align*}
\partial_t [z_r v^z + \nabla_r \cdot (z_r u^z)] + \partial_r [z_r v^z z_r] &= 0 \quad (42) \\
\partial_t [z_r u^z(z_r c^z) + \nabla_r \cdot (z_r u^z u^z)] + \partial_r [z_r \mathbf{v}^z z_r] &= -z_r \nabla \cdot \mathbf{J}^z + z_r c^z \\
\partial_t [z_r u^z(z_r v^z)] + \nabla_r \cdot (z_r u^z u^z) + \partial_r [z_r \mathbf{v}^z z_r] &= -z_r \nabla \cdot \mathbf{J}^z - \rho_0^{-1} z_r \partial_x \bar{p}^z + \mathbf{F}_x^z \\
\partial_t [z_r u^z(z_r v^z)] + \nabla_r \cdot (z_r u^z u^z) + \partial_r [z_r \mathbf{v}^z z_r] &= -z_r \nabla \cdot \mathbf{J}^z - \rho_0^{-1} z_r \partial_y \bar{p}^z + \mathbf{F}_y^z \\
\partial_t \bar{p}^z &= -g z_r \bar{p}^z \\
\end{align*}
\]

with \( \mathbf{J}^z = \mathbf{V}^z c^z, \mathbf{J}^u = \mathbf{V}^u u^z, \mathbf{J}^v = \mathbf{V}^v v^z \) (where \( \mathbf{v} \) denotes the 3-dimensional velocity vector),

\[
\mathbf{r}^\# = \frac{\mathbf{D}^z}{\mathbf{D}t} \quad (47)
\]

and \( \nabla \) (without subscript) represents the 3-dimensional divergence and gradient operators. The averaged horizontal pressure gradient can be expressed in \( r \)-coordinates as

\[
\partial_x \bar{p}^z = \partial_x [\bar{p}^z + \bar{p}^z \partial_x \Phi]. \quad (48)
\]

Aside from the additional eddy contributions, Eqs. (42) to (46) are again the same as the unaugmented equations in section 2.1 with \( u, c, p, \rho \), and \( \mathbf{F}_{x/y} \) replaced by their respective Eulerian averages, and \( \mathbf{r} \to \mathbf{r}^\# \), which gives the Lagrangian rate of change of \( r \) following the Eulerian-mean flow. Noticeably, the thickness remains \( z_r \), i.e. it is not affected by the Eulerian averaging, unless \( r \) itself is chosen to depend on the averaged quantities (which may be desirable or even necessary in practice). The eddy contributions appear as the Eulerian eddy flux convergences of tracer and momentum.

**The Eulerian mean equations in z-coordinates: \( a \to z, r \to z \)**

With \( r \to z \), equations (42) to (46) reduce to the well-known z-coordinate Eulerian mean equations:

\[
\begin{align*}
\nabla \cdot \mathbf{v}^z &= 0 \quad (49) \\
\partial_t [z_r v^z + \nabla_z \cdot (z_r u^z)] + \partial_z [z_r \mathbf{v}^z z_r] &= -\nabla \cdot \mathbf{J}^z + \mathbf{c}^z \\
\partial_t [z_r u^z(z_r v^z)] + \nabla_z \cdot (z_r u^z u^z) + \partial_z [z_r \mathbf{v}^z z_r] &= -\nabla \cdot \mathbf{J}^z - \rho_0^{-1} \partial_x \bar{p}^z + \mathbf{F}_x^z \\
\partial_t \bar{p}^z &= -g \bar{p}^z. \\
\end{align*}
\]

**The Eulerian mean equations in isopycnal coordinates: \( a \to z, r \to b \)**

The Eulerian mean equations in isopycnal coordinates are simply given by equations (42) to (46) with \( r \) replaced by a suitable buoyancy coordinate, \( b \).

4.2 The isopycnal TWA equations in generalized vertical coordinates: \( a \to b \)

The isopycnal TWA equations in generalized vertical coordinates are given by Eqs. (37) to (41) with \( a \to b \), where \( b \) is a suitably defined buoyancy. For the most general
case of a nonlinear equation of state and with buoyancy, \( b \), defined with respect to potential density, the equations do not obviously simplify (in an exact manner) and are hence not repeated here for the special case of \( a \to b \).

For a linear equation of state, where we can define buoyancy in the form

\[
\dot{b} = -g \left( \rho - \rho_0 \right) / \rho_0, \tag{54}
\]

we can use that \( \dot{\rho} = \rho \) and hence \( \rho'' = 0 \), such that the first eddy term in the pressure gradient in Eq. (20d) vanishes. This simplification is useful for comparison to previous work (and we will return to it in the next two subsections). The ocean components of climate models, however, generally use a nonlinear equation of state, in which case there is no materially conserved (for adiabatic flow) buoyancy variable that allows for this simplification.

**The isopycnal TWA equations in isopycnal coordinates: \( a \to b \) and \( r \to \bar{b} \)**

The isopycnal TWA equations in isopycnal coordinates are given directly by the generalized TWA equations (13,14,15/24, 16/25 and 17) with \( r \to \bar{b} \). Again, no obvious exact simplification is obtained for a buoyancy variable based on potential density with a realistic equation of state.

For a linear EOS where a buoyancy variable can be defined as in Eq. (54), it is useful to define a Montgomery potential \( M = p + \rho \Phi \), which allows us to express the pressure gradient at fixed height in terms of the Montgomery potential gradient at fixed buoyancy, such that

\[
\partial_x |_{z} \bar{b} = \frac{1}{\bar{\rho}} \frac{z_0}{\bar{z}_0} \partial_x |_{\bar{z}} \bar{b} = \frac{1}{\bar{\rho}} \frac{z_0}{\bar{z}_0} \partial_x |_{\bar{z}} \bar{M}^b \tag{55a}
\]

\[
= \frac{1}{\bar{\rho}} \frac{z_0}{\bar{z}_0} \partial_x |_{\bar{z}} \bar{M}^b \tag{55b}
\]

\[
= \partial_x |_{\bar{z}} \bar{M}^b + \frac{1}{\bar{\rho}} \frac{z_0}{\bar{z}_0} \partial_x |_{\bar{z}} \bar{M}^b \tag{55c}
\]

\[
= \partial_x |_{\bar{z}} \bar{M}^b + \frac{1}{\bar{\rho}} \frac{z_0}{\bar{z}_0} \partial_x |_{\bar{z}} \left( \rho_0 \frac{z_0^2}{2} \right) + \frac{1}{\bar{\rho}} \partial_0 \left( \bar{z} \partial_x |_{\bar{z}} \bar{M}^b \right) \tag{55d}
\]

(c.f. section 6 in Young, 2012).

**The “residual mean” equations: \( a \to b \) and \( r \to \bar{z}^b \)**

With \( a \to b \) and \( r \to \bar{z}^b \) we recover the “residual mean” equations as discussed e.g. in Young (2012). This choice gives \( \bar{\varpi}_r \to \partial_{\varpi} \bar{z}^b = 1 \), and equations (37) to (41) simplify to

\[
\nabla_{\bar{z}^b} \cdot \bar{u}^b + \partial_{\bar{z}^b} w^{\# b} = 0 \tag{56}
\]

\[
\partial_t |_{\bar{z}^b} \bar{c}^b + \nabla_{\bar{z}^b} \cdot \left( \bar{u}^b \bar{c}^b \right) + \partial_{\bar{z}^b} w^{\# b} \bar{c}^b = -\nabla_{\bar{z}^b} \cdot \bar{E} \bar{c}^b + \bar{c}^b \tag{57}
\]

\[
\partial_t |_{\bar{z}^b} \bar{v}^b + \nabla_{\bar{z}^b} \cdot \left( \bar{u}^b \bar{v}^b \right) + \partial_{\bar{z}^b} w^{\# b} \bar{v}^b - f \bar{c}^b = -\nabla_{\bar{z}^b} \cdot \bar{E} \bar{v}^b - \rho_0^{-1} \partial_{\bar{z}^b} \bar{z} \bar{z} \bar{p}^b + \bar{F}_x^b \tag{58}
\]

\[
\partial_t |_{\bar{z}^b} \bar{w}^b + \nabla_{\bar{z}^b} \cdot \left( \bar{u}^b \bar{w}^b \right) + \partial_{\bar{z}^b} w^{\# b} \bar{w}^b + f \bar{c}^b = -\nabla_{\bar{z}^b} \cdot \bar{E} \bar{w}^b - \rho_0^{-1} \partial_{\bar{z}^b} \bar{y} \bar{z} \bar{p}^b + \bar{F}_y^b \tag{59}
\]

\[
\partial_{\bar{z}^b} \bar{\rho}^b = -g \bar{\rho}^b \tag{60}
\]

where

\[
w^{\# b} = \partial_t |_{\bar{z}^b} \bar{z}^b + \bar{u}^b \cdot \nabla_{\bar{z}^b} \bar{z}^b + \bar{c}^b \partial_{\bar{z}^b} \bar{z}^b. \tag{61}
\]

Save for the eddy terms, these equations have the same form as the unaveraged equations in \( z \)-coordinates. Eddy terms appear as eddy flux convergences in the tracer and
Figure 2. Illustration of the averaging in the "residual" mean equations, where the average is taken along surfaces of constant buoyancy, \( b \), and the averaged equations are expressed using the averaged height \( \bar{z}^b \) as the vertical coordinate. The figure on the left shows an example field \( c(x, z) \) (shading) in physical space, together with isolines of \( b \) (solid) and \( z \) (dashed). The second panel from the left shows the same fields expressed in \((x, b)\)-space. Taking an average in the \( x \)-direction (along surfaces of constant \( b \)) yields \( \bar{c}^b(x, b) \), which is shown in the third panel together with isolines of \( b \) (solid), \( z(x, b) \) (dashed) and \( \bar{z}^b(x, b) \) (dotted). Transforming \( \bar{c}^b(x, b) \) into \((x, \bar{z}^b)\)-space yields the residual mean field, \( \bar{c}^b(x, \bar{z}^b) \), shown on the bottom right. This field is constant along the averaging dimension (here \( x \)), and we notice that this result holds generally if the averaged field is expressed using a vertical coordinate that is itself an averaged quantity. For comparison, the top right panel shows \( \bar{c}^b(x, z) \), which is not constant in \( x \).
Figure 3. Illustration of $b^\#$ and the corresponding average, here sketched for the case of a domain-wide average in $x$. The solid lines represent buoyancy surfaces in $(x, z)$-space. The quantity $\bar{c}^b(x', z') \equiv \bar{c}^b(x', b^\#(x', z'))$, where $(x', z')$ is the point marked by the “+” sign, represents an average along the red line, which in turn marks the buoyancy surface with $b = b^\#(x', z')$ (i.e., the buoyancy surface whose averaged height is equal to $z'$). For comparison, $\bar{c}^b(x', z') = \bar{c}^b(x', b(x', z'))$ would be the average taken along the buoyancy surface marked in green. Notice also that the red surface is still a surface of constant $b$ with $b = b^\#(x', z')$ not a surface of constant $b^\#$ (which in turn would be given by the thin red dashed line).

momentum equations, with the eddy momentum fluxes including advective as well as pressure contributions (Eqs. 22 & 23).

Notice that the choice of $z^b$ (rather than the unaveraged $z$) as the vertical coordinate is necessary to obtain a closed set of equations. The choice further guarantees that the averaged fields remain constant along the averaging dimension in the case of a Reynolds average - as sketched in Fig 2. Similarly, for a spatial filter, the choice guarantees that the filtered fields remain horizontally smooth along the model coordinate. The notation of Young (2012) drops the overbar on the $z$-coordinate in the averaged equation. As the residual mean equations (56 to 60) do not depend on $z$ but only $\bar{z}$, this difference may be viewed simply as a notational re-definition: $z^b \rightarrow z$. This modified definition of the $z$-coordinate, however, is important to keep in mind when interpreting model variables, parameters, and boundary conditions. E.g. boundary fluxes can affect model quantities at all $\bar{z}$-levels for which the corresponding isopycnal outcrops into the boundary anywhere within the averaging region (i.e., boundary conditions cannot technically be applied strictly at $z^b = z_B$, where $z_B$ is the height of the boundary). A simple re-definition of the vertical coordinate, moreover, cannot be applied in the mixed Eulerian/TWA interpretation of the $z$-coordinate equations advocated by McDougall & McIntosh (2001), as will be discussed in the following section.

4.3 Mixed Eulerian/TWA equations

McDougall & McIntosh (2001) argue that the tracers in $z$-coordinate models employing the GM parameterization should be interpreted as representing isopycnal thickness-weighted averages, while the model velocities represent Eulerian averages (with GM parameterizing the difference between the Eulerian and isopycnal TWA velocities). The corresponding model equations are obtained by combining the Eulerian mean momentum equations and residual mean tracer equations. However, since the model coordinate can only represent either $\bar{z}$ or $z$, their approach requires a new definition of the residual mean quantities in actual $z$-space. Specifically, one can define

$$\bar{c}^b(x, y, z, t) \equiv \bar{c}^b(x, y, b^\#(x, y, z, t))$$

(62)
where \( b^\#(x, y, z, t) \) is defined as in De Szoiske & Bennett (1993), McDougall & McIntosh (2001) and Young (2012) such that \( \overline{\pi}(x, y, b^\#(x, y, z), t) = z \), i.e. \( b^\#(x, y, z, t) \) describes the buoyancy surface whose mean height is \( z \). Notice that \( \overline{\pi} \) is here not an average along surfaces of constant \( b^\# \), but an average along the \( b \)-surface with \( b = b^\#(x, y, z, t) \) as sketched in Fig. 3.

We can also express \( b^\# \) and \( \overline{\pi} \) in an arbitrary vertical coordinate, as \( b^\#(x, y, z, r, t) = b^\#(x, y, z(x, y, r), t) = \overline{\pi}(x, y, z(x, y, r), t) = \overline{\pi}(x, y, b^\#(x, y, r, t)) \).

With analog definitions for \( \overline{\pi}^\# \), \( \overline{\pi}^\# \) and \( \dot{\overline{\pi}}^\# \), and using that \( \overline{\pi}^\#(x, y, b^\#, t) = z \), Eqs. (37) and (38) imply

\[
\partial_t |_r (z_r^\#) + \nabla_r \cdot (z_r \overline{\mathbf{u}}^\#) + \partial_r (z_r \dot{z}^\#) = 0 \tag{63}
\]

\[
\partial_t |_r (z_r \overline{\mathbf{c}}^\#) + \nabla_r \cdot (z_r (\overline{\mathbf{u}}^\# \cdot \overline{\mathbf{c}}^\#) + \partial_r (z_r \dot{r}^\#) \overline{\mathbf{c}}^\#) = z_r \overline{\mathbf{c}}^\# - z_r \nabla \cdot \tilde{\mathbf{j}}^\#. \tag{64}
\]

Defining the “quasi-Stokes” velocities as \( \mathbf{u}^* \equiv \overline{\mathbf{u}}^\# - \overline{\mathbf{u}} \) and \( \dot{\mathbf{r}}^* = \dot{r}^\# - \dot{r}^\# \), Eqs. (63) and (64) together with the Eulerian mean momentum equations form the following set of equations:

\[
\partial_t |_r (z_r^\#) + \nabla_r \cdot ((\overline{\mathbf{u}}^\# + \mathbf{u}^*) z_r) + \partial_r (z_r (\dot{\pi}^\# + \dot{\mathbf{r}}^*)) = 0 \tag{65}
\]

\[
\partial_t |_r (z_r \overline{\mathbf{c}}^\#) + \nabla_r \cdot (z_r (\overline{\mathbf{u}}^\# + \mathbf{u}^*) \overline{\mathbf{c}}^\#) + \partial_r (z_r (\dot{\pi}^\# + \dot{\mathbf{r}}^*)) \overline{\mathbf{c}}^\#) = z_r \overline{\mathbf{c}}^\# - z_r \nabla \cdot \tilde{\mathbf{j}}^\# \tag{66}
\]

\[
\partial_t |_r (z_r \mathbf{v}^\#) + \nabla_r \cdot (z_r (\overline{\mathbf{u}}^\# \cdot \mathbf{v}^\#) + \partial_r (z_r (\dot{\pi}^\# + \dot{\mathbf{r}}^*)) \mathbf{v}^\#) - \partial_t |_r (z_r \mathbf{v}^\#) = -z_r \nabla \cdot \mathbf{u}^\# - \rho_0^{-1} z_r \partial_r |_r \mathbf{p}^\# + z_r \mathbf{f}^\#_r \tag{67}
\]

\[
\partial_t |_r (z_r \mathbf{v}^\#) + \nabla_r \cdot (z_r (\overline{\mathbf{u}}^\# \cdot \mathbf{v}^\#) + \partial_r (z_r (\dot{\pi}^\# + \dot{\mathbf{r}}^*)) \mathbf{v}^\#) + \partial_t |_r (z_r \mathbf{v}^\#) = -z_r \nabla \cdot \mathbf{v}^\# - \rho_0^{-1} z_r \partial_r |_r \mathbf{p}^\# + z_r \mathbf{f}^\#_y \tag{68}
\]

\[
\partial_r |_r \mathbf{p}^\# = -g z_r \mathbf{p}^\#, \tag{69}
\]

where \( \partial_r |_r \mathbf{p}^\# \) can be expressed in \( r \)-coordinates as given in Eq. (48). For Eqs. (65) to (69) to be a closed set of equations, we need closures for the quasi-Stokes velocities and the eddy tracer and momentum flux divergences, as well as an equation of state that provides the Eulerian-averaged density \( (\overline{\rho}) \) as a function of the isopycnal TWA temperature and salinity \( (\overline{T}^\# \) and \( \overline{S}^\# \)). The latter issue has previously been addressed by McDougall & McIntosh (2001), who argue that the required approximation is no more inaccurate than computing the Eulerian mean density from the Eulerian mean temperature and salinity (see their Appendix B).

Notice also that the continuity equation can alternatively be replaced with the Eulerian mean continuity equation (42), and combining (65) with (42) moreover yields

\[
\nabla_r \cdot (\mathbf{u}^* z_r^\#) + \partial_r (z_r \dot{\mathbf{r}}^*) = 0, \tag{70}
\]

i.e., the three-dimensional quasi-Stokes velocity is divergence free (as previously pointed out by McDougall & McIntosh (2001)). Notice that the quasi-Stokes velocity differs from the 2D divergent “bolus” velocity \( (\mathbf{u}_{bolus} = \overline{\mathbf{u}}^\# - \overline{\mathbf{u}}) \).

**Mixed Eulerian/TWA equations in z coordinates**

In the z-coordinate representation, i.e., with \( r \rightarrow z \), Eqs. (65) to (69) reduce to the equations proposed by McDougall & McIntosh (2001) as an interpretation for z-coordinate ocean models that employ the GM parameterization (although the full set of equations...
is never explicitly written out in McDougall & McIntosh (2001):

$$\nabla_z \cdot \mathbf{u} + \partial_z \mathbf{w} = 0 \quad (71)$$

$$\partial_t \mathbf{c}^b + \nabla_z \cdot (\mathbf{u}^b \mathbf{c}^b) + \partial_z (\mathbf{w}^z + \mathbf{u}^* \mathbf{c}^b) = -\mathbf{c}^b \cdot \mathbf{J}^b_c \quad (72)$$

$$\partial_t \mathbf{u}^z + \nabla_z \cdot (\mathbf{u}^z \mathbf{w}^z) + \partial_z (\mathbf{w}^z \mathbf{u}^z) - f \mathbf{v}^z = -\nabla \cdot \mathbf{J}^u - \rho_0^{-1} \partial_z \mathbf{p}^z + \mathbf{F}_x \quad (73)$$

$$\partial_t \mathbf{v}^z + \nabla_z \cdot (\mathbf{v}^z \mathbf{v}^z) + \partial_z (\mathbf{w}^z \mathbf{v}^z) + f \mathbf{v}^z = -\nabla \cdot \mathbf{J}^v - \rho_0^{-1} \partial_y \mathbf{p}^z + \mathbf{F}_y \quad (74)$$

$$\partial_z \mathbf{p}^z = -g \mathbf{p}^z \quad (75)$$

where we used that $\mathbf{c}^b = \mathbf{w}$ and $\mathbf{v}^* = \mathbf{c}^b - \mathbf{w}^z = \mathbf{w} - \mathbf{w}^z$.

### 5 Comparison to existing model formulations

We now compare the averaged equations derived in the previous section to the equations solved in numerical ocean models, to infer which interpretations (if any) are consistent with the formulation of the equations and parameterizations in existing models, or which changes may be necessary to achieve consistency. Although the main focus here is on generalized vertical coordinate models, we start with a brief review of z-coordinate models. We will not explicitly consider isopycnal coordinate models, as purely isopycnal coordinates are rarely used in realistic global ocean models, although we note that the generalized vertical coordinate results carry over directly to the isopycnal coordinate case, as discussed in the previous subsection.

#### 5.1 Z-coordinate models

The equations solved by z-coordinate numerical ocean models can usually be written in the following form:

$$\nabla_z \cdot \mathbf{u} + \partial_z \mathbf{w} = 0 \quad (76)$$

$$\partial_t \mathbf{c} + \nabla \cdot (\mathbf{v} + \mathbf{v}_{GM}) \mathbf{c} = \mathbf{c} + \nabla \cdot (\mathbf{D} \nabla \mathbf{c}) \quad (77)$$

$$\partial_t \mathbf{u} + \nabla \cdot (\mathbf{v} \mathbf{u}) = \nabla \cdot \mathbf{v}^u - \rho_0^{-1} \partial_z \mathbf{p} + \mathbf{F}_x \quad (78)$$

$$\partial_t \mathbf{v} + \nabla \cdot (\mathbf{v} \mathbf{v}) + f \mathbf{u} = \nabla \cdot \mathbf{v}^v - \rho_0^{-1} \partial_y \mathbf{p} + \mathbf{F}_y \quad (79)$$

$$\partial_z \mathbf{p} = -g \mathbf{p} \quad (80)$$

where the tilde denotes a model variable (which may be interpreted in terms of a suitable average), $\mathbf{D}$ represents a diffusivity tensor, and $\mathbf{v}_{GM}$ represents the zonal/meridional component of a viscous stress tensor. $\mathbf{v}_{GM}$ represents a divergence-free eddy advection, which, if included, is typically parameterized following Gent & McWilliams (1990). The GM tracer tendency can alternatively be expressed in terms of a “skew flux” as

$$\nabla \cdot (\mathbf{v}_{GM} \mathbf{c}) = -\nabla \cdot (\mathbf{A} \nabla \mathbf{c}) \quad (81)$$

where $\mathbf{A}$ is an antisymmetric tensor (see Griffies, 1998, for details). Although numerical implementations may use either the “advective” or “skew flux” representation of the GM parameterization, the two formulations are analytically equivalent. As previously noted in section 2.1, the non-parameterized terms in the tracer and momentum equations may also be expressed in multiple analytically equivalent forms, which does not affect our conclusions here.

McDougall & McIntosh (2001) argued that z-coordinate models using the GM parameterization should be interpreted in terms of the mixed Eulerian (momentum) and isopycnal TWA (tracer) interpretation, as given by Eqs. (71) to (75). In this interpretation $\mathbf{v} = \mathbf{v}^z, \mathbf{c} = \mathbf{c}^b, \mathbf{v}^* = \mathbf{v}^z, \mathbf{p} = \mathbf{p}^z$, $\mathbf{F}_{x/y} = \mathbf{F}_{x/y}^{\pm}$, the viscous stress tensor parameterizes the Eulerian mean eddy momentum fluxes (i.e., $\nabla \cdot \mathbf{v}^u = -\nabla \cdot \mathbf{J}^{u,v}$), the GM advection represents the “quasi-Stokes drift” ($\mathbf{v}_{GM} = \mathbf{v}^*$) (which, by
construction, is not tracer dependent) and the diffusive tracer flux represents the isopycnal TWA eddy flux (i.e., \( \nabla \cdot (D \nabla c^z) = -\nabla \cdot \mathbf{J}^b \)) (with any possible skew-flux component to the isopycnal TWA eddy flux neglected).

As discussed in the introduction, it is, however, still common for z-coordinate models to be interpreted in terms of the Eulerian mean equations. In this case the tilde simply denotes an average at fixed depth (i.e., \( \tilde{\mathbf{f}} = \mathbf{f} \)), the eddy momentum fluxes are parameterized via a viscous stress (i.e., \( \nabla \cdot \boldsymbol{\tau}_{u/v} = -\nabla \cdot \mathbf{J}^u \nabla \tilde{c}^z \)), and the eddy tracer flux divergence is parameterized as

\[
\nabla \cdot \mathbf{J}^c = -\nabla \cdot [(A + D) \nabla c^z]
\]

where \( A \) is the antisymmetric GM skew flux tensor, and \( D \) is a symmetric diffusivity tensor, representing along-isopycnal mixing associated with mesoscale eddies and (the much smaller) diapycnal mixing by small-scale turbulence (Redi, 1982).

As pointed out by McDougall & McIntosh (2001), the main caveat of the Eulerian mean interpretation is that, although parameterized such, the Eulerian mean eddy tracer flux is in fact not strictly along isopycnals, even in fully adiabatic flow. In addition, both the symmetric diffusivity tensor and the antisymmetric skew flux tensor (or, equivalently, the eddy advective velocity) are not necessarily tracer independent (e.g. Sun et al., 2021; Kamenkovich et al., 2021). However, the variance budget does require that the eddy flux is on-average along isopycnals for adiabatic flow (e.g. Eden et al., 2007). The simplifying assumption to represent the eddy tracer flux as locally along isopycnals is therefore not fundamentally different from the assumption that along-isopycnal eddy fluxes can be represented as locally down-gradient (which also can only be justified in a mean sense). Similarly, the possible tracer dependence also applies to diffusive closures, which are widely used (e.g. Gnanadesikan et al., 2015; Petersen et al., 2015; Adcroft et al., 2019). Given the much greater simplicity of the Eulerian mean interpretation, we therefore consider this interpretation to remain a reasonable alternative to the mixed Eulerian/residual-mean interpretation suggested by McDougall & McIntosh (2001).

Alternatively, Eqs. (56) to (60) provide a full “residual mean” interpretation of the z-coordinate model equations, where \( \tilde{\mathbf{f}} = \tilde{\mathbf{u}}^b, \tilde{w} = w^b, \tilde{c} = c^b, \tilde{\rho} = \rho^b \) and \( \tilde{F}_{x/y} = F_{x/y}^b \). However, this interpretation is only consistent with Eqs. (76) to (80) if the GM parameterization is not used (i.e. \( \mathbf{v}_{GM} = 0 \)). Moreover, the stress tensor then needs to represent both the eddy momentum advection and the eddy form stress (\( \nabla \cdot \tau_{u/v} = -\nabla \cdot \mathbf{E}^b \)). In the currently used model formulations, where the eddy stress is represented via a viscous stress tensor with relatively weak vertical viscosity, this interpretation is only justified if the eddy form stress is assumed negligible. However, the eddy form stress can readily be incorporated into the viscous stress tensor following Rhines & Young (1982) and Greatbatch & Lamb (1990), as has been done by Ferreira & Marshall (2006); Zhao & Vallis (2008); Saenz et al. (2015).

5.2 Hybrid and generalized vertical coordinate models

**Model equations**

We here focus primarily on the model equations in MOM6, which we are most familiar with, and then briefly compare to other models. The MOM6 model equations are
(Adcroft et al., 2019):

\[
\begin{align*}
\partial_t |_{\tilde{r}} \tilde{z}_r + \nabla_r \cdot (\tilde{z}_r (\tilde{u} + u_{GM})) + \partial_r (\tilde{z}_r \tilde{v}) &= 0 \quad (83) \\
\partial_t |_{\tilde{r}} (\tilde{z}_r \tilde{c}) + \nabla_r \cdot (\tilde{z}_r (\tilde{u} + u_{GM}) \tilde{c}) + \partial_r (\tilde{z}_r \tilde{v} \tilde{c}) &= \tilde{z}_r \nabla \cdot (D \nabla \tilde{c}) + \tilde{z}_r \tilde{c} \quad (84) \\
\partial_t |_{\tilde{r}} \tilde{u} + \tilde{v} \partial_r \tilde{u} - (f + \tilde{\zeta}) \tilde{v} + \partial_z |_{\tilde{r}} (|\tilde{u}|^2/2) &= \nabla \cdot \tau^u - \rho_0^{-1} \partial_z |_{\tilde{r}} \tilde{p} + \tilde{F}_x \quad (85) \\
\partial_t |_{\tilde{r}} \tilde{v} + \tilde{v} \partial_r \tilde{v} + (f + \tilde{\zeta}) \tilde{u} + \partial_y |_{\tilde{r}} (|\tilde{u}|^2/2) &= \nabla \cdot \tau^v - \rho_0^{-1} \partial_y |_{\tilde{r}} \tilde{p} + \tilde{F}_y \quad (86) \\
\partial_r \tilde{p} &= -\rho \partial_r \tilde{\Phi}, \quad (87)
\end{align*}
\]

where the tilde denotes a model variable (which may be interpreted in terms of a suitable average), \( \nabla \) (without subscript) represents the 3-dimensional (coordinate-system independent) divergence and gradient operators, \( D \) represents a symmetric anisotropic diffusivity tensor, with a large isopycnal and much smaller diapycnal component, and \( \tau^u/v \) represents the zonal/meridional component of the viscous stress tensor. \( u_{GM} \) represents an eddy advection, which, if included, is typically parameterized broadly following Gent & McWilliams (1990), although notice that, unlike in the original GM parameterization for \( z \)-coordinate models, \( u_{GM} \) here is a two-dimensional divergent velocity field rather than a non-divergent three-dimensional velocity.

For comparison with the equations discussed in this manuscript we can use the continuity equation to re-write the model’s momentum equation in flux form as

\[
\begin{align*}
\partial_t |_{\tilde{r}} (\tilde{z}_r \tilde{u}) + \nabla_r \cdot (\tilde{z}_r \tilde{u} \tilde{u}) + \partial_r (\tilde{z}_r \tilde{v} \tilde{u}) + \tilde{u} \nabla_r \cdot (\tilde{z}_r u_{GM}) - f \tilde{z}_r \tilde{v} &= \tilde{z}_r \nabla \cdot \tau^u - \rho_0^{-1} \tilde{z}_r \partial_x |_{\tilde{r}} \tilde{p} + \tilde{F}_x \quad (88) \\
\partial_t |_{\tilde{r}} (\tilde{z}_r \tilde{v}) + \nabla_r \cdot (\tilde{z}_r \tilde{v} \tilde{v}) + \partial_r (\tilde{z}_r \tilde{v} \tilde{v}) + \tilde{v} \nabla_r \cdot (\tilde{z}_r u_{GM}) + f \tilde{z}_r \tilde{u} &= \tilde{z}_r \nabla \cdot \tau^v - \rho_0^{-1} \tilde{z}_r \partial_y |_{\tilde{r}} \tilde{p} + \tilde{F}_y 
\end{align*}
\]

Notice that the divergence of the GM advection appears here as a result of its appearance in the continuity equation. This formulation illustrates that the inclusion of a divergent GM advection in the continuity equation but not the momentum equation violates the conservation of momentum.

The HYCOM model solves principally the same equations as MOM6 (83 to 87), except that the GM advective velocity is computed based on a (biharmonic) interface height diffusion (Bleck, 2002). The MPAS ocean model, instead uses a 3-dimensional divergence-free eddy advective velocity more directly following Gent & McWilliams (1990) and Gent et al. (1995).

Notice that MOM6 and HYCOM employ a semi-Lagrangian temporal discretization (e.g. Durran, 2010, Chapter 7), where the vertical coordinate follows the flow during time-stepping (i.e. \( \tilde{r} = 0 \) in Eqs. 83 to 86) and the model state is then re-mapped onto a target grid between time steps (Bleck, 2002; Griffies et al., 2020).
Interpretation in terms of coordinate-following generalized TWA

Figure 4. Sketch of coordinate-following average interpretation. The thin black lines show iso-surfaces of the vertical coordinate “\( r \)” in \((x-z)\) space, with the cyan line indicating the \( r \)-surface with \( r = r' \). The average is here assumed to be a spatial filter along the \( x \)-dimension with width \( 2\delta x \). In the coordinate-following average interpretation, the model variable \( \tilde{p}(x', r') = \bar{p}'(x', r') \), would then represent the average of \( p \) along the thick red line. Similarly \( \tilde{z}(x', z') = \bar{z}'(x', r') \) would be the average height along the red line, \( \tilde{c}(x', r') = \bar{c}'(x', r') \) would indicate the thickness-weighted average of \( c \) along the red line (and similarly for \( \tilde{\dot{c}} = \bar{\dot{c}}', \tilde{\dot{r}} = \bar{\dot{r}}', \tilde{\dot{F}}_{x/y} = \bar{\dot{F}}_{x/y}' \) and \( \tilde{\rho} = \bar{\rho}' \)).

One plausible interpretation of generalized vertical coordinate models is in terms of the generalized TWA equations with the average following the model coordinate (i.e. equations 13 to 17). In this case \( \tilde{z} = \bar{z}', \tilde{u} = \bar{u}', \tilde{r} = \bar{r}', \tilde{c} = \bar{c}', \tilde{\dot{c}} = \bar{\dot{c}}', \tilde{\dot{r}} = \bar{\dot{r}}', \tilde{\dot{F}}_{x/y} = \bar{\dot{F}}_{x/y}' \), and the diffusive tracer flux parameterizes the generalized TWA eddy tracer transport (i.e., \( \nabla \cdot (D \nabla \tilde{c}) = -\nabla \cdot \tilde{J}^c \)). However, as the generalized TWA equations do not include an eddy contribution in the continuity equation, this interpretation is only consistent with the model equations is (83) to (87) if no GM parameterization is used (i.e. \( u_{GM} = 0 \)). Moreover, the stress tensor in this interpretation needs to represent both the eddy momentum advection and the generalized eddy form stress \((\nabla \cdot \tau^u/v = -\nabla \cdot \tilde{E}^{u/v} \)). Finally, we note that \( \tilde{J}^c \) may in general have a skew-flux component in addition to a diffusive component, which would need to be assumed negligible when using a symmetric diffusivity tensor.

A challenge with the generalized TWA interpretation of a hybrid coordinate model is that the physical interpretation of the variables, and, perhaps more importantly, of the eddy terms that need to be parameterized, changes throughout the domain (e.g., as the coordinate transitions from being isopycnal to Eulerian), thus requiring parameterizations to be “coordinate system aware”. This situation poses a challenge for parameterization development, although it is not necessarily an unsurmountable problem. Indeed the interface height diffusion in HYCOM is coordinate aware as it acts on the layer interface height, and is thus automatically turned off where the coordinate follows \( z \)-surfaces, although the appearance of this closure in the continuity equation is not consistent with the equations derived here. At present, we are not aware of any any existing hybrid vertical coordinate model that employs coordinate-aware parameterizations consistent with the coordinate-following generalized TWA equations in Eqs. 13 to 17.
Interpretation in terms of Eulerian mean equations

Figure 5. Sketch of the Eulerian average interpretation. The thin black lines show iso-surfaces of the vertical coordinate “r” in (x-z) space, with the cyan line indicating the r-surface with \( r = r' \). The average is here assumed to be a spatial filter along the x-dimension with width 2δx.

In the Eulerian-mean interpretation, the quantity \( \tilde{c}(x', r') = c(z(x', r')) \) would represent the average of \( c \) along the height-surface \( z = z(x', r') \) marked by the red line. The same average applies to \( \tilde{\dot{c}} = \dot{c}(z), \tilde{\dot{\rho}} = \dot{\rho}, \tilde{\mathcal{F}}_{x/y} = \mathcal{F}_{x/y} \) and \( \tilde{\rho} = \rho \). The model's cross-coordinate velocity, \( \tilde{\dot{r}} \), is defined as the rate of change of \( r \) following the Eulerian-mean flow, that is \( \tilde{\dot{r}} = \dot{r}' \) as defined in Eq. (47).

If we regard the choice of model coordinate system as independent of the averaging, the Eulerian average may provide a simple interpretation also for generalized vertical coordinate models, although this interpretation is consistent with the current model equations in Eqs. (83) to (87) only if no GM parameterization is used (i.e. \( u_{GM} = 0 \)).

In the Eulerian mean interpretation the model variables are interpreted as \( \tilde{z} = z, \tilde{u} = u, \tilde{\dot{r}} = \dot{r}', \tilde{\dot{c}} = \dot{c}, \tilde{\dot{\rho}} = \dot{\rho}, \tilde{\mathcal{F}}_{x/y} = \mathcal{F}_{x/y} \) and \( \tilde{\rho} = \rho \). The viscous stress tensor needs to represent the effect of the Eulerian eddy momentum fluxes (i.e., \( \nabla \cdot \mathcal{F} = \nabla \cdot \mathcal{F} = \nabla \cdot (u/v) = 0 \)) and the tracer diffusion needs to capture the Eulerian eddy tracer flux (i.e. \( \nabla \cdot (D \nabla \tilde{c}) = \nabla \cdot (D \nabla c) \)). Notice that in the semi-Lagrangian time discretization we set \( \tilde{\dot{r}} = \dot{r}' = 0 \), which in this interpretation implies that the vertical coordinate follows the Eulerian mean flow (i.e. between remapping steps the coordinate is Lagrangian with respect to the Eulerian mean flow rather than the full unaveraged flow). Notice also that in hybrid coordinate models where the coordinate follows isopycnals in the interior, we set \( r = b(\theta, S, z) = b(\theta, S, z) \), that is the vertical coordinate is itself defined in terms of averaged quantities (which guarantees that model variables will be smooth along the model coordinate surfaces).

However, there is no eddy term in the Eulerian mean continuity equation (42), which is inconsistent with the implementation of GM in MOM6 and HYCOM. Instead, the “advective” effect of eddies appears as a skew flux contribution to the eddy term in the tracer equations in this interpretation, which could be parameterized as in z-coordinate models via either an antisymmetric component to the eddy diffusivity tensor or a 3D eddy advection, as done in the MPAS ocean model.

Notice that for a 3D divergence-free GM velocity (i.e. \( \nabla \cdot (z, u_{GM}) + \partial_z (z, \dot{r}_{GM}) = 0 \)) we can add this divergence to the LHS of the continuity equation (42) which gives an equation that is consistent with the model equation (83) if we interpret \( \tilde{\dot{r}} = \dot{r}' + \dot{r}_{GM} \). This interpretation is also consistent with the tracer equations where the advection with
the 3D divergence-free GM velocity would be interpreted as representing the skew flux component of the Eulerian eddy tracer flux. However, the interpretation that \( \tilde{r} = r^\# + r_{GM} \) is not consistent with the formulation of the momentum equations.

The interpretation of the model variables as Eulerian averages is arguably desirable for the justification of boundary conditions and boundary layer parameterizations, which are generally formulated assuming the Eulerian mean flow as given. Implementing the GM parameterization in such a way that it is consistent with this interpretation is therefore worthy of consideration. In the meantime, the Eulerian mean interpretation is appropriate for existing semi-Lagrangian models only in higher resolution configurations that do not employ a GM parameterization.

**Interpretation in terms of isopycnal TWA equations**

If the GM parameterization is not used (i.e., \( u_{GM} = 0 \)), Eqs. (83) to (87) can also be interpreted in terms of the isopycnal TWA Equations (37 to 41 with \( a \to b \)). In this interpretation, \( \tilde{z}(x', z') = \tilde{z}^b(x', r') \) would indicate the thickness-weighted average of \( c \) along the red line and similarly for \( \tilde{c} = \tilde{c}^b \), \( \tilde{u} = \tilde{u}^b \), \( \tilde{F}_{x/y} = \tilde{F}_{x/y}^b \) and \( \tilde{\rho} = \tilde{\rho}^b \). The model’s cross-coordinate velocity, \( \tilde{r} \), would need to be defined as the rate of change of \( r \) following the isopycnal thickness weighted mean flow, that is \( \tilde{r} = r^\# \) as defined in Eq. (30).

![Figure 6. Sketch of isopycnal average interpretation. The thin black lines show iso-surfaces of the vertical coordinate “r” in (x-z) space, with the cyan line indicating the r-surface with \( r = r' \). The average is here assumed to be a spatial filter along the x-dimension with width 2δx.](image-url)

In the isopycnal thickness-weighted-mean interpretation, the model variable \( \tilde{p}(x', r') = \tilde{p}^b(x', r') \), would represent the average of \( p \) along the isopycnal-surface \( b = b(x', r') \) marked by the red line. Similarly \( \tilde{z}(x', r') = \tilde{z}^b(x', r') \) would be the average height along the red line. \( \tilde{c}(x', r') = \tilde{c}^b(x', r') \) would indicate the thickness-weighted average of \( c \) along the red line and similarly for \( \tilde{c}^b \), \( \tilde{u}^b \), \( \tilde{F}_{x/y}^b \) and \( \tilde{\rho}^b \). The tracer diffusion needs to capture the isopycnal TWA eddy tracer flux, that is \( \nabla \cdot (D \nabla \tilde{c}) = -\nabla \cdot \tilde{J}_{c}^b \). Treating the diffusivity tensor \( D \) as symmetric then implies that we are ignoring any potential skew-flux contribution to \( \tilde{J}_{c}^b \). Finally, the viscous stress tensor needs to generally represent both the advective isopycnal TWA eddy momentum fluxes and the eddy form stress (i.e., \( \nabla \cdot \mathbf{F}^{u/v} = -\nabla \cdot \tilde{E}_{c}^b \)). When using semi-Lagrangian time-stepping, where we set \( \tilde{r} = 0 \), the implication of the isopycnal-TWA interpretation is that the vertical coordinate is Lagrangian with respect to the isopycnal TWA flow.
A potential caveat of the isopycnal TWA interpretation is that bulk formulas and boundary layer parameterizations tend to be formulated in terms of Eulerian mean quantities, which are not available in this formulation.

Perhaps more importantly, however, the implementation of the “GM” parameterization in Eqs. (83) to (87) is again generally inconsistent with the isopycnal TWA interpretation. Specifically the appearance of a “GM” eddy advection term in the continuity is again inconsistent with the isopycnal TWA interpretation. Instead, the effect of mesoscale eddies in reducing isopycnal slopes would need to be parameterized via the eddy form stress in the momentum equations (which in turn drives an ageostrophic circulation that tends to flatten isopycnals). A form-stress parameterization (that achieves a similar effect as the GM parameterization) can be implemented via an enhanced vertical viscosity (see Rhines & Young., 1982; Greatbatch & Lamb, 1990; Loose et al., 2023).

Interpretation in terms of mixed Eulerian / isopycnal TWA

Without the GM parameterization, and assuming the eddy advection to be negligible, the model equations (83) to (87) can also be interpreted in terms of the mixed Eulerian/isopycnal average in Eqs. (65) to (69). In this interpretation \( \tilde{z} = z \), \( \tilde{u} = \mathbf{u}^z \), \( \tilde{\rho} = \rho^z \), \( \tilde{\mathbf{F}}_{x/y} = \tilde{\mathbf{F}}_{x/y}^z \), and \( \tilde{\mathbf{J}}^{b^#} = \tilde{\mathbf{J}}^{b^#} \). The tracer diffusion then needs to capture the isopycnal TWA eddy tracer flux (i.e. \( \nabla \cdot (D \nabla \tilde{c}) = -\nabla \cdot J^{b^#} \)). In semi-Lagrangian time-stepping, where we set \( \tilde{\mathbf{r}} = 0 \), the implication is that the coordinate follows the Eulerian mean flow.
Unfortunately, the mixed interpretation is still not consistent with the model equations (83) to (87) when the GM advection is included. While the appearance of the horizontal eddy advection in the continuity and tracer equations (65 and 66) is consistent with the GM implementation in MOM6, the continuity and tracer equations in the mixed interpretation also include the vertical eddy advection, while the momentum equations do not. I.e. in Eqs. (83) and (84) we would need to assume \( \tilde{v} = \tilde{v}^{\#z} + \tilde{v}_{GM} \) to achieve consistency with the mixed interpretation, but in Eqs. (85) and (86) we would need to assume \( \tilde{v} = \tilde{v}^{\#z} \), so there is no choice of \( \tilde{v} \) that leads to a complete consistent set of equations.

Similar to the Eulerian mean interpretation, the model equations could be made consistent with the mixed Eulerian/isopycnal TWA interpretation by including a 3-dimensional divergence-free eddy advection (representing \( u^* \) and \( \tilde{v}^* \) in Eq. 66) in the tracer equations, and either including the same 3-dimensional eddy advection in the continuity equation, or removing the eddy advection in the continuity equation altogether. (A divergence-free eddy advection has no effect in the continuity equation.)

### 6 Conclusions

We derived the arbitrarily averaged equations in generalized vertical coordinates (Eqs. 37 to 41). The equations can be written in a form that mirrors the unaveraged equations, with a relatively straightforward interpretation of the “resolved” variables (which, however, are not all directly equal to the average of the respective variable) plus additional eddy forcing terms that can be written in the form of eddy flux divergences. These eddy flux divergences are fundamentally coordinate-system independent, but instead depend on the averaging coordinate. The implication is that eddy parameterizations need to be developed specific to the choice of average but not the choice of the model coordinate. (Although certain averaging choices may be more natural and/or numerically advantageous for different model coordinate systems.)

We also considered special cases for common averages (Eulerian and isopycnal) and formulated the resulting equations in generalized and specific coordinate systems, which allows us to (a) recover known results for isopycnal and z-coordinate equations and (b) consider candidates for the interpretation of existing generalized vertical coordinate models.

Various interpretations (Eulerian mean, isopycnal TWA, or a mixed Eulerian/isopycnal interpretation) are consistent with the existing generalized vertical coordinate model formulations if no GM parameterization (or interface-height diffusion) is used, and the eddy form stress and/or skew fluxes are assumed to be negligible (as may be appropriate for “eddy resolving” models).

However, no interpretation has been found that is consistent with the common implementation of the GM parameterization (or interface height diffusion) in existing generalized vertical coordinate models that employ a 2D divergent eddy advection in the continuity and tracer equations. We therefore suggest that implementation of the “GM” parameterization (or a dynamically similar parameterization of the form stress—e.g., Greatbatch & Lamb, 1990; Loose et al., 2023) in generalized vertical coordinate models should be modified to match one of the interpretations discussed in this paper. The candidates are essentially the same as for z-coordinate models, although numerical considerations may make an isopycnal TWA interpretation particularly desirable for semi-Lagrangian discretizations where the vertical coordinate approximately follows isopycnals in the interior.

Consistency with both the Eulerian mean and the mixed Eulerian/TWA interpretation can be achieved by implementing the GM parameterization via a 3D divergence-free advection (or, equivalently, a skew flux) in the tracer equation only, as done in the
MPAS ocean model. In semi-Lagrangian models, this implementation would imply that the vertical coordinate follows the Eulerian mean flow.

For models that use isopycnal target-coordinates in the interior, it is, however, numerically advantageous for the Lagrangian coordinate to follow the residual flow (as this requires no re-interpolation in an adiabatic isopycnal limit). This goal can be achieved by using the isopycnal TWA interpretation, in which case the GM parameterization needs to be replaced with a closure in the momentum equations, e.g. following Greatbatch & Lamb (1990). Such a closure has recently been implemented in MOM6 and tested successfully in a purely isopycnal configuration (Loose et al., 2023). Since the eddy form stress depends on the choice of averaging, and not the model coordinate system, the same closure should readily be applicable in generalized vertical coordinates.

The interpretation of generalized vertical coordinate models in terms of a coordinate-following average is also interesting to entertain, as it most naturally conforms to the model’s numerics. Indeed, the finite volume discretization naturally implies a volume-weighted grid-box average following the model coordinate (Griffies et al., 2020). For a hybrid isopycnal-z-coordinate model, the coordinate-following average also has the advantage that it naturally reduces to an Eulerian average near the surface and where stratification vanishes (which is useful for parameterizations in those regions). The major hurdle, however, is that the parameterizations will have to be “coordinate system aware”, which is expected to significantly complicate parameterization development.

We focus in this paper on conservative “thickness-weighted” averages with the generalized thickness defined as $\partial_a z$ for any averaging coordinate, $a$. As discussed in Loose et al. (2023) for the specific case of isopycnal averaging, any non-thickness weighted average following a coordinate surface with non-constant thickness (i.e., $\partial_a z \neq \text{const.}$) is non-conservative and hence leads to non-conservative equations for the mean quantities. Accepting this major limitation, one could interpret the velocities in isopycnal and perhaps generalized vertical coordinate models as non-thickness-weighted isopycnal averages, which then introduces the bolus transport into the thickness weighted continuity and tracer equations. As discussed in Appendix A, this interpretation is arguably the most consistent with the implementation of the “GM” parameterization in existing isopycnal coordinate models, although we consider the non-conservative nature of the equations to be highly undesirable. For generalized vertical coordinate models this interpretation moreover leads to an inconsistency in the treatment of the vertical velocity between the continuity and momentum equations.

We end by noting that consistency of the model equations with the analytically averaged equations is not just desirable for theoretical reasons but is fundamental for parameterization development, which relies on a clear definition of the eddy terms that need to be parameterized. The need for a consistent and agreed upon definition is particularly urgent given the recent rise in data-driven parameterization development, where parameterizations are “trained” offline based on filtered high-resolution data sets (e.g. Bachman et al., 2015, 2020; Zanna & Bolton, 2020; Guillaumin & Zanna, 2021; Perezhigin et al., 2023; Zhang et al., 2023). When applied online, these parameterizations can only be expected to be successful if the numerical model formulation is consistent with the filtering operation assumed in the training of the parameterization.

**Appendix A  Semi-thickness-weighted averaging**

We here derive the semi-thickness weighted equations where the continuity and tracer equations are averaged along isopycnals with thickness-weighting, but the velocities represent non-thickness-weighted averages. For simplicity we here assume a Reynolds average. The generalization to a non-Reynolds average works analogously to the results in the main manuscript and does not affect any of our conclusions.
Combining the thickness-weighted average continuity and tracer equations (13 and 14), with a non-thickness weighted average of the momentum equations (3,4), we can obtain a set of $b$-averaged equations of the following form (where for discussion purposes we will here assume $b$ to represent a suitably defined buoyancy variable, but formally it can be an arbitrary field that is monotonic in depth):

\[
\begin{align*}
\partial_t [\overline{\mathbf{u}^b}] + \nabla_b \cdot (\overline{\mathbf{u}^b} (\mathbf{u}_b + \mathbf{u}_b)) + \partial_b (\overline{\mathbf{u}^b} (b^b + b_b)) &= 0 \tag{A1} \\
\partial_t [\overline{\mathbf{v}^b}] + \nabla_b \cdot (\overline{\mathbf{v}^b} (\mathbf{v}_b + \mathbf{v}_b)) + \partial_b (\overline{\mathbf{v}^b} (b^b + b_b))^2 &= -\overline{\mathbf{u}^b} \cdot \nabla_b \mathbf{u}_b - b^b \partial_b \mathbf{u}_b^b - f \mathbf{v}_b^b - \rho_0 \overline{\partial_z [\overline{\mathbf{p}^b}]} + \mathbf{f}_x^b \tag{A2} \\
\partial_t [\overline{\mathbf{v}^b}] + \nabla_b \cdot (\overline{\mathbf{v}^b} (\mathbf{v}_b + \mathbf{v}_b)) + b_b \partial_b \mathbf{v}_b^b + f \mathbf{v}_b^b &= -\overline{\mathbf{u}^b} \cdot \nabla_b \mathbf{u}_b^b - b^b \partial_b \mathbf{v}_b^b - \rho_0 \overline{\partial_z [\overline{\mathbf{p}^b}]} + \mathbf{f}_y^b \tag{A3} \\
\partial_t [\overline{\mathbf{v}^b}] + \nabla_b \cdot (\overline{\mathbf{v}^b} (\mathbf{v}_b + \mathbf{v}_b)) - f \mathbf{v}_b^b &= -g \overline{\mathbf{b}^b} - \overline{\mathbf{p}^b} \tag{A4} \\
\partial_t [\overline{\mathbf{p}^b}] &= -g \overline{b^b} - \overline{\mathbf{p}^b} \tag{A5}
\end{align*}
\]

where $\mathbf{u}_b = \frac{\overline{\mathbf{u}^b}}{\overline{\mathbf{v}^b}} \mathbf{z}_b^b$ is the bolus velocity, $b_b = \frac{\overline{\mathbf{b}^b}}{\overline{\mathbf{v}^b}}$, and the pressure gradient term can be expressed in $b$-coordinates as

\[
\overline{\partial_z [\overline{\mathbf{p}^b}]} = \overline{\partial_x [\overline{\mathbf{b}^b}]} + \rho \overline{\partial_x [\overline{\mathbf{b}^b}]} - \rho \overline{\partial_x [\overline{\mathbf{b}^b}]} , \tag{A6a}
\]

and similarly for $\overline{\partial_y [\overline{\mathbf{b}^b}]}$. The eddy term in Eq. (A6b) vanishes if $\rho$ is constant along $b$-surfaces (as is the case for a linear equation of state with a buoyancy variable defined via Eq. (51)).

Notice that the eddy momentum flux contributions in Eqs. (A3) and (A4) are not in the form of a flux-derivative and are hence not conservative, as pointed out by Loose et al. (2023). This is in disagreement with the usual implementation of the eddy stress in numerical ocean models. Moreover, the appearance of the bolus transport in the continuity and tracer equations is consistent with the usual implementation of mesoscale eddy advection in isopycnal models, as long as we assume that $b_b \approx 0$, which is a reasonable assumption in the ocean interior, where mesoscale eddies are assumed to be largely adiabatic.

Despite these shortcomings, Eqs. (A1) to (A5) offer arguably the most obvious interpretation of existing isopycnal coordinate models (where—as pointed out in section 5.2—the parameterized eddy advection indeed does not conserve momentum). Specifically, the appearance of the bolus transport in the continuity and tracer equations is consistent with the usual implementation of mesoscale eddy advection in isopycnal models, as long as we assume that $b_b \approx 0$, which is a reasonable assumption in the ocean interior, where mesoscale eddies are assumed to be largely adiabatic.
To express the momentum equations (A3) and (A4) in an arbitrary vertical coordinate \( r \), we use that
\[
\frac{D}{Dt} \Phi^b = \partial_t \Phi^b + \mathbf{u}^b \cdot \nabla_b \Phi^b + \tilde{b} \partial_b \Phi^b
\]
\[
= \partial_t \Phi^b - \partial_t \tilde{b} (\partial_b \tilde{b})^{-1} \partial_b \Phi^b + \mathbf{u}^b \cdot \nabla_r \Phi^b
\]
\[
- (\mathbf{u}^b \cdot \nabla_r \tilde{b}) (\partial_b \tilde{b})^{-1} \partial_b \Phi^b + \tilde{b} (\partial_b \tilde{b})^{-1} \partial_b \Phi^b
\]
\[
= \partial_t \Phi^b + \mathbf{u}^b \cdot \nabla_r \Phi^b - (\partial_b \tilde{b})^{-1} \left[ \partial_t \tilde{b} + \mathbf{u}^b \cdot \nabla_r \tilde{b} - \tilde{b} \right] \partial_b \Phi^b
\]
\[
= \partial_t \Phi^b + \mathbf{u}^b \cdot \nabla_r \Phi^b + \tilde{r}^{tb} \partial_b \Phi^b
\]
where
\[
\tilde{r}^{tb} = \frac{D r}{Dt}
\]
\[
= \partial_t r + \mathbf{u}^b \cdot \nabla_b r + \tilde{b} \partial_b r
\]
\[
= - (\partial_b \tilde{b})^{-1} \left[ \partial_t \tilde{b} + \mathbf{u}^b \cdot \nabla_r \tilde{b} - \tilde{b} \right]
\]
is the Lagrangian rate of change of \( r \) following the non-thickness-weighted \( b \)-averaged flow.

Using Eq. (A9) to express the non-thickness-weighted momentum equations (A3, A4) in generalized vertical coordinates, and combining with the thickness-weighted continuity equation, tracer equation, and hydrostatic balance in arbitrary vertical coordinates (Eqs. 37, 38 and 41), we can write the semi-thickness-weighted \( b \)-averaged equations in generalized vertical coordinates as
\[
\partial_t [\tilde{z}_r^b] + \nabla_r \cdot ((\tilde{u}^b + u_b) \tilde{z}_r^b) + \partial_r [\tilde{z}_r^b \tilde{r}^{tb}] = 0
\]
\[
\partial_t [\tilde{z}_r^b \tilde{c}^b] + \nabla_r \cdot ((\tilde{u}^b + u_b) \tilde{c}^b) + \partial_r [\tilde{z}_r^b \tilde{r}^{tb} \tilde{c}^b] = - \tilde{z}_r^b \tilde{c}^b \cdot \nabla_b \tilde{c}^b + \tilde{z}_r^b \tilde{c}^b
\]
\[
\partial_t [\tilde{x}_r^b] + \tilde{u}^b \cdot \nabla_r \tilde{x}_r^b + \tilde{r}^{tb} \partial_b \tilde{x}_r^b - f \tilde{x}_r^b = - \tilde{u}^b \cdot \nabla_b \tilde{u}^b - \tilde{b} \partial_b u^b
\]
\[
- \rho_0^{-1} \frac{\partial}{\partial z} \tilde{p}^b + \tilde{p}^b
\]
\[
\partial_t [\tilde{v}_r^b] + \tilde{u}^b \cdot \nabla_r \tilde{v}_r^b + \tilde{r}^{tb} \partial_b \tilde{v}_r^b + f \tilde{v}_r^b = - \tilde{u}^b \cdot \nabla_b \tilde{v}^b - \tilde{b} \partial_b v^b
\]
\[
- \rho_0^{-1} \frac{\partial}{\partial y} \tilde{p}^b + \tilde{p}^b
\]
\[
\partial_r \tilde{p}^b = - \tilde{b} \partial_r \tilde{p}^b
\]
where the pressure gradient can be written in \( r \)-coordinates as
\[
\frac{\partial}{\partial z} \tilde{p}^b = \partial_z \tilde{p}^b + \tilde{b} \partial_{\tilde{z}} \tilde{p}^b + \rho \partial \tilde{b} \partial_{\tilde{b}} \tilde{p}^b.
\]
We again keep the eddy terms in \( b \)-coordinates as their effects need to be parameterized.

The semi-thickness-weighted \( b \)-averaged equations in (A11) to (A15) closely resemble the equations solved by MOM6 and HYCOM (Eqs. 83 to 87). However, the effective “vertical” velocities appearing the continuity and tracer versus momentum equations again differs. To interpret the existing model equations in terms of the semi-thickness weighted isopycnal average, we would need to assume that \( \tilde{r} = \tilde{r}^{tb} = \tilde{r}^{tb} \), but generally \( \tilde{r}^{tb} \neq \tilde{r}^{tb} \). Notice that the formulation of HYCOM and MOM6 evolved from isopycnal coordinate models (i.e. \( r = b \)) in which case \( \tilde{b}^{tb} = \tilde{b}^{tb} \) as long as the flow is adiabatic — this is the case discussed above. For a general vertical coordinate, however, \( \tilde{r}^{tb} \neq \tilde{r}^{tb} \), even for adiabatic flow. When using semi-Lagrangian time-stepping, setting \( \tilde{r}^{tb} = 0 \) in the continuity equation implies that the coordinate follows the thickness-weighted average flow. We then cannot also set \( \tilde{r}^{tb} = 0 \) in the momentum equation (which in turn would require the coordinate to follow the non-thickness-weighted flow), unless we assume the two flows to be the same (i.e. the bolus transport is not parameterized at all).
This inconsistency, together with the non-conservative form of the momentum equations and the eddy momentum flux, leads us to conclude that an interpretation of the generalized vertical coordinate model equations in terms of the semi-thickness-weighted isopycral average equations is neither desirable nor fully consistent with the existing model implementations.

Open Research

Scripts to create the plots in Figs. 1 and 2 can be found at doi.org/10.5281/zenodo.11509777.

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$c(x, z)$

$a(x, z)$

$r(x, z)$

$ar{c}^\alpha(x, r)$

$x$

express in $a$-coordinates

$c(x, a)$

$x$

express in $r$-coordinates

average

$ar{c}^\alpha(x, a)$
Figure 4.
The diagram illustrates a function $r(x, z)$ with a domain $x$ and a range $z$. The function is defined over a range $x' - \delta x$ to $x' + \delta x$ along the $x$-axis. The diagram shows how the function changes over this interval, with $r = r'$ indicating a constant value within the region $x' - \delta x$ to $x' + \delta x$. The vertical dashed lines at $x' - \delta x$ and $x' + \delta x$ denote the boundaries of the interval.
Figure 6.
\[ b = b^\#(x', z') \]
\[ b = b(x', z') \]
Figure 5.
Figure 2.
Figure 7.
\[ r(x, z) \]
\[ b(x, z) \]
\[ r = r' \]
\[ b = b^*(x', r') \]