Equations of Enhanced Phase-Locked Loop

Shafayat Abrar¹, Muhammad Mubeen Siddiqui¹, and Azzedine Zerguine¹

¹Affiliation not available

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Abstract—This work presents the design and analysis of an enhanced phase-locked loop (EPLL) for single-phase power systems. This paper examines two methods for developing an EPLL: gradient-flow optimization and Lyapunov theory. The suggested EPLL nonlinear differential equations are analyzed for stability using dynamic system theory. Computer simulations show that the proposed EPLL structure is highly efficient at extracting amplitudes, phases, and frequencies that match actual values, even in the presence of harmonics.

Index Terms—Enhanced phase-locked loop; gradient flow; natural gradient; Lyapunov stability; Poincaré map

I. INTRODUCTION

P

HASE-locked loop (PLL) is an important component in modern power systems. Its major role is to estimate the parameters of the line voltages and track any variations. The information on the frequency and phase of the line voltages is important in several tasks including synchronization and control of grid-connected power converters [1]. The enhanced phase-locked loop (EPLL) is an extension of PLL where, unlike a standard PLL, the amplitude of the sinusoidal excitation is also tracked down. The basic structure of EPLL was originally mentioned in a patent document in 1982 [2]. The patent documentation contained a non-mathematical description of EPLL, which received little or no attention from the community. Some twenty years later, around the same time, an EPLL, mathematically equivalent to that of [2], was rediscovered independently by Karimi-Ghartemani and Iravani [3], and Wu and Bodson [4]. They obtained the EPLL update expressions by minimizing a squared error cost. Consider that the system is given a sinusoidal input \( u(t) \), fed at the nominal frequency \( \omega_n \) and nominal voltage \( \rho_n \), giving \( u(t) = \rho_n \sin \delta_n \), where \( \delta_n = \omega_n t + \delta_n \). The EPLL provides \( \rho(t) \) and \( \delta(t) \) as estimates of \( \rho_n \) and \( \delta_n \), respectively, by minimizing the square of the error \( e(t) := u(t) - \rho(t) \sin \delta(t) \), using the steepest-descent (the standard gradient flow) leading to the following equations [5], [6]:

\[
\begin{align*}
\dot{\rho}(t) &= \mu_1 e(t) \sin \delta(t) \\
\dot{\delta}(t) &= \mu_2 \rho(t) e(t) \cos \delta(t), \quad \omega(-\infty) = \omega_n \\
\dot{\phi}(t) &= \omega(t) + \mu_3 \dot{\phi}(t)
\end{align*}
\]

Later, the above idea was applied straightforwardly in a three-phase system in [7]. In a sequel, intending to speed up the convergence, Karimi-Ghartemani et al. explored the potential of Hessian-based Newton’s gradient method (a.k.a. Newton-Raphson flow) to obtain a modified EPLL [8]. The following EPLL update was obtained:

\[
\begin{align*}
\dot{\rho}(t) &= \frac{1}{\sin^2 \phi (1 + \cos^2 \phi) - \frac{\delta}{\rho} \rho \sin \phi} \\
\dot{\delta}(t) &= \frac{e \sin \phi}{\rho \cos \phi \sin^2 \phi}
\end{align*}
\]

The above update, however, exhibited undesirable oscillations. To rectify the problem, the authors suggested replacing the common denominator with a constant and doing some other simplifications to obtain the following ad hoc solution:

\[
\begin{align*}
\dot{\rho}(t) &= \mu_1 e(t) \sin \delta(t) \\
\dot{\delta}(t) &= \mu_2 \rho(t) \sin(\delta(t) \cos \delta(t)), \quad \omega(-\infty) = \omega_n \\
\dot{\phi}(t) &= \omega(t) + \mu_3 \dot{\phi}(t)
\end{align*}
\]

The immediate difference between the two EPLLs, (1) and (3), is in the update \( \omega(t) \) where \( \rho(t) \) goes down from numerator to denominator. This unintentional finding proved to be immensely beneficial because it not only resulted in fast convergence [6], but also made it possible to express the EPLL structure (3) as a linear time-invariant system in Cartesian coordinates [9]. In recent articles on EPLL design, [10, Eq. (2)] and [11, Eq. (2)-(4)], authors have considered the use of factor \( \rho^{-1} \) in the phase update as one of the major steps. Nevertheless, the open literature lacks a proper explanation of this factor, and it is not evident what optimization problem may be formulated around the update process (3); the present work fills this gap. Here, we provide two approaches to account for the reciprocal of modulus in the phase update; we specifically use gradient flow-based optimization and Lyapunov stability theory to provide mathematical justification of and insight into the EPLL. Furthermore, this paper provides a complete analysis using nonlinear system theory to demonstrate the desirable performance, bounds on the step sizes, and stability of the proposed EPLL.

II. DERIVATION OF EPLL EQUATIONS

Consider a single-phase AC system where \( u(t) \) is the signal under consideration and it is expressed as:

\( u(t) = \rho_n \sin(\omega_n t + \delta_n) + u_b(t) \)

where \( \omega_n \) and \( \rho_n \) are nominal frequency and voltage, respectively; \( \delta_n \) is an arbitrary phase; and \( u_b(t) \) comprises sum of undesirable signal components like dc, harmonics, noise, and transients; for convenience, we denote \( \phi_n = \omega_n t + \delta_n \). The primary objective of an EPLL is to estimate the frequency, amplitude, and phase angle of the fundamental component of a given periodic signal \( u(t) \) for measurement and synchronization.

To achieve this goal well, we must reduce the undesired component \( u_b(t) \) of \( u(t) \) to the greatest extent possible which obviously can be done by some filtering process. One of the
best options is to use an orthogonal signal generator to obtain a pair of clean orthogonal signals \((u_α, u_β)\) from \(u(t)\) where \(u_α\) lags \(u_β\) by \(\pi/2\). This requires an estimate of the nominal frequency, which is provided by the EPLL; see Fig. 1. Using the orthogonal pair \((u_α, u_β)\), the EPLL produces another set of orthogonal signals \((y_α, y_β)\), where \(y_α\) and \(y_β\) are supposedly the estimates of \(u_α\) and \(u_β\), respectively. In the sequel, we explain two methods to implement the EPLL.

Fig. 1. The structural block diagram of the EPLL preceded by an OSG.

### A. Lyapunov Stability-Based Derivation

Abramovitch [12] initially published the guidelines for Lyapunov theory-based PLL design. Accordingly, the second method of Lyapunov [13] was used as a principle that provided a stability guarantee of nonlinear differential equations of the resulting phase-locked loop (without requiring to solve the differential equation). Lyapunov’s second method uses the notion of energy to establish stability. So, to achieve asymptotic stability of the system in the Lyapunov sense, all that is required is an energy-like function of the system state that is positive definite (nonvanishing as long as the state is not zero) and continuously decreasing. To put this in mathematical form, consider a generic dynamic system defined as follows:

\[
ed(t) = f(e), \quad e(0) = c, \quad \tag{4}\]

**La Salle’s Theorem** [14, Th. 4.1]: For the system (4), suppose there exists a positive definite scalar function, \(V(e)\), such that \(\dot{V}(e)\) is negative semi-definite, i.e.,

\[
\begin{align*}
V(e) &> 0, \quad \frac{dV}{dt} = \frac{\partial V}{\partial e} \dot{e} = \dot{V}(e) \leq 0 \quad \forall e \neq 0,
V(e) & = 0, \quad \frac{dV}{dt} = \frac{\partial V}{\partial e} \dot{e} = \dot{V}(e) = 0 \quad e = 0.
\end{align*} \tag{5}\]

Suppose also that the only solution of \(\dot{e} = f(e), \dot{V}(e) = 0\) is \(e(t) = 0\) for all \(t \geq 0\), then \(e = f(e)\) is globally asymptotically stable.

This theorem will be highly beneficial in the sequel. \(V\) is considered a Lyapunov function if it follows La Salle’s theorem. Let \((u_α(t), u_β(t))\) be the pair of orthogonal signals generated from the input signal \(u(t)\) using an orthogonal signal generator. Also, let \((y_α(t), y_β(t))\) be the pair of orthogonal signals synthesized by the EPLL using \((u_α(t), u_β(t))\). The aim is to make \(y_α(t)\) and \(y_β(t)\) match \(u_α(t)\) and \(u_β(t)\), respectively, by ensuring that the EPLL polar outputs \(\rho(t)\) and \(\delta(t)\) respectively converge to the desired values \(\rho_0\) and \(\delta_0\). We define a vector of errors as follows:

\[
e = \begin{bmatrix} e_α(t) \\ e_β(t) \end{bmatrix} = \begin{bmatrix} u_α(t) - y_α(t) \\ u_β(t) - y_β(t) \end{bmatrix} \tag{6}\]

We introduce the energy-like Lyapunov function as follows:

\[
V(e) = \frac{1}{2} \|e\|^2 = \frac{1}{2} (e_α^2 + e_β^2) \tag{7}\]

which is nonvanishing as long as the error terms are not zero. Since the purpose of an EPLL is to track the change in amplitude and phase, the task in hand is to obtain expressions for \(\dot{\rho}(t)\) and \(\dot{\delta}(t)\) using the Lyapunov theory. Differentiating \(V(e)\) w.r.t. \(t\), we get

\[
\dot{V}(t) = e_α \dot{e}_α + e_β \dot{e}_β
= -\dot{\rho}(e_α \sin \phi + e_β \cos \phi) +
\omega_n(e_α u_β - e_β u_α) - \dot{\phi}(e_α y_β - e_β y_α) \tag{8}\]

Let \(\dot{V}_0\) denote the term \(-\dot{\rho}(e_α \sin \phi + e_β \cos \phi)\), and \(\dot{V}_1\) denotes the term \(\omega_n(e_α u_β - e_β u_α) - \dot{\phi}(e_α y_β - e_β y_α)\). To ensure \(\dot{V}_0 \leq 0\), \(\dot{\rho}\) may be selected as follows:

\[
\dot{\rho} = \mu(e_α \sin \phi + e_β \cos \phi), \quad \text{for some } \mu > 0, \tag{9}\]

which leads to satisfy the negative semi-definite condition:

\[
\dot{V}_0(t) = \mu(e_α \sin \phi + e_β \cos \phi)^2 \leq 0. \tag{10}\]

Next, consider \(\dot{V}_1(t)\); exploiting the identity

\[
e_α u_β - e_β u_α = e_α y_β - e_β y_α, \tag{11}\]

we get \(\dot{V}_1(t) = (\omega_n - \dot{\phi})(e_α y_β - e_β y_α)\). Next, we select \(\dot{\phi}\) such that \(\dot{V}_1 \leq 0\); an immediate solution is

\[
\dot{\phi}(t) = \omega_n + \eta(e_α y_β - e_β y_α), \quad \text{for some } \eta > 0, \tag{12}\]

which leads to

\[
\dot{V}_1(t) = -\eta(e_α y_β - e_β y_α)^2 \leq 0. \tag{13}\]

Combining (10) and (13), we get

\[
\dot{V}(t) = -\mu(e_α \sin \phi + e_β \cos \phi)^2 - \eta(e_α y_β - e_β y_α)^2 \tag{14}\]

To ensure globally exponential Lyapunov stability, \(\mu\) and \(\eta\) must be selected such that \(\dot{V}(t) = -\kappa \|e\|^2, \kappa > 0\); refer to [15, Th. 4.10].

Let \(\eta = \mu/\rho^2\), we establish that

\[
\dot{V}(t) = -\mu(e_α \sin \phi + e_β \cos \phi)^2 - \frac{\mu}{\rho^2}(e_α y_β - e_β y_α)^2
= -\mu(e_α \sin \phi + e_β \cos \phi)^2 - \mu(e_α \cos \phi - e_β \sin \phi)^2
= -\mu \cdot (e_α^2 + e_β^2) = -2\mu V(t) \tag{15}\]

The cost \(V(t)\) thus satisfies La Salle’s theorem as both \(\dot{V}(t)\) and \(\dot{V}(t)\) become zero when \(e_α = e_β = 0\), and \(\dot{V}(t) > 0\) when \(e_α \neq 0, e_β \neq 0\). Substituting \(\eta = \mu/\rho^2\) in \(\dot{\phi}(t)\), we obtain

\[
\dot{\phi}(t) = \omega_n + \frac{\mu}{\rho^2}(e_α y_β - e_β y_α) = \omega_n + \frac{\mu}{\rho}(e_α \cos \phi - e_β \sin \phi)
\]

where \(\dot{\phi} = \omega_n + \dot{\phi}\). Substituting \(e_α\) and \(e_β\), the following amplitude-phase updates for the proposed EPLL are obtained:

\[
\dot{\rho}(t) = \mu(u_α(t) \sin \phi(t) + u_β(t) \cos \phi(t)) - \mu \rho(t) \tag{16a}\]

\[
\dot{\delta}(t) = \frac{\mu}{\rho(t)}(u_α(t) \cos \phi(t) - u_β(t) \sin \phi(t)) \tag{16b}\]
Note that Eq. (15) yields $V(t) = V(t_0)e^{-2\mu(t-t_0)}$ for $t \geq t_0$. So, $V(t) \to 0$, as $t \to \infty$. This shows that $V(t)$ constantly decreases along any solution of (16). Another finding is that two separate step sizes are not required for amplitude and phase updates rather a single controlling parameter $\mu$ is enough to ensure the exponential stability.

### B. Gradient Flow-Based Derivation

Gradient-based adaptation is a well-known practice for adjusting the parameters of dynamic systems by optimizing a candidate cost function. Here, we describe how an EPLL may be designed by exploiting the theory of gradient adaptation. Let $y(t) := [y_\alpha(t)\; y_\beta(t)]^T$ be the EPLL output in Cartesian coordinates. A change in $y$ due to incremental changes in $\rho(t)$ and $\delta(t)$ is

$$
\Delta y = [\Delta y_\alpha(t)\; \Delta y_\beta(t)]^T
= \left[ (\rho + \Delta \rho) \sin(\omega_n t + \delta + \Delta \delta) \overline{C} (\rho + \Delta \rho) \cos(\omega_n t + \delta + \Delta \delta) \right] - \left[ \rho \sin(\omega_n t + \delta) \overline{C} \rho \cos(\omega_n t + \delta) \right]
$$

Expanding, we get

$$
\Delta y_\alpha(t) =
\Delta \rho \cos(\Delta \delta) \cos(\delta) \sin(\omega_n t) - \rho \cos(\omega_n t) (\sin(\delta) - \rho \cos(\omega_n t)) \sin(\delta) + \Delta \rho \cos(\Delta \delta) \cos(\omega_n t) \sin(\delta) + \Delta \rho \sin(\Delta \delta) \cos(\omega_n t) \sin(\delta) + \rho \cos(\omega_n t) \sin(\delta) - \Delta \rho \sin(\Delta \delta) \sin(\omega_n t) - \rho \sin(\Delta \delta) \sin(\omega_n t)
$$

$$
\Delta y_\beta(t) =
\rho \sin(\delta) \sin(\omega_n t) - \rho \cos(\omega_n t) \cos(\omega_n t) + \Delta \rho \cos(\Delta \delta) \cos(\omega_n t) \sin(\omega_n t) - \Delta \rho \sin(\Delta \delta) \cos(\omega_n t) \sin(\delta) - \rho \cos(\omega_n t) \sin(\delta) - \Delta \rho \sin(\Delta \delta) \sin(\omega_n t) - \rho \sin(\Delta \delta) \sin(\omega_n t)
$$

Considering $\sin(\Delta \delta) \approx \Delta \delta$, $\cos(\Delta \delta) \approx 1$, and ignoring the terms containing the product $\Delta \rho \Delta \delta$, we obtain

$$
\Delta y \approx \left[ \begin{array}{c}
\Delta \rho \cdot \sin(\omega_n t + \delta) + \Delta \delta \cdot \rho \cos(\omega_n t + \delta) \\
\Delta \rho \cdot \cos(\omega_n t + \delta) - \Delta \delta \cdot \rho \sin(\omega_n t + \delta)
\end{array} \right]
$$

The squared Euclidean norm is $\|\Delta y\|^2 = (\Delta \rho)^2 + \rho^2(\Delta \delta)^2$. Define an incremental parameter vector $\Delta \theta := [\Delta \rho, \Delta \delta]^T$, where $\theta := [\rho, \delta]^T$, the following matrix form is obtained

$$
\|\Delta y\|^2 = \Delta \theta^T G \Delta \theta \quad (17)
$$

where the matrix $G$ is commonly known as the Riemannian metric tensor for polar space. The norm $\|\Delta y\|$ represents the Euclidean distance between the polar points $(\rho, \delta)$ and $(\rho + \Delta \rho, \delta + \Delta \delta)$. The optimization task in hand is thus to find the parameter space $\theta \in \mathbb{R}^2$ on which a (suitably smooth) gradient-decent searchable cost function $V(\theta(t))$ is defined subject to bounding the squared Euclidean norm $\|\Delta \theta\|^2_G := \Delta \theta^T G \Delta \theta \leq \epsilon^2$ when we wish to update $\theta$ to $\theta + \Delta \theta$ in time $\Delta t$. We formulate a natural gradient flow problem as follows [16, Chap. 13]-[17, Chap. 4]:

$$
\Delta \theta_t^\dagger = \arg\min_{\Delta \theta \in \mathbb{R}^2} V(\theta + \Delta \theta), \text{ s.t. } \Delta \theta^T G \Delta \theta \leq \epsilon^2 \quad (18a)
$$

$$
\frac{d \theta}{dt} = \Delta \theta^\dagger \quad (18b)
$$

where the cost $V(\theta, t)$ is the same as specified in (8). To optimize (18), we introduce a Lagrangian multiplier $\lambda > 0$, and consider the following unconstrained formulation:

$$
\Delta \theta^\dagger = \arg\min_{\Delta \theta \in \mathbb{R}^2} \left\{ V(\theta + \Delta \theta) + \frac{\lambda}{2} (\Delta \theta^T G \Delta \theta - \epsilon^2) \right\} \quad (19)
$$

Expanding Taylor’s series around $V(\theta)$, we get

$$
V(\theta + \Delta \theta) \approx V(\theta) + \Gamma^T \Delta \theta,
$$

where $\Gamma := \nabla_\theta V(\theta)$ is gradient vector. With the above approximation in consideration, we minimize the right-hand side of (19) over $\Delta \theta$. Setting the gradient vector relative to $\Delta \theta$ to zero and assuming the full rank, we obtain

$$
\Delta \theta = -\frac{\lambda}{2} (G + G^T)^{-1} \Gamma
$$

Exploiting the facts that $G = G^T$, we determine $\lambda$ by seeking the maximum disturbance, i.e., $\|\Delta y\|^2 = \Delta \theta^T G \Delta \theta = \epsilon^2$, where $\Delta \theta = -\left(\lambda G\right)^{-1} \Gamma$. We get

$$
\Gamma^T (\lambda G)^{-1} G (\lambda G)^{-1} \Gamma = \epsilon^2
$$

The optimal value of the Lagrangian multiplier and the corresponding gradient flow are obtained as

$$
\lambda_\alpha = \frac{\|\Gamma\|}{\epsilon} \quad (20a)
$$

$$
\dot{\theta}(t) = - (\lambda_\alpha G)^{-1} \Gamma = -\frac{1}{\lambda_\alpha} G^{-1} \Gamma \quad (20b)
$$

The quantity $G^{-1} \Gamma$ in $\dot{\theta}(t)$ is evaluated as

$$
G^{-1} \Gamma = \text{diag} \left( [1, \rho^2] \right) \left[ \frac{\partial}{\partial \rho} \frac{\partial}{\partial \delta} \right]^T V(\theta)
$$

where

$$
\frac{\partial}{\partial \delta} V(\theta) = \frac{1}{2} \frac{\partial}{\partial \delta} \left( e_\alpha^2 + e_\beta^2 \right) = e_\alpha \frac{\partial e_\alpha}{\partial \delta} + e_\beta \frac{\partial e_\beta}{\partial \delta}
$$

$$
= - e_\alpha \sin \phi - e_\beta \cos \phi = - u_\alpha \sin \phi - u_\beta \cos \phi + \rho. \quad (21)
$$

and

$$
\frac{\partial}{\partial \rho} V(\theta) = \frac{1}{2} \frac{\partial}{\partial \rho} \left( e_\alpha^2 + e_\beta^2 \right) = e_\alpha \frac{\partial e_\alpha}{\partial \rho} + e_\beta \frac{\partial e_\beta}{\partial \rho}
$$

$$
= - e_\alpha \sin \phi - e_\beta \cos \phi = - u_\alpha \sin \phi - u_\beta \cos \phi + \rho. \quad (22)
$$

Denoting $\theta_* := [\rho_*, \delta_*]$, we see that $\nabla_\theta V(\theta) = 0$ when $\theta = \theta_*$. Further, simple manipulations yield

$$
\|G\|^{-1}_\alpha = (\rho - u_\alpha \sin \phi - u_\beta \cos \phi)^2 + \frac{1}{\rho^2} (u_\alpha y_\beta - u_\beta y_\alpha)^2
$$

$$
= \rho_*^2 + \rho^2 - 2 \rho_* \rho \cos(\delta - \delta_*)
$$

$$
e_\alpha^2 + e_\beta^2 = \|e\|^2 = \epsilon_*^2 \quad (23)
$$
This means that the ODE $\dot{\theta}(t)$ may be written as
\[
\dot{\theta}(t) = -\frac{1}{\lambda_n} G^{-1} \Gamma = -\frac{\|\Delta y\|}{\|e\|} G^{-1} \Gamma = -\frac{\epsilon_\star}{\epsilon_x} G^{-1} \Gamma \tag{24}
\]
which signifies that the ODE is not only aware of the actual distance ($\|e\|$) between EPLL outputs ($\rho$, $\delta$) and the required nominal values ($\rho_n$, $\delta_n$) but it is also aware of the allowed step ($\|\Delta y\|$) it can take. Finally, denoting $\mu := \epsilon_\star/\epsilon_x$, we obtain
\[
\dot{\rho} = -\mu \frac{\partial}{\partial \rho} V(\theta) = \mu (u_a \sin \phi + u_\beta \cos \phi) - \mu \rho \tag{25a}
\]
\[
\dot{\delta} = -\frac{\mu}{\rho^2} \frac{\partial}{\partial \delta} V(\theta) = \frac{\mu}{\rho} (u_a \cos \phi - u_\beta \sin \phi) \tag{25b}
\]
which is equivalent to the ODE obtained earlier using the Lyapunov method; it establishes that the simplified gradient flow update (25) exhibits Lyapunov stability guarantee. Therefore, when $\mu$ is small, it means that the parameter vector $\theta$ deviates from its present value as little as possible while seeking the steepest direction to minimize $V(\theta)$, and getting closer to the required nominal values.

III. AVERAGING AND LINEARIZATION

This section discusses the behavior of the proposed EPLL after averaging and linearization; it establishes that the proposed EPLL can track down the amplitude and phase of the input sinusoid irrespective of the initial conditions. The simplest form of averaging is periodic averaging, which is concerned with solving a perturbation problem in the standard form [18, Chapter 2]
\[
\dot{\theta}(t) = \epsilon \tilde{f}(\theta, t, \epsilon), \quad \theta(0) = c, \tag{26}
\]
where $f$ is $T$-periodic in $t$. So, averaging over $t$ (while holding $\theta$ constant), we consider the averaged equation
\[
\tilde{\theta}(t) = \epsilon \tilde{f}(\tilde{\theta}), \quad \tilde{\theta}(0) = \alpha, \tag{27a}
\]
with $\tilde{f}(\tilde{\theta}) = \frac{1}{T} \int_T^T f(\tilde{\theta}, t) dt \tag{27b}$

The basic result is that the solutions of these systems remain close (of order $\epsilon$) for a time interval of order $1/\epsilon$:
\[
\|\theta(t) - \tilde{\theta}(t)\| \leq \epsilon c \quad \text{for} \quad 0 \leq t \leq L/\epsilon \tag{28}
\]
for some positive constants $c$ and $L$.

To apply averaging to EPLL (25), we first consider the input to the system is given by $u(t) = u_n(t) + g(t)$, where $u_n(t) := \rho_n \sin (\omega_n t + \delta_n)$ is the nominal $T$-periodic signal, and $g(t) = \sum_{k \neq 1} A_k \sin(\omega_n t + \theta_k)$ is an arbitrary bounded continuous function with no frequency component at $\omega_n = 2\pi/T$. Secondly, we consider a first-order integrator [19] as shown in Fig. 2 and described as follows:
\[
\dot{x}(t) = u(t) - K_F \bar{\omega} x(t) = \frac{1}{K_F} (\bar{\omega} x(t) + u_\beta(t)) \tag{29}
\]
where $u(t)$ and $u_\beta$ are input and output, respectively; $\bar{\omega}$ is an estimate of $\omega_n$; $x(t)$ is auxiliary, and $K_F > 0$.

Considering pairs $u_\beta(t) \leftrightarrow U_\beta(s)$ and $u(t) \leftrightarrow U(s)$, and eliminating $\dot{x}(t) \leftrightarrow \dot{X}(s)$, we get
\[
U_\beta(s) = \frac{K_F s - \bar{\omega}}{s + K_F \bar{\omega}} \cdot U(s) \tag{30}
\]

When $u(t) = \rho_n \sin (\omega_n t + \delta_n)$ and $\bar{\omega} = \omega_n$ as a good estimate, we get $U_\beta(j \omega_n) = U(j \omega_n) \exp(j \pi/2)$; this gives $u_\beta(t) = \rho_n \cos \delta_n \forall K_F \in \mathbb{R}^+$. Let $M(\omega)$ and $p(\omega)$ be the magnitude and phase responses of FOI, respectively, then the input $g(t)$ yields a response, loosely denoted as $g_\beta(t)$:
\[
g_\beta(t) := \sum_{k \neq 1} A_k M(\omega_n) \sin(k \omega_n t + \theta_k + p(k \omega_n)) \tag{31}
\]

Note $g_\beta(t)$ is not necessarily an orthogonal copy of $g(t)$ except for some values of $k$, where $\theta_k + p(k \omega_n) = \pm \pi/2$ might be true. Next, we express our proposed EPLL (16) in the standard form suitable for the averaging analysis. Define $\mu = \epsilon T$, where $0 < \epsilon \ll 1$. Let $\theta = [\rho, \delta]^T = \epsilon f(\theta, t, \epsilon)$, where $f : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^2$ is of period $T = 2\pi/\omega_n$ in $t$; we get
\[
f(\theta, t, \epsilon) = \begin{bmatrix} \pi(\sin \phi(t) (u_n + g) + \cos \phi(t) (u_\beta + g_\beta) - \rho) \\ \pi(\cos \phi(t) (u_n + g) - \sin \phi(t) (u_\beta + g_\beta))/\rho \end{bmatrix} \tag{32}
\]

where $\phi(t) = \omega_n t + \delta(t)$. The associated average system is defined as (where overline is used to denote average variables):
\[
\tilde{\theta}(t) = \begin{bmatrix} \tilde{\rho} \\ \tilde{\delta} \end{bmatrix} = \frac{\epsilon}{T} \int_0^T f(\bar{\theta}, t, 0) dt := \epsilon \bar{f}(\bar{\theta}), \tag{33}
\]

where $\bar{\theta}$ is the average value of $\theta$; thus,
\[
\tilde{\rho}(t) = \frac{\mu}{\bar{\rho}} \int_0^T [-\bar{\rho} \sin (\omega_n t + \delta) (u_n + g) + \cos (\omega_n t + \delta) (u_\beta + g_\beta)] dt, \\
\tilde{\delta}(t) = \frac{\mu}{\bar{\rho}} \int_0^T [\cos (\omega_n t + \delta) (u_n + g) - \sin (\omega_n t + \delta) (u_\beta + g_\beta)] dt \tag{34}
\]

Owing to facts: $\int_0^T \cos (k \omega_n t) \cos (\ell \omega_n t) dt = 0, k \neq \ell$, $\int_0^T \sin (k \omega_n t) \sin (\ell \omega_n t) dt = 0, k \neq \ell$, and $\int_0^T \cos (k \omega_n t) \sin (\ell \omega_n t) dt = 0, \forall k, \ell$, all integrals involving $g$ or $g_\beta$ are equal to zero, and we obtain an average system
\[
\tilde{\rho}(t) = -\mu \bar{\rho} + \mu \rho_n \cos (\delta - \delta_n), \tag{35a}
\]
\[
\tilde{\delta}(t) = -\frac{\mu \rho_n \bar{\rho}}{\bar{\rho}} \sin (\delta - \delta_n). \tag{35b}
\]

To examine the convergence behavior for different initial conditions of the averaged form (31) of the proposed EPLL (25). This is done by obtaining the trajectories and quiver plot of (31). We assume $\rho_n = 1$ and $\delta_n = \pi/8$; we consider $\mu = 1$, and obtain quivers and phase diagram using the MATLAB tool [20]. Results are shown in Fig. 3; clearly, stable equilibrium
points are located at \( \overline{p}(\infty) = \pm \rho_n \) and \( \overline{\delta}(\infty) = \delta_n \) and \( \delta_n \pm \pi \). So, based on this result, obtained for the averaged-linearized system, we deduce that the system (25) exhibits desirable stable nodes.

After some standard manipulations, the following approximate solution for \( t \geq t_0 \) is obtained for the average-linearized system (proof is simple and thus skipped):

\[
\overline{p}(t) \approx \rho_n + (\overline{p}(t_0) - \rho_n) e^{-\mu(t-t_0)} + \frac{\rho_n (\overline{\delta}(t_0) - \delta_n)^2}{2} (e^{-2\mu(t-t_0)} - e^{-\mu(t-t_0)}) \tag{33a}
\]

\[
\overline{\delta}(t) \approx \delta_n + (\overline{\delta}(t_0) - \delta_n) \exp [-\mu(t-t_0) - \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{\rho_n - \overline{p}(t_0)}{\rho_n} \right)^k \left( 1 - e^{-\mu k(t-t_0)} \right)] \tag{33b}
\]

Note that the modulus and angle of the average-linearized system are bounded and converge to the desired values, i.e., \( \overline{p}_0(\infty) \to \rho_n \), and \( \overline{\delta}(\infty) \to \delta_n \), irrespective of the initial conditions, \( \overline{p}_0(t_0) \) and \( \overline{\delta}(t_0) \), where the step-size \( \mu \) governs the speed of convergence. Under the present context of periodic orbit analysis, we consider \( \overline{\theta}_\ell := [\overline{p}_0, \overline{\delta}]^T \) with \( \overline{p}_0(t_0) =: \rho_0 \) and \( \overline{\delta}_0(t_0) =: \delta_0 \). If \( \Sigma \) is the ray \( \overline{\theta}_0(t) \) through the origin, then \( \Sigma \) is perpendicular to \( \Gamma \) and the trajectory through the point \( (\rho_0, \delta_0) \in \Sigma \cap \Gamma \) at \( t = t_0 \) intersects the ray \( \overline{\theta}_0(t_0) \) again at \( t = 2\pi/\omega_n \); cf. Fig. 4, where the instantaneous phase is considered equal to \( \omega_n t + \delta_0 \). It follows that the Poincaré map is given by (assuming \( t_0 = 0 \))

\[
P(\overline{p}_0) \approx \rho_n + (\overline{p}_0 - \rho_n) e^{-2\pi\mu/\omega_n} + \frac{\rho_n (\overline{\delta}_0 - \delta_0)^2}{2} \left( e^{-4\pi\mu/\omega_n} - e^{-2\pi\mu/\omega_n} \right) \tag{34}
\]

Note \( P(\rho_n; \overline{\delta}_0 = \delta_0) = \rho_n \) corresponds to the cycle \( \Gamma \), and

\[
\frac{\partial P(\overline{p}_0)}{\partial \overline{p}_0} = e^{-2\pi\mu/\omega_n} < 1 \tag{35}
\]

establishes that the periodic solution \( \overline{\theta}_\ell \) of the linearized averaged EPLL has a stable limit cycle (refer to Theorem B and Corollary in Section B of the appendix.).

IV. Periodic Orbit Analysis

This section provides a periodic orbit analysis of the proposed EPLL using Poincaré’s map theory. Poincaré’s map theory is an effective tool for assessing the stability of periodic orbits and limit cycles in \( n \)-dimensional dynamical systems (where the system’s trajectories are relatively simple to integrate). Poincaré’s theorem, in particular, establishes necessary and sufficient conditions for the stability of periodic orbits built from a Poincaré return map.

In particular, for a given candidate periodic trajectory, an \((n-1)\)-dimensional hyperplane is created that is transverse to the periodic trajectory and determines the Poincaré return map [21]. The idea of the Poincaré map is: if \( \Gamma \) is a periodic orbit of \( \overline{\theta} = T(\theta) \) through the point \( \theta_0 \) and \( \Sigma \) is a hyperplane perpendicular to \( \Gamma \) at \( \theta_0 \), then for any point \( \theta \in \Sigma \) sufficiently near \( \theta_0 \), the solution of \( \overline{\theta} \) through \( \theta \) at \( t = 0, \psi(\theta) \), will cross \( \Sigma \) again at a point \( P(\theta) \) near \( \theta_0 \). The mapping \( \theta \to P(\theta) \) is called the Poincaré map [22]; this is briefly discussed in Section A of the appendix.

Consider the average system (31). We linearize it around \((\rho_n, \delta_n)\) using Taylor’s series as shown below, where the subscript \( \ell \) and the overline denote the linearization and averaging, respectively:

\[
\dot{\overline{p}}_\ell(t) \approx -\mu (\overline{p}_\ell - \rho_n) - \frac{\mu \rho_n}{2} (\overline{\delta}_\ell - \delta_n)^2 \tag{32a}
\]

\[
\dot{\overline{\delta}}_\ell(t) \approx -\mu (\overline{\delta}_\ell - \delta_n) \sum_{k=0}^{\infty} \left( \frac{\rho_n - \overline{p}_\ell}{\rho_n} \right)^k \tag{32b}
\]

After some standard manipulations, the following approximate solution for \( t \geq t_0 \) is obtained for the average-linearized system.

\[1\] We assumed \( \overline{\delta}_0 = \delta_0 \) to establish \( P(\rho_n) = \rho_n \); though, this is true when \( t \to \infty \), this assumption is not strictly required. Let us assume that \( \overline{\delta}_0 \) is in a close vicinity of \( \delta_0 \), and not necessarily equal to \( \delta_0 \) such that \( \cos(\overline{\delta} - \delta_0) \approx 1 \) and \( \sin(\overline{\delta} - \delta_0) \approx \overline{\delta} - \delta_0 \), and as a result, the averaged system (31) reduces to \( \overline{p}(t) = -\mu \overline{p} + \mu \rho_n \), and \( \overline{\delta}(t) = -\mu \rho_n (\overline{\delta} - \delta_0) / \overline{p} \), that leads to \( \overline{p}(t) \approx \rho_n + (\overline{p}_0 - \rho_n) e^{-\mu t} \), giving \( P(\rho_n) = \rho_n \) and \( P^*(\rho_n) = e^{-2\pi\mu/\omega_n} < 1 \) without requiring \( \delta_0 \) to be necessarily equal to \( \delta_n \) or to assume a large \( t \).
Because \( y_\alpha(t) = \rho(t) \sin(\phi(t)), y_\beta(t) = \rho(t) \cos(\phi(t)) \), and \( \phi(t) = \omega_n t + \delta(t) \), we may express the proposed EPLL (25) in Cartesian form as follows:

\[
\begin{align*}
\dot{y}_\alpha(t) &= -\mu (y_\alpha(t) - u_\alpha(t)) + \omega_n y_\beta(t) \\
\dot{y}_\beta(t) &= -\mu (y_\beta(t) - u_\beta(t)) - \omega_n y_\alpha(t)
\end{align*}
\]  

where \( u_\alpha(t) = \rho_n \sin(\delta_n) \) and \( u_\beta(t) = \rho_n \cos(\delta_n) \). This finding establishes that when the OSG knows \( \omega_n \) precisely, the EPLL outputs converge to the desired periodic orbit regardless of the initial condition; cf. Fig. 5. The steady-state orbit is \( \Gamma : \gamma(t) = (\sin(\omega_n t + \delta_n), \cos(\omega_n t + \delta_n))^T \), and is stable for all positive values of \( \mu \) no matter how large.

So far, we have discussed the existence of stable periodic orbit in the linearized averaged system and the nonlinear amplitude-phase system where the OSG has the perfect knowledge of nominal frequency. Next, we continue to discuss the periodic orbit analysis of the proposed EPLL under the following considerations:

(C1) Firstly, we assume that the primary input signal \( u(t) \) may change. To track such changes in \( \omega_n \), an additional integral update is incorporated in the proposed EPLL (16) resulting in an amplitude-phase-frequency model:

\[
\begin{align*}
\dot{\rho}(t) &= \mu P (u_\alpha(t) \sin \phi(t) + u_\beta(t) \cos \phi(t)) - \mu P \rho(t) \\
\dot{\omega}(t) &= \frac{\mu I}{\rho(t)} (u_\alpha(t) \cos \phi(t) - u_\beta(t) \sin \phi(t)) \\
\dot{\phi}(t) &= \omega(t) + \frac{\mu P}{\mu I} \dot{\omega}(t)
\end{align*}
\]

where \( \mu P \) and \( \mu I \) are gains of proportional and integral paths, respectively.

(C2) Secondly, we consider that the orthogonal pair \((u_\alpha, u_\beta)\) are synthesized from the primary input \( u(t) \) which may be contaminated with harmonics and noise. For that purpose, we consider a second-order generalized integrator (SOGI) [23], [24]; the dynamics of SOGI are given as:

\[
\begin{align*}
\dot{u}_\alpha(t) &= -\tilde{\omega} (K_S u_\alpha(t) - K_S u(t) - u_\beta(t)) \\
\dot{u}_\beta(t) &= -\tilde{\omega} u_\alpha(t)
\end{align*}
\]

where \( \tilde{\omega} = \omega_n \) is an estimate (provided by the proposed EPLL) of the present value of input frequency \( \omega_n \), and \( K_S \) is a suitable positive gain.

(C3) Thirdly, we consider that the input signal \( u(t) \) is contaminated with a harmonic element, therefore,

\[
u(t) = \rho_n \sin(\omega_n t + \delta_n) + \epsilon \sin(k \omega_n t)
\]

where \( k \neq 1 \) and \( \epsilon \) is small.

A suitable cross-section to obtain the Poincaré map is

\[
\Sigma = \{(\rho, \omega, \phi) : |\rho - \rho_n| < \epsilon, |\omega - \omega_n| < \epsilon, \phi = \delta_n\}
\]

So far, we have discussed the existence of stable periodic orbit in the linearized averaged system and the nonlinear amplitude-phase system where the OSG has the perfect knowledge of nominal frequency. Next, we continue to discuss the periodic orbit analysis of the proposed EPLL under the following considerations:

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\[
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\dot{\rho}(t) &= \mu P (u_\alpha(t) \sin \phi(t) + u_\beta(t) \cos \phi(t)) - \mu P \rho(t) \\
\dot{\omega}(t) &= \frac{\mu I}{\mu P} (u_\alpha(t) \cos \phi(t) - u_\beta(t) \sin \phi(t)) \\
\dot{\phi}(t) &= \omega(t) + \frac{\mu I}{\mu P} \dot{\omega}(t)
\end{align*}
\]

where \( \mu P \) and \( \mu I \) are gains of proportional and integral paths, respectively.

(C2) Secondly, we consider that the orthogonal pair \((u_\alpha, u_\beta)\) are synthesized from the primary input \( u(t) \) which may be contaminated with harmonics and noise. For that purpose, we consider a second-order generalized integrator (SOGI) [23], [24]; the dynamics of SOGI are given as:

\[
\begin{align*}
\dot{u}_\alpha(t) &= -\tilde{\omega} (K_S u_\alpha(t) - K_S u(t) - u_\beta(t)) \\
\dot{u}_\beta(t) &= -\tilde{\omega} u_\alpha(t)
\end{align*}
\]

where \( \tilde{\omega} = \omega_n \) is an estimate (provided by the proposed EPLL) of the present value of input frequency \( \omega_n \), and \( K_S \) is a suitable positive gain.

(C3) Thirdly, we consider that the input signal \( u(t) \) is contaminated with a harmonic element, therefore,

\[
u(t) = \rho_n \sin(\omega_n t + \delta_n) + \epsilon \sin(k \omega_n t)
\]

where \( k \neq 1 \) and \( \epsilon \) is small.

A suitable cross-section to obtain the Poincaré map is

\[
\Sigma = \{(\rho, \omega, \phi) : |\rho - \rho_n| < \epsilon, |\omega - \omega_n| < \epsilon, \phi = \delta_n\}
\]

Consider the initial point \((\rho_0, \omega_0, \phi_0 = \delta_n) \in \Sigma_e\). After a period \( t^* \), this point will be transformed to
\( \rho(t) = \rho_0 + \int_0^t \dot{\rho}(\tau) \, d\tau = \rho_0 + O(\epsilon), \)

\( \omega(t) = \omega_0 + \int_0^t \dot{\omega}(\tau) \, d\tau = \omega_0 + O(\epsilon), \)

\( \phi(t) = \delta_n + \int_0^t \dot{\phi}(\tau) \, d\tau = \delta_n + \int_0^t (\omega(\tau) + \frac{\mu_p}{\mu_l} \omega(\tau)) \, d\tau \)

\( = \delta_n + \omega_0 t + O(\epsilon) + \frac{\mu_p}{\mu_l} (\omega_0 + O(\epsilon) - \omega_0) \)

\( = \delta_n + \omega_0 t + O(\epsilon), \quad t \in [0, t^*], \)

where \( t^* = 2\pi/\omega_0 = 2\pi/\omega_0 + O(\epsilon). \) Considering \( K_S = 2 \)

Assuming \( \omega = \omega_0 + O(\epsilon), \) we expand amplitudes and phases of \( u_\alpha(t) \) and \( u_\beta(t) \) as follows:

\( u_\alpha(t) = \rho_0 (1 - \frac{e^{\omega_0 t}}{\omega_0} + O(\epsilon^2)) \sin(\omega_0 t + \delta_n + \frac{\omega_0}{\omega_0} + O(\epsilon^2)) + \epsilon \left( \frac{2k}{k^2 + 1} + O(\epsilon) \right) \sin(k\omega_0 t + \delta_k + \frac{2k}{k^2 + 1}) + O(\epsilon^2) \)

\( = \rho_0 \sin(\omega_0 t + \delta_n + O(\epsilon)) + \epsilon \left( \frac{2k}{k^2 + 1} + O(\epsilon) \right) \sin(k\omega_0 t + \delta_k + \frac{2k}{k^2 + 1}) + O(\epsilon^2) \) \hspace{1cm} (43)

\( u_\beta(t) = \rho_0 (1 + \frac{\omega_0}{\omega_0} + O(\epsilon^2)) \cos(\omega_0 t + \delta_n + \frac{\omega_0}{\omega_0} + O(\epsilon^2)) + \epsilon \left( \frac{2k}{k^2 + 1} + O(\epsilon) \right) \cos(k\omega_0 t + \delta_k + \frac{2k}{k^2 + 1}) + O(\epsilon^2) \)

\( = \rho_0 \cos(\omega_0 t + \delta_n + O(\epsilon)) + \epsilon \left( \frac{2k}{k^2 + 1} + O(\epsilon) \right) \cos(k\omega_0 t + \delta_k + \frac{2k}{k^2 + 1}) + O(\epsilon^2) \) \hspace{1cm} (44)

Making use of the following facts: \( [t^* = 2\pi/\omega_0 + O(\epsilon)] \)

\( \sin((\omega_0 - \omega_0)\epsilon) = \frac{2\pi}{\omega_0} + O(\epsilon) \)

\( \sin((\omega_0 + \omega_0)\epsilon) = \frac{2\pi}{\omega_0} + O(\epsilon^2) \)

\( \sin((k\omega_0 \pm \omega_0)\epsilon) = \frac{2(2\pi/(k+1))}{\omega_0} + O(\epsilon^2) \)

\( \epsilon \left( \frac{2\pi}{\omega_0} \cos(2\pi/(k+1)) + \frac{2(2\pi/(k+1))}{\omega_0} \right) + O(\epsilon^2) \)

\( \delta_n + \omega_0 t + O(\epsilon), \quad t \in [0, t^*], \)

and \( \sin(a+b) - \sin(b) \approx \sin(a) \) for \( b \ll a, \) we obtain

\( \rho_1 = \rho_0 - \frac{2\pi \mu_p}{\omega_0} (\rho_0 - \rho_n) + O(\epsilon) \) \hspace{1cm} (45)

which indicates that \( \rho_1 \) is the weighted sum of \( \rho_0 \) and \( \rho_n, \)

i.e., \( \rho_1 = (1 - \eta) \rho_0 + \eta \rho_n + O(\epsilon), \)

where \( \eta = 2\pi \mu_p/\omega_0, \) this implies that, when \( \rho_0 < \rho_n, \rho_1 \) grows up; it grows down, otherwise. Given the Poincare map \( \rho_1 = P(\rho_0), \) we obtain a discrete-time system as follows: (cf. [21, Sec. 4.10])

\( \rho_{k+1} = P(\rho_k), \quad \rho_0 \in \mathcal{U}, \quad k \in \mathbb{Z}^+ \) \hspace{1cm} (46)

where \( \mathcal{U} \) is the neighborhood of a fixed point of (46); we get

\( \rho_{k+1} = (1 - \eta) \rho_k + \eta \rho_n + O(\epsilon) \)

\( = (1 - \eta)^k \rho_0 + \eta (1 + (1 - \eta) + \cdots + (1 - \eta)^k) \rho_n + O(\epsilon) \)

As \( k \) gets large, \( \rho_\infty \to \rho_n + O(\epsilon) \) provided \( |1 - \eta| < 1, \)

satisfaction of which provides the bound:

\( 0 < \mu_p \leq \frac{\omega_n}{2\pi} = 2f_n \) \hspace{1cm} (47)

Satisfying the condition on the derivative of the Poincare map, \( 0 < \partial P(\rho_0)/\partial \rho_0 < 1, \) however, we get

\( 0 < \mu_p \leq \frac{\omega_n}{2\pi} = f_n \) \hspace{1cm} (48)

which is more restrictive than (47). The integral evaluation of \( \dot{\omega}(t) \) and \( \dot{\phi}(t) \) may be carried out like that of \( \dot{\rho}(t); \) the final expressions, however, are long and cumbersome which are difficult to interpret. Readers may refer to [25] for the trajectory expressions obtained for the phase and frequency of the traditional EPLL. Alternatively, as suggested in [26], the phase tracking problem may be linearized under the assumption of small perturbations, and the asymptotic stability of the actual nonlinear system may be inferred from that of the linear system. So, considering (39c), we obtain the following, where \( \phi(t) \leftrightarrow \Phi(s) \) and \( \omega(t) \leftrightarrow \Omega(s) \) are transform pairs:

\( s \Phi(s) = \Omega(s) + \zeta \cdot \mu_p \cdot \frac{s \Omega(s)}{\zeta \mu_l} =: \Omega(s) + \zeta \cdot \mu_p \cdot \Phi(s), \) \hspace{1cm} (49)

where \( s \Omega(s) = \zeta \cdot \mu_l \cdot \Phi(s) \) yields a linear system

\[
\begin{bmatrix}
\dot{\omega}(t) \\
\dot{\phi}(t)
\end{bmatrix} =
\begin{bmatrix}
0 & \mu_p \zeta \\
1 & \mu_l \zeta
\end{bmatrix}
\begin{bmatrix}
\omega(t) \\
\phi(t)
\end{bmatrix} =: A
\begin{bmatrix}
\omega(t) \\
\phi(t)
\end{bmatrix}
\] \hspace{1cm} (50)

If we assume that the sinusoidal excitation at the input of EPLL is not subject to large errors, then the parameters estimated by the EPLL will remain close to the true values; this implies that the stability of the actual nonlinear system...
(39b-c) can be inferred from that of the aforementioned linear system in (50). Solving (50), we obtain
\[
\begin{bmatrix} \omega_1 \\ \phi_1 \end{bmatrix} = \exp (\Lambda t^*) \begin{bmatrix} \omega_0 \\ \phi_0 \end{bmatrix} + \mathcal{O}(\epsilon)
\] (51)
The eigenvalues, \( \lambda_a \) and \( \lambda_b \), of \( \exp(\Lambda t^*) = M \exp(\Lambda^* t) M^{-1} \), where \( \Lambda = \text{diag}([\lambda_a, \lambda_b]) \), are
\[
\begin{bmatrix} \lambda_a \\ \lambda_b \end{bmatrix} = \begin{bmatrix} 2\mu \rho \zeta - \frac{1}{2}\sqrt{2\rho^2 \zeta^2 + 4\zeta \lambda} \\ 2\mu \rho \zeta + \frac{1}{2}\sqrt{2\rho^2 \zeta^2 + 4\zeta \lambda} \end{bmatrix}
\] (52)
which ensures guaranteed asymptotic stability of the estimates provided \( \zeta < 0, \mu_P > 0, \) and \( \mu_I > 0 \). For, \( \zeta = -1, \) a critically damped response (\( \lambda_a = \lambda_b \)) may be ensured by selecting \( \mu_P = 2\sqrt{\mu_I} \). So, given \( \mu_P \), the \( \mu_I \) is obtained as:
\[
\mu_I \leq \frac{1}{2} \mu_P^2
\]
(53)
Based on the finding (53), this study reveals that all three updates (amplitude-frequency-phase) of EPLL depend on a single controlling parameter \( \mu \) as follows:
\[
\begin{align*}
\dot{\rho}(t) &= \mu (u_\alpha(t) \sin \phi(t) + u_\beta(t) \cos \phi(t)) - \mu \rho(t) \\
\dot{\omega}(t) &= \frac{\mu^2}{4\rho(t)} (u_\alpha(t) \cos \phi(t) - u_\beta(t) \sin \phi(t)) \\
\dot{\phi}(t) &= \omega(t) + \frac{4}{\mu} \dot{\omega}(t)
\end{align*}
\] (54a-b-c)
which simplifies the operation of the proposed EPLL.

V. SIMULATION RESULTS

**Experiment 1:** This experiment is about comparing the proposed EPLL (16) with the traditional EPLL [6] which does not make the use of a (synthesized) orthogonal copy of input signal \( u(t) \), and has one single output \( y(t) = \rho(t) \sin(2\pi f_n t + \delta(t)) \); the traditional EPLL is expressed as:
\[
\begin{align*}
\dot{\rho}(t) &= 2\mu (u(t) - y(t)) \sin (2\pi f_n t + \delta(t)), \\
\dot{\delta}(t) &= 2\mu (u(t) - y(t)) \cos (2\pi f_n t + \delta(t)).
\end{align*}
\] (55a-b)
Moreover, we also superimpose the traces of averaged EPLL (31) and approximate solution of linearized averaged EPLL (33). We assume nominal values \( \delta_n = 0.5, f_n = 2 \) Hz, and \( \rho_n = 2.0; \) this gives \( u(t) = 2 \sin(4\pi t + 0.5) \). The initial values of \( \rho(t) \) and \( \delta(t) \) at \( t = 0 \) are assumed to be 0.25 and 0, respectively, which are not equal to the nominal values; the gain parameter is selected to be \( \mu = 5 \). The convergence behaviors are summarized in Fig. 7. We note that the proposed EPLL (16), the averaged version (31), and the averaged-linearized solution (33) successfully converge to desired nominal values without double-frequency effect, and in a nearly identical manner. In contrast, the traditional EPLL (55) converges to the nominal values while suffering from the double-frequency effect; the frequency of the harmonic component is 2\( f_n \) Hz.

**Experiment 2:** We vary the amplitude of the input sinusoidal signal in a step-manner multiple times and observe the behavior of \( \rho(t) \) and \( f(t) = \omega(t)/(2\pi) \) of the proposed EPLL (25). The nominal amplitude is \( \rho_n = 1 \) pu, however, it goes up to 1.2 pu for 0.4 sec at \( t = 0.4 \) and goes down to 0.8 pu for 0.4 sec at \( t = 1.2 \). The tracking behavior of the proposed EPLL is obtained where orthogonal signals are synthesized by FOI and SOGI (as two separate cases). A critically damped response is obtained in both cases, see Fig. 8, where \( \rho(t) \) tracked step-variation in \( \rho_n \) very well and the convergence is attained within 0.05 sec. The estimated frequency \( f(t) \) may have little disturbances (less than 0.25-Hz) around the nominal 50-Hz. Values of FOI, SOGI, and EPLL parameters are mentioned in the figure, where \( \mu_P \) and \( \mu_I \) are selected according to (47) and (53), respectively.

**Experiment 3:** We vary the frequency of the input sinusoidal signal in a step-manner multiple times and observe the behavior of \( \rho(t) \) and \( f(t) \). The nominal frequency is \( f_n = 50 \) Hz \( (f_n = \omega_n/(2\pi)) \), however, it goes up to 60 Hz for 0.4 sec at \( t = 0.4 \) and goes down to 40 Hz for 0.4 sec at \( t = 1.2 \). Again, a critically damped response is obtained in both cases, see Fig. 10, where \( f(t) \) tracked step-variation in \( f_n \) very well and the convergence is attained within 0.1 sec. The \( \rho(t) \) may be seen to exhibit some undershoots which are less than 0.15-pu on average.

**Experiment 4:** We vary the phase angle of the input signal. We estimate the instantaneous phase using the EPLL outputs as \( \delta(t) = \phi(t) - \omega_n t \). The amplitude and frequency are kept at their nominal values, 1 pu and 50 Hz, respectively. The phase of the input sinusoidal signal goes up to \( \pi/8 \) radian from zero for 0.4 sec at \( t = 0.4 \); and then goes down to \(-\pi/8\)
Fig. 8. Performance of the proposed EPLL with FOI and SOGI with abrupt changes in amplitude under sinusoidal excitation.

Fig. 9. Performance of the proposed EPLL with FOI and SOGI with abrupt changes in frequency under sinusoidal excitation.

radian from zero for 0.4 sec at $t = 1.2$. We notice that the phase is tracked by the proposed EPLL satisfactorily with a little overshoot. We note sharp disturbances in modulus $\rho(t)$ at rising and falling edges of the phase. Similarly, the estimated frequency $f(t)$ rises and drops off at the rising and falling edges of the phase, respectively. The transient in frequency estimate takes 0.1 sec to settle down, which is the same time as taken by the EPLL phase to align with the input signal’s phase fully.

Experiment 5: A noisy 50-Hz cropped sinusoidal excitation is considered to demonstrate the robustness of the proposed EPLL. The signal-to-noise ratio is maintained at 30 dB. The orthogonal signals generated by FOI and SOGI are illustrated in Fig. 11. Clearly, SOGI is more capable of removing noise than FOI. The phase portraits of EPLL outputs $(y(t), y_\beta(t))$

are obtained for two cases of amplitude initial conditions — less than the desired value $\rho(0) = 0$, and greater than the desired value $\rho(0) = 2$. Trajectories of modulus converged at 1.2 unit in the steady-state as expected.

VI. CONCLUSIONS

We obtained an enhanced phase-locked loop (EPLL) system by solving a gradient flow problem, that can estimate the amplitude, phase, and frequency of the sinusoidal excitation. Mathematical derivation of the differential equations, solution, Lyapunov stability analysis, and performance evaluation were included. It has been noted that the suggested EPLL can lock into the fundamental signal’s amplitude, phase, and frequency. It is thus well suited for power systems where the filtering and parameter estimation of polluted signals are required.

APPENDIX

A. Poincare Map:

Theorem A [22, Sec. 3.4]: Let $E$ be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$. Suppose that $\psi_t(x_0)$ is a periodic solution of $\dot{x} = f(x)$ of period $T$ and the cycle is

$$\Gamma = \{x \in \mathbb{R}^n | x = \psi_t(x_0), 0 \leq t \leq T\}$$

Let $\Sigma$ be the hyperplane orthogonal to $\Gamma$ at $x_0$:

$$\Sigma = \{x \in \mathbb{R}^n | (x - x_0) \cdot f(x_0) = 0\}$$

Then there is a $\delta > 0$ and a unique function $\tau(x)$, defined and continuously differentiable for $x \in N_\delta(x_0)$, such that
\( \tau(x_0) = T \) and \( \psi_{\tau(x)}(x) \in \Sigma \) for all \( x \in N_\delta(x_0) \).

**Definition** [22, Sec. 3.4]: Let \( \Gamma, \Sigma, \delta \) and \( \tau(x) \) be defined as in Theorem A. Then for \( x \in N_\delta(x_0) \cap \Sigma \), the function \( P(x) = \psi_{\tau(x)}(x) \) is called the Poincaré map for \( \Gamma \) at \( x_0 \).

**B. Derivative of Poincare Map:**

**Theorem B** [22, Sec. 3.4]: Let \( E \) be an open subset of \( \mathbb{R}^2 \) and suppose that \( f \in C^1(E) \). Let \( \gamma(t) \) be a periodic solution of \( \dot{x} = f(x) \) of period \( T \). Then the derivative of the Poincaré map \( P(s) \) along a straight line \( \Sigma \) normal to \( \Gamma \) is
\[
P'(0) = \exp \left( \int_0^T \nabla \cdot f(\gamma(t)) \, dt \right)
\]

**Corollary** [22, Sec. 3.4]: Under the hypotheses of Theorem B, the periodic solution \( \gamma(t) \) is a stable limit cycle if
\[
\int_0^T \nabla \cdot f(\gamma(t)) \, dt < 0
\]

and it is an unstable limit cycle if
\[
\int_0^T \nabla \cdot f(\gamma(t)) \, dt > 0
\]

It may be a stable, unstable, or semi-stable limit cycle or belong to a continuous band of cycles if this quantity is zero.

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**REFERENCES**


