The THz Catastrophe. Part 2: the EM Degrees of Freedom due to Thermal Agitation

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Abstract—The energy associated to the electromagnetic (EM) field due to thermal agitation in homogeneous ohmic media is evaluated. To this goal I propose an expansion of electromagnetic fields in terms of orthogonal modal functions that are eigen-vector of Maxwell’s. These fields are recognized as the degrees of freedom of the field and the energy associated to each of them is represented using the Johnson thermal voltage sources associated to the dissipative part of the impedance of the modes. At low frequency the conduction currents are dominant and in phase with the agitation of the electrons: the thermal energy is thus efficiently transferred to the EM field. At higher frequencies the radiated displacement currents overshadow the conduction currents. However, since the displacement currents are out of phase with the agitation of the electrons, the thermal energy is not efficiently transferred to the EM field. The model, which is entirely analytical, first provides the thermally generated EM energy per unit of volume. Then, relying on a number of simplifications commonly adopted in radiometry, provides the energy radiated outside the warm bodies. This latter radiated energy is finally compared with recent measurements for moderate conductivity silicon presented in a companion paper [11]. The model’s accuracy with respect to measurements is within 1 dB, while the application of Planck’s law, even mediated by the emissivity, is clearly off target.

Index Terms—Thermal noise, radiometry, degrees of freedom

I. INTRODUCTION

The only widely used model for describing thermal radiation accounting for the properties of the considered medium was proposed by Rytov [1][2] in the 60’s. It was a hybrid classic-quantum model that relied on solving Maxwell’s Equations (ME) classically, but with a forcing term represented by impressed currents. The spectrum of these currents was described by introducing, ad hoc, the number of photons, so that the thermal radiation from a large body would equal Planck’s law for a black body, if the interface between the material and free space does not give rise to reflections [3]. In turn reflections induced by the material discontinuities would be accounted for by the emissivity being different from unity. Since then the model has been used by scientists in countless applications: [4]-[10] are just some of the important references that I found tracking some of the key literature to these decade. However, a verification of the validity of Planck’s law for non conductive ohmic medium is absent. In the companion paper [11] some important discrepancies have been recently observed in the measured spectra for silicon samples: the results in [11] could not be explained resorting only to Planck’s or Rytov theories. Those discrepancies were not unexpected. In the last few years I have worked toward a new model for a classic electromagnetic description of thermal radiation. Here, I concisely present this theory, aiming specifically at explaining the problems raised by the results in [11]. Fig.1 shows the measured spectral energy density received at the detectors used in [11], when a specific wafer YW3 – 4, was heated to $T_e = 425 \text{K}$. Fig.1 also shows the results of the theory presented in this paper, the DoF (degrees of Freedom) line. The agreement is outstanding especially when compared with the prediction deriving from the use of Planck’s law. This should encourage the readers to pursue the reading of the following theoretical paper.

![Fig 1 Measured Spectral Energy Densities vs predictions from Planck’s law corrected with estimated emissivity and the Degrees of Freedom (DoF) presented in this paper. The estimated doping was $n = 5.18 \times 10^{11}$](image)

This paper applies the Equipartition Theorem for the evaluation of the electromagnetic energy rendered available from the thermal agitation of the electrons in the medium. To this goal in section II the field in every observation point is expanded in terms of eigen modes of the electric fields and currents that represent the solution of ME in absence of sources in an infinite ohmic medium. The properties of the EM solutions are used to construct an equivalent impedance for each modal function. The availability of such impedance allows us to introduce a Johnson [12]-[13] like thermal source in an equivalent circuit that represents the current and the field for the modal functions. This is, in fact the application of the Equi-partition Theorem to each and all radiative components of the EM fields. Each eigen vector current is excited in proportion to the thermal agitation of the electrons.

In section II, it is also clarified that the degrees of freedom (DoF) of the electromagnetic field are limited in number, as only those functions which are centred around spatial points located at half the effective wavelength one from the other give rise to mutual coupling equal to zero and thus can be considered electromagnetically independent. As a result the actual number of independent degrees of freedom is estimated...
In section III the EM energy provided by the sources is explicitly provided. At low frequency the conduction currents are dominant and in phase with the agitation of the electrons: the thermal energy is thus efficiently transferred to the EM field. At higher frequencies the displacement currents overshadow the conduction currents. However, the displacement currents are out of phase with the agitation of the electrons and accordingly the thermal energy is NOT efficiently transferred to the EM field. This means the energy drops above a certain frequency for any material. This is the reason of the suggestive title of the paper.

In section IV the energy radiated outside a finite body is discussed: to this goal a simplified approach that resorts to reciprocity as well as the emissivity is used. Both reciprocity and emissivity are also adopted in classic radiometry, in fact they are proper electromagnetic procedures, once the appropriate radiating currents are used. To identify the new appropriate equivalent currents, in section V the autocorrelation of the DoF currents introduced in section II and III is evaluated. The resulting procedure for the evaluation of the radiated energy is eventually compared with the measurements of [11] in section VI. It is apparent that the method is very accurate in estimating the radiated energies. Conclusions are drawn in Section VII. Some lengthy EM proofs are also reported in the appendix.

II. A CLASSICAL DESCRIPTION OF THERMAL FIELDS AND CURRENTS

Given the enormous number of electrons and interactions involved per unit of volume, it is still now days impossible to track the evolution in time of the EM field due to thermal sources. The Equipartition Theorem (ET) is a non deterministic, and less ambitious, strategy that can be followed to obtain information about the energetic content of the EM fields in ohmic media at known temperatures. The version of the ET often used in electrical engineering was introduced by Johnson [12]-[13] already in 1928. It states that the thermal energy rendered available to an electromagnetic mode can be quantified with an equivalent voltage or current generator and its corresponding impedance (Thevenin like or Norton like Johnson circuits). When using the Johnson circuit it is explicitly assumed that one is only interested in the energy generated due to thermal fluctuations and that the EM mode considered is an eigen vector of ME, for which the ratio between voltage and current can be characterized by a single complex impedance. For this reason the phases of the noise sources are undefined.

Here, it is proposed to express the EM field as the superposition of orthogonal field modal contributions: the Degrees of Freedom (DoF) of the field. The total electric field, \( \vec{e}(\vec{r}) \) is then expressed as the superposition of the fields modal contributions emerging from different locations \( \vec{r}_n \)

\[
\vec{e}(\vec{r}) = \sum_{n=1}^{N_{\text{pos}}} \vec{e}_n(\vec{r}, \vec{r}_n) \quad (1)
\]

In detail each of the fields \( \vec{e}_n(\vec{r}, \vec{r}_n) \) will be the superposition of different modes that can share the same location \( \vec{r}_n \), while still being orthogonal. Figure 2 presents a qualitative picture of the modal current superposition in eq. (1), assuming for simplicity only one mode per location \( \vec{r}_n \). Each of the modes associated to \( \vec{r}_n \), is taken to be a solution of Maxwell’s equations in absence of sources and the modes are orthogonal to each other. The modes, already well known, are those associated to infinite homogeneous ohmic media. Since all the different modes are orthogonal, the mutual phase of each of these modal fields is also irrelevant to calculate the EM energy. Moreover, assuming that the entire body is at a uniform temperature, the amplitudes of all modes, located at different locations is the same independently from the location, \( \vec{r}_n \). Equation (1) can also be expressed per unit of volume. In this case, the field can be expressed as

\[
\vec{e}(\vec{r}) = \vec{e}_n(\vec{r}, \vec{r}_n) \quad (2)
\]

where \( N_{\text{pos}} \) indicates the number of points per unit of volume, around which the modal functions are located.

As anticipated, to calculate the amplitudes of the modes I propose the application of the Johnson’s circuit, Fig. 3, to each modal function. Specifically, each modal function associated to a specific location, \( \vec{r}_n \), will be associated an impedance, \( Z_p \), where \( p \) will indicate the polarization characterizing the different co-located modes. For a Thevenin like Johnson source

\[
|\nu_{\theta \nu}^{\text{np}}| = \sqrt{4 k_B T R_p} \quad (3)
\]

where \( R_p \) is the ohmic resistive part of the impedance \( Z_p \). The amplitude of the voltage source is proportional to \( k_B T \), and I willingly do not include the Plankian characteristic frequency dependence \( \frac{f^3}{e^{\frac{f}{k_B T}}-1} \) that Nyquist [13] suggested should complement the amplitude of Thermal sources for high frequencies. Note that, while [13] is more readable than [12] its conclusions about the Planckian frequency dependence have not been experimentally validated yet, to my knowledge.

II.A Modal Solution of Maxwell’s Equations

The infinite ohmic and homogeneous medium that I consider is assumed to be characterized by a Drude like, [14]-[16],
frequency dependent conductivity \( \sigma(f) = \sigma_e(f) + j\sigma_i(f) \),
electric permittivity, \( \epsilon_r \), and free space magnetic permeability, \( \mu_0 \). A standard modal expansion for such medium can be obtained associating the lowest order, \((m = 0, n = 1)\)
spherical waves [17] to each of the specific spatial points, \( \vec{r}_n \).

![Fig. 4. Electric field lines of the lowest order TM spherical mode. Inside the source region, of characteristic dimension \( \Delta \), there is the superposition of an inward and an outward wave. In the source less region there are only outward propagating waves.](image)

Fig. 4, as an example, represents a \( \hat{z} \) oriented electric current, centered in the origin, \( \vec{r}_n = \vec{0} \), that generates an outward spherical wave, of the lowest order. The picture represents a realistic field configuration, with the central region of space of diameter \( \Delta \), that includes the superposition of an inward and an outward wave, with equal amplitudes. Outside the source region, only the TM (transverse magnetic with respect to \( r \)) outward propagating wave of the lowest order is seen emerging from the origin. The dual TE wave shares the same volume, is associated to a magnetic elementary source and its fields are orthogonal to the fields generated by the TM wave. The magnetic currents play a role but it will only be addressed later on in the paper. The explicit expressions for the electric fields of the TM lowest order modes spherical modes are well known, [17]. For instance one can express the outward electric fields radiated outside the source region as, \( e^{\text{in}}_{\text{out}}(\vec{r}) = e^{\text{in}}_{\text{out}}(\vec{r})\hat{r} + e^{\text{in}}_{\text{out}}(\vec{r})\hat{\theta} \). Assuming that the dipole is characterized by the length \( \Delta \) and current \( i_z \),

\[
e^{\text{in}}_{\text{out}}(\vec{r}, i_z \Delta) = I_z e^{-jkr} \frac{\cos \theta}{kr} \left[ 1 + \frac{1}{jk} \right] e^{\text{in}}_{\text{out}}(\vec{r})\hat{r} + e^{\text{in}}_{\text{out}}(\vec{r})\hat{\theta}
\]

One should note that in (3), the propagation constant will be later expressed in terms of its real and imaginary parts \( k(\omega) = \beta - j\alpha \). Instead of the full field in (3) one can focus only on the components that depend from the radial distance as \( \frac{e^{\text{in}}_{\text{out}}(\vec{r})}{r} \). Selecting only this spherical spreading is equivalent to isolating the visible field component, [18], which still is a solution of ME. It can be expressed as \( e^{\text{in}}_{\text{out}}(\vec{r}, i_z \Delta) = I_z e^{-jkr}\frac{\cos \theta}{kr} \left[ 1 + \frac{1}{jk} \right] e^{\text{in}}_{\text{out}}(\vec{r})\hat{r} \), where the normalized visible field is defined as

\[
\tilde{e}^{\text{in}}_{\text{out}}(\vec{r}, r_0) = \Delta e^{-jkr}\frac{\cos \theta}{kr} \left[ 1 + \frac{1}{jk} \right] e^{\text{in}}_{\text{out}}(\vec{r})\hat{r}
\]

This last expression highlights the location, \( r_0 \) of the originating dipole, which can be different from the origin. Also in (4) \( \hat{\theta} \) represents the elevation unit vector with respect to this new origin of radiation. The most important quantity to characterize the outgoing field in (4) is the propagation constant, \( k = k_d \frac{1 - \frac{j\alpha}{\omega\epsilon_r\mu_0}}{1 - \frac{j\alpha}{\omega\epsilon_r\mu_0}} \) (and possibly the related characteristic impedance \( \zeta = \frac{\mu_0}{\sqrt{1 - \frac{j\alpha}{\omega\epsilon_r\mu_0}}} \)). The effective dielectric constant accounting for the impulsive response of the materials can be characterized via the Drude’s model [14][15] for waves propagating in ohmic media characterized by the presence of a number of free electrons. As clarified in [16] good and bad conductors as well as many semi-conductors including silicon can be studied, until a few THz, resorting to this model only. Fig. 5 shows the propagation constant of phosphorus doped silicon, normalized to that of undoped silicon, \( k_d = k_0\sqrt{\epsilon_r \epsilon_r} = 11.7 \), as a function of the frequency in the sub-THz range. The curves are parametrized with respect to the doping \( n \) which indicates the free electrons per cubic meter. Solid lines represent the real part and dashed lines represent the imaginary part. The scattering time of the silicon is assumed to be \( \tau_s = 2 \times 10^{-13} \) for all dopings.

![Fig. 5. Propagation constant of phosphorus doped silicon, normalized to that of undoped silicon, \( k_d \), as a function of the frequency in the sub-THz range, curves are parametric with doping density](image)

\[\text{II.B Eigen values of the modal solutions}\]

Solutions of Maxwell’s equation as those in eq. (3)-(4), that is outside the domain of the generating sources, can alternatively be represented as radiated by a set of equivalent volumetric electric currents distributed only over the volume external to the actual sources, in absence of generating source. To this goal, it is worth observing a property of ME in homogeneous space, demonstrated in Appendix A, following the proof that originally provided in the excellent articles [19] [20]. Specifically the property is that if \( \tilde{e}(\vec{r}) \) is an outward propagating solution of ME in a homogeneous space, \( \tilde{e}(\vec{r}) \) can be expressed as radiated by auxiliary currents \( \tilde{j}_s \)

\[
\tilde{e}(\vec{r}) = \int_{\infty}^{-\infty} \int_{\infty}^{\infty} \tilde{g}_{-}^{\text{el}}(\vec{r}, \vec{r}') \cdot \tilde{j}_s(\vec{r}') d\vec{r}'
\]

where \( \tilde{g}_{-}^{\text{el}}(\vec{r}, \vec{r}') \) the Green’s function dyadic that provides the electric field from the electric currents in the homogeneous space and the auxiliary currents are defined as
\[ \tilde{j}_x (\vec{r}') \equiv -(\sigma + j\omega\varepsilon_0\varepsilon_r) \tilde{e}(\vec{r}') \] (5b)

Since the field in (4) is solution of ME one can apply eqs. (5a) and (5b) to it, and obtain that

\[ \iiint_{-\infty}^{\infty} \tilde{g}^{ij} (\vec{r}, \vec{r}') \cdot \tilde{e}_{no}^{ij} (\vec{r}', \vec{r}_0) \, d\vec{r}' = \frac{\tilde{e}_{no}^{ij} (\vec{r}, \vec{r}_0)}{v} \] (6)

Thus one can recognize \( \tilde{e}_{no}^{ij} (\vec{r}, \vec{r}_0) \) as an electric field eigen vector, and \( j_{no}^{ij} (\vec{r}, \vec{r}_0) \), defined as

\[ \tilde{j}_{no}^{ij} (\vec{r}, \vec{r}_0) = v \tilde{e}_{no}^{ij} (\vec{r}, \vec{r}_0) \] (7)

as an electric current eigen vector that are related by a corresponding eigenvalue of ME

\[ v = -(\sigma + j\omega\varepsilon_0\varepsilon_r) \] (8)

It will be apparent that the length parameter, \( \Delta \), that was introduced to define \( \tilde{e}_{no}^{ij} (\vec{r}, \vec{r}_0) \) in eq. (4), is only needed to have dimensionally congruent quantities but does not play a role in the following, as long as it is small with respect to the distance between free electrons.

II.C Extension to 3 electric dipole born DoF per point

According to the theory of spherical modes, [17], three electric currents sources can share the same location, \( \vec{r}_n \), and be independent degrees of Freedom. The 3 independent solutions are associated to electric currents polarized along \( p = x, y \) or \( z \), Accordingly, I be will using the compact notation \( \tilde{e}_{no}^{jp} (\vec{r}, \vec{r}_n) \) to indicate the desired normalized electric field. Extending eq. (7) and (8) to the three orthogonal modes associated to a point \( \vec{r}_n \) it results

\[ \iiint_{-\infty}^{\infty} \tilde{g}^{ij} (\vec{r}, \vec{r}') \cdot \tilde{e}_{no}^{jp} (\vec{r}', \vec{r}_n) \, d\vec{r}' = \frac{\tilde{e}_{no}^{jp} (\vec{r}, \vec{r}_n)}{v} \] (9)

and also

\[ \tilde{j}_{no}^{jp} (\vec{r}, \vec{r}_n) = v \tilde{e}_{no}^{jp} (\vec{r}, \vec{r}_n) \] (10)

II.D Number of Degrees of Freedom

Every point, \( \vec{r}_n \), in the infinite medium can be associated to 3 outgoing sources. However, these eigen modal fields are not necessarily orthogonal to the fields emerging from different locations, \( \vec{r}_m \neq \vec{r}_n \).

The modal fields emerging from \( \vec{r}_m \) could be electromagnetically coupled to the modal fields emerging from \( \vec{r}_n \). Let us for instance consider two eigen-vectors, \( j, q \) associated to \( \vec{r}_n \) and \( j, p \), associated to \( \vec{r}_m \), that is electric currents \( p \) and \( q \) polarized (with \( p \) and \( q \) being either \( x, y \), or \( z \)). Also let us assume that the spatial points \( \vec{r}_n \) and \( \vec{r}_m \) are related by \( \vec{r}_m = \vec{r}_n + \vec{d} \), as in Fig. 6. The coupling, \( Z_{mn}^{ij,pq} \), between the electric eigen currents can be estimated using a conjugate projection. Since the outgoing sources are defined over an infinite domain, the mutual coupling includes infinite extreme of integrations. Accordingly, the mutual coupling between modal distributions can be expressed as

\[ Z_{mn}^{ij,pq} \equiv -\iiint_{-\infty}^{\infty} \tilde{j}_{no}^{ij} (\vec{r}, \vec{r}_n) \cdot \tilde{e}_{no}^{pq} (\vec{r}, \vec{r}_m) \, d\vec{r} \] (11)

Recognizing (9)-(10), the inner integral in (11) can be evaluated analytically and (11) can be re-written as

\[ Z_{mn}^{ij,pq} = -\iiint_{-\infty}^{\infty} \tilde{j}_{no}^{ij} (\vec{r}, \vec{r}_n) \cdot \tilde{e}_{no}^{pq} (\vec{r}, \vec{r}_m) \, d\vec{r} \] (12)

In turn substituting in (12), eq. (10) provides

\[ Z_{mn}^{ij,pq} = (\sigma + j\omega\varepsilon_0\varepsilon_r) \iiint_{-\infty}^{\infty} \tilde{e}_{no}^{ij} (\vec{r}, \vec{r}_n) \cdot \tilde{e}_{no}^{pq} (\vec{r}, \vec{r}_m) \, d\vec{r} \] (13)

The mutual coupling integrals could be evaluated numerically. However, if one focuses only on distances where the mutual coupling is dominated by visible components of the fields, the integrals can be simplified analytically. This is shown in Appendix B, where also the meaning of the visible field is clarified. Eventually, for \( d > \frac{\lambda}{2\pi} \) one finds

\[ Z_{mn}^{ij,pq} \propto (\sigma^* - j\omega\varepsilon_0\varepsilon_r) e^{-\alpha d} \text{sinc}(\beta d) \text{ Pol}(\hat{p}, \hat{q}) \] (14)

where \( \alpha = \text{Im}[k] \) and \( \beta = \text{Re}[k] \), and \( d = |\vec{d}| \). In (14) the function \( \text{Pol}(\hat{p}, \hat{q}) \) is a real function that depends on the mode that is being considered but not on the distance \( d \). The striking property emerging from (14) is that the mutual coupling integrals are null for distances, \( d \), between source centres that are multiple of half a wavelength in the considered medium. For instance \( Z_{pq}^{pq} = 0 \) for \( d = \frac{\lambda}{2} \), with \( \lambda = \frac{2\pi}{\beta} \). Accordingly, when \( d = \frac{\lambda}{2} \) the visible fields radiated by three orthogonal electric currents, oriented in any direction, \( p = x, y, z \), can be assumed to be the independent eigen vectors associated to the solutions of Maxwell’s equations since their coupling is zero. Another reason why the mutual coupling between eigen vectors can be extremely small is that the ohmic losses are so strong that attenuation factor \( e^{-\alpha d} \) in (14) renders the coupling negligible even for \( d < \frac{\lambda}{2} \). In the following of this paper the distance \( d \) will indicate the smallest distance at which the coupling is negligible. It will be a function of the frequency, as it depends from \( \alpha \) and \( \beta \), which are both functions of the frequency and it will be defined as the solution of the following dispersion equation

\[ e^{-\alpha f} d \text{sinc}(\beta f) d = 0.01 \] (15)
The threshold value, 0.01, has been taken arbitrarily but it does not impact the validity of the conclusions in this paper in any significant way.

The number of independent positions per unit of volume $D_{pos}$ is shown in Fig.8, for the same parametric silicon case addressed in Fig. 5 and Fig.7.

The multiplication $\times 6$ (corresponding to $3 \times 2$) between the number of positions and degrees of freedom is due to the fact that each location, $\vec{r}_n$, there are three electric currents oriented in the three cartesian directions and also three magnetic currents that also provide fields orthogonal one to the ones of the other and to all the fields due to the electric currents. This number of degrees of freedom is congruent with the existing antenna literature that uses the concept of degrees of freedom to evaluate the properties of the far field radiation from distributions [21][22][23].

III. ELECTROMAGNETIC ENERGY PER UNIT OF VOLUME

The Johnson equivalent circuit is now used to estimate the amplitude of the various electric current induced modes identified in section II. Modes associated to magnetic currents will be addressed toward the end of the section. Recalling the electric field representation in (2), we can express $\vec{e}_{\nu}(\vec{r},\vec{r}_n)$ associated to each location $\vec{r}_n$ as superposition of three modal distributions associated to orthogonal directions $\vec{p}$

$$\vec{e}(\vec{r}) = \sum_{n=1}^{N_{pos}} \sum_{p=1}^{3} i \phi_{n,p}^{\mu}(\vec{r},\vec{r}_n)$$

(17)

Specifically each electric field contributions $\vec{e}_{\nu}^{\mu}(\vec{r},\vec{r}_n)$ is expressed as due to a generating electric current $j_{gen}^\nu(\vec{r},\vec{r}_n)$ by means of the Green’s functions $\vec{e}_\nu^{\mu}(\vec{r})$ representing the electric fields due to electric currents in the presence of the infinite ohmic medium

$$i p \vec{e}^{\mu}_{n,0}(\vec{r},\vec{r}_n) = \vec{e}_\nu^{\mu}(\vec{r}) * i p j_{gen}^\nu(\vec{r},\vec{r}_n)$$

(18)

Each of the generating electric currents, oriented along a different direction $\vec{p}$ and centred in $\vec{r}_n$, is imagined to restricted to a rectangular domain with characteristic dimensions, $\Delta$, also centred in $\vec{r}_n$: 

$$j_{gen,p}^\nu(\vec{r},\vec{r}_n) = \vec{p} \frac{1}{\Delta^2} \text{rect}(\vec{r} - \vec{r}_n,\Delta^2) e^{ij\vec{p}\vec{r}(\vec{r})}$$

(19)

where $\text{rect}(\vec{r} - \vec{r}_n,\Delta^2)$ is a domain function equal to zero everywhere except for $\vec{r}$ in Cube, with Cube identifying a cube of side $\Delta$ and centered in $\vec{r}_n$. With (17)-(19) the explicit relation between the electric field modes and the current in the equivalent circuit in Fig.3 is established.

A. Energy per unit of volume

In order to evaluate the energy per unit of volume one can first express the total electric current as

$$j(\vec{r}) = \sum_{n=1}^{N_{pos}} j_n(\vec{r},\vec{r}_n) = \sqrt{\sum_{n=1}^{N_{pos}} \sum_{p=1}^{3} (i p \vec{e}^{\mu}_{n,0}(\vec{r},\vec{r}_n))}$$

(20)

These currents include both conduction and displacement currents since they are defined by (7) and (8). Once the currents are known, the EM energy delivered to the body per unit of volume associated to the currents in (20) can be expressed as the integration over the volume of the scalar product between the currents and the total electric field

$$E^{tot} = -Re \int \int \int_{-\infty}^{\infty} \vec{e}(\vec{r}) \cdot j(\vec{r}) d\vec{r}$$

(21)

Substituting expression (17) and (20) the integration in (21) is formally complex

$$E^{tot} = -Re \left\{ \int \int \sum_{m=1}^{N_{pos}} \sum_{p=1}^{3} i p \vec{e}^{\mu}_{n,0}(\vec{r},\vec{r}_m) \cdot \sqrt{\sum_{n=1}^{N_{pos}} \sum_{q=1}^{3} i q \vec{e}^{\mu}_{n,0}(\vec{r},\vec{r}_n)} \right\}$$

(22)
However, since the locations of the elementary dipoles have been chosen such that the mutual coupling is equal to zero, for currents that are not co-located, only contributions to the double summation that derive from the self-coupling terms, \( m, p = n, q \), need to be considered.

\[
E^{\text{Tot}} = - \sum_{n=1}^{N_{\text{pos}}} \sum_{p=1}^{3} \Re \{ |l^p| \} \int \int \int_{-\infty}^{\infty} \tilde{e}^{jp}_n (\vec{\tau}, \vec{\tau}_n) \cdot \tilde{e}^{jp}_n (\vec{\tau}, \vec{\tau}_n) d\vec{\tau} \quad (23)
\]

All the reaction integrals in (23) are all equal, independently from the location \( \vec{\tau}_n \). One can then recall that for the normalized fields, the modal self impedances of the p-th mode was defined as

\[
Z^p = -\nu^* \int \int \int_{-\infty}^{\infty} \tilde{e}^{jp}_n (\vec{\tau}, \vec{\tau}_n) \cdot \tilde{e}^{jp}_n (\vec{\tau}, \vec{\tau}_n) d\vec{\tau} \quad \forall n, p
\]

(24)

Accordingly the electromagnetic energy, per unit of volume, associated to the \( p - th \) mode can be indicated as

\[
E^{\text{Tot}} = \sum_{n=1}^{N_{\text{pos}}} \sum_{p=1}^{3} |l^p| \Re \{ Z^p \}
\]

(25)

The self impedance can be evaluated analytically as shown at the end of Appendix B.

**B. Solution of the Johnson Circuit**

The current \( i^p \) that defines the amplitude of the \( p - th \) eigen vector mode can be evaluated using the Johnson-generator in Fig.3. The generator is connected to the load. The load is the eigen mode impedance itself. In this case the thermal energy that the Johnson noise circuit provides is directly dissipated in the isolated component as electromagnetic energy. It is immediate to verify that a matched load, i.e. \( Z_{\text{load}} = Z^p \) connected to the terminals of the source would lead to a maximum transfer of spectral energy \( E(\nu) = k_B T \) to the load. This does not happen in an infinite medium because \( Z^p \) is actually a complex function and thus \( Z_{\text{load}} = Z^p \neq Z^p \). The explicit expression of the impedance in equation (24) can be simplified recalling (8) so that

\[
Z^p = (\sigma^* - j \omega \varepsilon_0 \varepsilon_r) \int \int \int_{-\infty}^{\infty} \tilde{e}^{jp}_n (\vec{\tau}, \vec{\tau}_n) \cdot \tilde{e}^{jp}_n (\vec{\tau}, \vec{\tau}_n) d\vec{\tau} \quad \forall n, p
\]

(26)

It is then simple to observe that independently from the distribution \( \tilde{e}^{jp}_n (\vec{\tau}, \vec{\tau}_n) \) the integration in (26) is a real function of the frequency and the dielectric parameters, so that

\[
Z^p = (\sigma^* - j \omega \varepsilon_0 \varepsilon_r) P^p
\]

(27)

where the factor \( P^p \) is evaluated analytically at the end of Appendix B and will be provided in the next section (eq.(41)).

The only imaginary contribution to the impedance arises from the pre-factor \( (\sigma^* - j \omega \varepsilon_0 \varepsilon_r) \) that accounts for the reactive energy stored in the field as it propagates though the electron loaded medium. Even if the equivalent impedance of the circuit was not fully known the amplitude of each current in (17) or (25) can be obtained solving the circuit in Fig.3

\[
i^p = \frac{v_0}{(2\pi^P)} = \frac{\sqrt{n k_B T \nu}}{2\pi^P}
\]

(28)

which suggest the energy EM spectral energy in the load, associated to one DoF, from (25) can be expressed

\[
E_1^{\text{DoF}} = \frac{k_B T (\varepsilon_0)^2}{4|Z|^2}
\]

(29)

And simplifying the real part of the impedance

\[
E_1^{\text{DoF}} = k_B T \frac{\sigma^2}{|\sigma - j \omega \varepsilon_0 \varepsilon_r|^2}
\]

(30)

The striking property emerging from (30) is that the energy rendered available to one degree of freedom does not depend from which mode one is considering. Modes emerging from electric or magnetic originating dipoles, render available the same EM energy, also independently from their orientation. The frequency dispersion from Drude’s model implies that the conductivity is mostly real at low frequency and mostly imaginary at high frequencies. Specifically

\[
\sigma_r = \frac{\sigma_{re}}{1+j \omega \tau_r} \quad \sigma_i = \frac{\sigma_{im}}{1+j \omega \tau_i}
\]

(31)

Looking at eq. (30) it is apparent that the energy available to each DoF is equal to \( k_B T \) only if \( \sigma_r \gg \sigma_i \) and \( \sigma_i \gg \omega \varepsilon_0 \varepsilon_r \). Fig. 9, shows the spectral energy delivered to a Degree of Freedom of the EM field as a function of the frequency for the same parametric cases studied in Fig.5, Fig.7 and Fig.8 and assuming the temperature \( T = 300 \text{ K} \).

![Fig. 9 Visible Electromagnetic energy, \( E_1^{\text{DoF}} \), at 300 K delivered to one eigen mode as a function of the frequency.](image-url)

For frequencies such that \( \omega \varepsilon_0 \varepsilon_r > \sigma_r \), the energy delivered to the EM mode begins to drop. This means that displacement currents, are dominant with respect to the conductivity currents. This drop occurs for lower frequencies when the doping is lower and thus corresponding to lower conductivities. As the frequencies grow even the real part of the conductivity drops with the square of the frequency and accordingly the limit for frequencies tending to infinity can be expressed analytically as follows:

\[
\lim_{\nu \to \infty} E_1^{\text{DoF}} \approx k_B T \frac{\sigma_{re}^2}{2(2\pi)^3 \varepsilon_0^2 \omega^2 \tau_r^4}
\]

(32)

The total EM energy per unit of volume associated to all the electric current born modes was given in (25). However since we have seen that the electromagnetic energy does not depend
on the specific impedance, it is apparent that all the modes contained within a unit cell are associate to the same energy. Recalling that $N_{\text{DoF}} = 6 \times N_{\text{pos}}$, the electromagnetic energy in a volume, $\text{Vol}$, is equal to the summation of the energies, $E^{\text{DoF}}_1$, associated to each degree of freedom, or eigen vector

$$E^{\text{Tot}} = \text{Vol} N_{\text{DoF}} E^{\text{DoF}}_1 = \frac{\text{Vol}}{4\pi(\frac{1}{4})^2} \times 6k_B T \frac{\sigma_r}{\sigma_a} \frac{\sigma_r}{\sigma_a} \frac{\Delta}{\Omega}$$

IV. RADIATED AND RECEIVED ENERGY

In this section a brief reminder of how radiated energy is evaluated in standard radiometry is provided. With appropriate modifications the same procedure can be applied also in the DoF based procedure proposed in this paper. In standard radiometry it is well known that the field radiated by a warm body (its emissivity) can be evaluated using reciprocity by evaluating the energy absorbed (its absorbivity) when the thermally equivalent sources are assumed to be located in the far field, [24]. The procedure was proposed by Rytov [1][2] and it has since then adopted by the international literature [4] and following literature [5]-[10]. It is a completely classical procedure, except for the definition (or postulation) of the currents themselves. Once the emissivity (equal to the absorbivity) is known, the spectral energy density received by a singly polarized, mono-modal, receiving antenna, characterized by an antenna solid angle, $\Omega_a$, when it is oriented toward a warm body that defines a solid angle, $\Omega_B$ larger than $\Omega_a$, can be expressed, [10], according to

$$E^{\text{Planck}}_\text{ant}(T, f) \approx k_BT e_m(f)\eta_a(f)$$

where $\eta_a$ is the antenna efficiency.

![Fig.10 Emittance of infinitely silicon dielectric slabs assumed to be of infinite thickness for different doping levels. The maximum emittance is reached the effective dielectric constant tends to be the one in absence of losses.](image)

Here it was assumed that the frequencies are such that $f \ll \frac{k_BT}{\hbar}$. In [11], this formula was used to provide a preliminary estimation of the expected energy in the case of radiation from silicon slabs. The emittance for the usual 3 parametric silicon doping considered in the previous cases is presented in Fig.10. It is apparent that the maximum emittance is reached, at frequencies for which the losses in the silicon have dropped significantly (these frequencies are higher for higher doping). According to (34) and neglecting the losses associated to the antenna, the emittance is the only important parameter to assess the spectral energy received at an antenna.

Unfortunately, as discussed in [11], the measurements associated to silicon samples at sub-THz frequencies do not validate eq. (37) as a useful expression to estimate the received energy. Accordingly, in the following the new classical currents derive in section II and III will be used to derive a new expression similar to (37) that provides much better agreements with experiments. The purpose is to be able to compare the impact in terms of radiated energy of using the new derived currents in comparison with the standard Planck’s (or Rytov’s) procedure. The only differences between the standard procedures and the newly proposed classic procedure based on the Degrees of Freedom will be in the estimation of the currents. The uncertainties associated to the emittance of the body and the antenna adopted will be exactly the same for the classic DoF and Quantum born procedure.
V. AUTOCORRELATION OF CURRENTS FROM DOF

In this section I will estimate a set of equivalent currents, \( \bar{J}_{eq,\text{DoF}} \) that could be assumed to be distributed over the warm body, as in Fig. 11, and would radiate outside the body, the same energy as the currents that were discussed in sections II and III. Their autocorrelation will be calculated over a cubic volume with specific discretization, \( \Delta \). The advantage of introducing these currents is that the evaluation of the energy received by an antenna in reception can then be obtained using equation (37) but just replacing the autocorrelation of the currents from Rytov with the autocorrelation of the currents based on the Degrees of Freedom. The model introduced in section II and III states that if one was able, with a magnifying lens as in Fig. 12, to zoom on any specific area of the body, and would select a specific frequency, \( f \), the currents would be expressed as superposition of modal distributions.

![Fig.12 Typical geometrical problem to be investigated by V-MoM](Image)

The field picture would be consistent of periodically displaced spheres of radius \( d/2 \) as in (33). The total electric field, \( \vec{E}(\vec{r}) \) would be expressed as the superposition of the fields due to the modal contributions that can be expressed as radiated by generating currents, \( \vec{J}_\text{gen}(\vec{r}) \), oriented along all three polarizations and located at different locations, the centers of the spheres, as expressed in (17)-(19). The generating currents per unit of volume would be expressed compactly as

\[
\vec{J}_\text{gen}(\vec{r}) = \sum_{n=1}^{n_\text{max}} \sum_{p=1}^{3} i^p \vec{J}_{\text{gen,p}}(\vec{r}, \vec{r}_n)
\]

where \( \vec{J}_{\text{gen,p}}(\vec{r}, \vec{r}_n) \) indicates the currents, polarized along \( p \) centred at location \( \vec{r}_n \) as in (19). These generating currents can be assumed to present aleatory phase, since the fields they generate have been already shown to be independent (the generating currents are uncoupled: their mutual coupling is zero). The amplitudes \( i^p \) of each of the generating currents in (38) was given in (28) and their amplitude squared is expressed by

\[
|i^p|^2 = k_B T \frac{R_p}{|\vec{Z}_p|} = k_B T \text{Re} \left\{ \frac{1}{\vec{Z}_p} \right\}
\]

as a function of the modal impedances evaluated analytically in Appendix B, and provided here

\[
\vec{Z}_p = (\sigma^* - j \omega \varepsilon_0 \varepsilon_r) \times C_p, \quad C_p = \frac{\varepsilon_r^2}{\omega_0} \frac{\Delta^3 \pi}{3 \lambda^3} \in R
\]

A. Energetically equivalent average generating currents

The specific location, \( \vec{r}_n \), Fig. 13a of a generating current within a warm body is indicated only formally. One can imagine that the generating current’s central location, \( \vec{r}_n \) is only one possible realization, in probabilistic sense, of the generating current. The generating current could be centred everywhere within the cell assumed to be a sphere of radius \( d/2 \) and volume \( \frac{4}{3} \pi \left( \frac{d}{2} \right)^3 \), that defines the domain of each of degrees of freedom, as in Fig. 13b. When averaging in time the energy radiated, the result would not change.

![Fig.13 Generating electric currents distributed on the \( m \)-th cell, a) one generating current, \( j_{\text{gen,}p} \) at the centre of the cell its centre, b) alternative realizations, \( j_{\text{eq,}p}^{\text{eq},\text{DoF}}(\vec{r}) \), for \( l_r : 1 \rightarrow L_r \)](Image)

This thinking can be further generalized. All the realizations in Fig. 13b are equally likely to occur at one time. Accordingly, if one is interested in the average (in time) energy radiated, rather than thinking the different realizations in Fig. 13b as alternative possibilities, one can think of them as coexisting incoherent realizations that all radiate the same energy simultaneously. The generating current would be distributed everywhere over the spherical cell of radius \( d/2 \), giving rise to a uniform amplitude distribution of the energetically equivalent generating currents with incoherent phases. Clearly the amplitudes of the currents would have to be reduced to maintain the same total radiated energy. The same energy would be radiated if each currents amplitude was scaled by a factor \( \frac{1}{\sqrt{\Delta^3}} \). The average, energetically equivalent currents, around a point \( \vec{r}_n \), can then be expressed as

\[
\bar{J}_{\text{eq,}p}^{\text{eq},\text{DoF}}(\vec{r}, \vec{r}_n) = \frac{1}{\sqrt{\Delta^3}} \sum_{l_r=1}^{L_r} \hat{p} i^p \frac{1}{\Delta^3} \text{rect} \left( \vec{r} - \vec{r}_n + \vec{r}_l, \Delta^3 \right)
\]

Considering as a volume any cell of size \( \Delta \) the auto-correlation \( A \) can simply evaluated for the currents in (41) and be expressed analytically as

\[
A^\Delta \left( \vec{r}_{\text{eq,}p}^{\text{eq},\text{DoF}} \right) = \Delta^3 \frac{1}{\Delta} \frac{|i^p|^2}{\Delta^3} = \Delta^2 \frac{|i^p|^2}{\Delta^3}
\]

It is finally interesting, at last, to express the electric current amplitude in (42) using (39) and (40).
in which the meaning of the terms in (44) remains the same as the terms in (37). However we also recall now that all that was said for the degrees of freedom associated to the electric currents could be said also for the degrees of freedom associated to the magnetic currents. Consequently

\[
E^{\text{DoF, mag}}(T, f) = E^{\text{DoF, el}}(T, f)
\]

and consequently

\[
E^{\text{DoF}}(T, f) = \frac{A^3}{2\sigma_r} e_{m_l}(f) \eta_a(f)
\]

While eq. (46) is correct it can be useful to express it as the multiplication of eq.(37) per a correction factor, \(N(\sigma, \epsilon_r, d, \alpha)\)

This correction factor would include the information emerging from to the fact that each Degree of Freedom is associated to electromagnetic energy (rather than mechanical energy as implicit in (34)) and how many of such degree of freedom should be accounted for per unit of volume.

\[
E^{\text{ant}}(T, f) = E^{\text{Planck}} N(\sigma, \epsilon_r, d, \alpha)
\]

where

\[
N(\sigma, \epsilon_r, d, \alpha) = Re \left( \frac{1}{\sigma^3 - j \omega \epsilon_0 \epsilon_r} \right) \frac{1}{\sigma^3 - j \omega \epsilon_0 \epsilon_r} \lambda_3^3 \frac{3\alpha}{\epsilon_0 \epsilon_r 2\sigma_r}
\]

Eq. (47) allows to focus only on the differences between the new DoF theory and the standard application of radiometry isolating the differences with a single function (48)

B. Applicability: light and strong doping

Assuming that the currents in (41), with their autocorrelation in (43), can distributed over a finite body, with the same amplitude in every cell as in Fig11, is an approximation that makes sense when the losses in the ohmic medium are important. However, when the losses are low, the assumption that the currents within the body are the same as in an infinitely extended medium, is less appropriate. With low losses the geometrical properties of the radiating body will start to be important, and the EM problems will have to be solved accounting for the geometrically dominated characteristic modes of the bodies [23]. Global standing waves within the body will have consequences, if one is interested in the accurate evaluations of thermally radiated fields. This was recently already recognized also in the international literature that builds on Rylov’s currents [25]. These EM problems will have to be part of possible future works, that specifically focus on advanced numerical modelling.

VI. COMPARISON BETWEEN DOF ENERGY AND MEASUREMENTS

In [11] a series of measurements were presented, pertinent to the spectral energy received at detectors connected to single polarized mono-mode horn antennas. In all cases the radiating samples were made of crystalline silicon doped with phosphorus atoms. Comparing the measurements with simulations based on an approximate estimation of the emissivities of the silicon samples suggested that the was a problem with the expectations from Planck’s law. As the readers could see in Fig.1 of the present paper, the measurement systematically seemed to be well below the expectations due to Planck’s law, represented by eq.(37) in this paper. Here we present the comparison between the present calculations, eq. (44), and the measurements from the representative specific sample that in [11], was indicated as YW3 – 4. The sample was composed by a stack of two wafers for a total thickness of 600 \(\mu m\), with a doping realized with phosphorus inclusions with an estimated density of \(n_{PH}^{best} = 5 \times 10^{21}\). Fig. 14, complementing Fig1, shows the measured spectral energy radiated by the stack, focusing on the frequencies below 500 GHz. The expected spectral densities are obtained using eq. (47) for the DoF procedure and (37) for the Planck’s curve.

Fig.14 Spectral energy from sample YW3-4,(at temperature \(T_{si} = 425K\)). Measurements vs calculations, in linear scale

Fig. 15 shows similar curves but expressed in dBJ and parametric with respect to the temperature. For each increasing temperature considered the conductivity of the silicon is scaled down to account for the higher scattering time, following [26]. An excellent agreement was obtained, considering that the analysis presented in this paper does not include the information of the specific geometry in defining the degrees of freedom modal functions.

VII. CONCLUSIONS

Here, I have presented a rigorous procedure to identify the Electromagnetic modes that represent the independent Degrees of Freedom in an infinite ohmic medium. A classical procedure
is then used to expand the electromagnetic field with the mentioned modal distributions, whose amplitude is determined by means of equivalent thermal sources. First, the analysis provides the EM energy rendered available to the material per unit of volume. Each eigen vector current is excited in proportion to the thermal agitation of the electrons: the thermal energy is thus efficiently transferred to the EM field. At low frequencies the conduction currents are dominant and in phase with the agitation of the electrons; the thermal energy is thus efficiently transferred to the EM field. At higher frequencies the radiated displacement currents overshadow the conduction currents. Since the displacement currents are out of phase with the agitation of the electrons, the thermal energy is not efficiently transferred to the EM field for higher frequencies. Secondly, the procedure is applied to estimate, in a simplified manner, the brightness of the warm ohmic body.

Even if the theoretical justification is completely different, the analytical procedure to evaluate the brightness can be related to the Planck’s one (or more correctly Rayleigh Jeans as follows)

\[ B^{DoF}(T, f) = B_{Planck}^{RJ} \times N(\sigma, \varepsilon_r, d, \alpha) \]

where the factor

\[ N(\sigma, \varepsilon_r, d, \alpha) = Re \left( \frac{1}{(\sigma - j \omega \varepsilon_0 \varepsilon_r)} \right) \frac{1}{2\pi} \lambda_0^3 \frac{3\pi}{\varepsilon_0^2} \frac{1}{2\sigma_r} \]

includes the parameters and the dispersivity of the radiating body. The comparison between the expected radiated energy with the measurements that were described in the companion paper [11] provides a convincing validation of the theory for the radiation from silicon slabs, explaining the THz catastrophe mentioned there.

It is important to observe that the model as it is suggests a discretization of an homogeneous space by means of Degrees of Freedom functions that fill the volume at a density of half of the effective wavelength or higher. However, the present analysis is only a first step. It does not account for the geometry of the bodies in defining the degrees of freedom, as the Degrees of Freedom are assumed to have distribution irrespectively of their position within the radiating body. The use of advanced numerical techniques will be required to describe accurately the radiation from bodies that present low conductivity or are small in terms of the wavelength.

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**APPENDIX A: THE EIGEN VALUES OF THE FREE SPACE GREEN’S FUNCTION**

In this appendix it is demonstrated that if \( \tilde{e}(\vec{r}') \) verifies ME in an homogeneous space, a volumetric current

\[ \tilde{J}(\vec{r}) \equiv -(\sigma + j \omega \varepsilon_0 \varepsilon_r) \tilde{e}(\vec{r}) \]  

is an eigen vector current for the homogeneous space Green’s function and thus

\[ \int_{-\infty}^{\infty} \tilde{g}^{\varepsilon}(\vec{r}, \vec{r}') \cdot \tilde{J}(\vec{r}) \, d\vec{r}' = \tilde{e}(\vec{r}) \]

**Proof.**

The Proof resorts heavily on properties of fields in homogeneous media that were highlighted first by Arthur Yaghjian [19]. Specifically in [19], eq. (5b), he showed that the
electric field radiated by a currents, \( \mathbf{j} \) can be expressed as the sum of a principal value integral and a term proportional to the current itself

\[
\ddot{\mathbf{e}}(\mathbf{r}) = \frac{\mathbf{j}(\mathbf{r})}{\omega} + j\omega\epsilon \mathbf{h}
\]  

(A.3)

However, in the appendix [20] Yaghjian also demonstrated that in an homogeneous space with the fields obeying the outward radiation condition, the Maxwell’s equation

\[
\nabla \times \mathbf{e} = -j\omega\mu \mathbf{h}
\]  

(A4)

\[
\nabla \times \mathbf{h} = j\omega\epsilon \mathbf{e} + \mathbf{j}
\]  

(A5)

can be replaced by the equivalent set of equations

\[
\nabla^2 \mathbf{h} + k^2 \mathbf{h} = -\nabla \times \mathbf{j}
\]  

(A6)

\[
\nabla^2 \left( \ddot{\mathbf{e}} - j\frac{\mathbf{j}}{\omega} \right) + k^2 \left( \ddot{\mathbf{e}} - j\frac{\mathbf{j}}{\omega} \right) = \frac{1}{\omega} \left( \nabla \times \nabla \times \mathbf{j} \right)
\]  

(A7)

In fact taking the curl of (A5)

\[
\nabla \times \nabla \times \mathbf{h} = j\omega\epsilon \nabla \times \ddot{\mathbf{e}} + \nabla \times \mathbf{j}
\]  

(A8)

and replacing (A4) in (A8)

\[
\nabla \times \nabla \times \mathbf{h} = k^2 \mathbf{h} + \nabla \times \mathbf{j}
\]  

(A9)

Using in (A9) the identity

\[
\nabla \times \nabla \times \mathbf{h} = \nabla (\nabla \cdot \mathbf{h}) - \nabla^2 \mathbf{h}
\]  

(A10)

And recognizing that \( \nabla \cdot \mathbf{h} = 0 \) in absence of magnetic charges leads to (A6). Moreover, taking again the curl of (A6) one can obtain

\[
\nabla \times \nabla \times \mathbf{h} = k^2 \mathbf{h} + \nabla \times \mathbf{j}
\]  

(A11)

However using again (A10) in (A11) one obtains

\[
\nabla \times \left[ -\nabla^2 \mathbf{h} \right] + k^2 \mathbf{h} = -\nabla \times \nabla \times \mathbf{j}
\]  

(A12)

Which, inverting the order of the derivations can be written as

\[
\nabla^2 \mathbf{h} + k^2 \nabla \times \mathbf{h} = -\nabla \times \nabla \times \mathbf{j}
\]  

(A13)

If one substitutes in (A13) \( \nabla \times \mathbf{h} = (j\omega\epsilon \mathbf{e} + \mathbf{j}) \) it results

\[
\nabla^2 (j\omega\epsilon \mathbf{e} + \mathbf{j}) + k^2 (j\omega\epsilon \mathbf{e} + \mathbf{j}) = -\nabla \times \nabla \times \mathbf{j}
\]  

(A14)

Accordingly if the current \( \mathbf{j} \) is such that

\[
\mathbf{j} = -j\omega\epsilon \mathbf{e}
\]  

(A13)

\[
-\nabla \times \nabla \times \mathbf{j} = 0
\]  

(A14)

And thus (A3) can be read as

\[
\ddot{\mathbf{e}}(\mathbf{r}) = j\frac{[\mathbf{r}]}{\omega\epsilon}
\]  

(A15)

This is already the results that needed to be proved once it is recognized that in the lossy media cases, the object of this paper, \( \epsilon = \epsilon_0 \epsilon_r \left( 1 - \frac{j}{\omega \sigma_0 \epsilon_r} \right) \). Introducing the lossy effective dielectric in eq. (A10)

\[
\ddot{\mathbf{e}}(\mathbf{r}) = -(\sigma + j\omega\epsilon_0 \epsilon_r) \ddot{\mathbf{e}}(\mathbf{r})
\]  

(A15)

reproduces is the condition in (A1)

APPENDIX B: MUTUAL COUPLING OF EIGEN VECTORS

In this appendix it is demonstrated that the mutual coupling, \( Z_{mn}^{pq} \), between \( \tilde{\mathbf{f}}_{\text{vis}}^m(\mathbf{r}) \) and \( \tilde{\mathbf{f}}_{\text{vis}}^q(\mathbf{r}) \) can be approximated analytically as

\[
Z_{mn}^{pq} \propto (\sigma^* - j\omega\epsilon_0 \epsilon_r) C_{mn}^{pq}
\]  

(B2)

where

\[
C_{mn}^{pq} = \iint_{-\infty}^{\infty} \tilde{e}^{mp'}(\mathbf{r}, \mathbf{\hat{r}}) \tilde{e}^{m'q}(\mathbf{\hat{r}}, \mathbf{\hat{r}}_m) d\mathbf{\hat{r}} d\mathbf{\hat{r}}_m
\]  

(B3)

The lowest order spherical wave, centred in \( \mathbf{\hat{r}}_m \), can be simply expressed as the field of a short electric dipole oriented along \( \hat{z} \) and centred in \( \mathbf{\hat{r}}_m \)

\[
\tilde{e}^{m}(\mathbf{\hat{r}}, \mathbf{\hat{r}}_m) = \tilde{g}(\mathbf{\hat{r}}, \mathbf{\hat{r}}_m) \cdot \hat{z}
\]  

(B4)

In turn the GF can be represented using the 3D Fourier representation as

\[
\tilde{g}(\mathbf{\hat{r}}) = \frac{i\epsilon}{k(2\pi)^3} \int \int \int_{-\infty}^{\infty} \tilde{B}(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{\hat{r}}} d\mathbf{k}
\]  

(B5)

where, \( \mathbf{k} = k_x \hat{x} + k_y \hat{y} + k_z \hat{z} \), and \( d\mathbf{k} = dk_x dk_y dk_z \). The kernel of the spectrum, includes

\[
k^2 = k_x^2 + k_y^2 + k_z^2
\]  

and with

\[
\tilde{B}(\mathbf{k}) = \begin{bmatrix}
k^2 - k_x^2 & -k_z k_y & -k_z k_x \\
-k_z k_y & k^2 - k_y^2 & -k_y k_x \\
-k_z k_x & -k_y k_x & k^2 - k_x^2
\end{bmatrix}
\]  

(B7)

The integral representation in (B5) is too complex to lead to an analytical expression of the mutual coupling in (B3) because it includes the reactive energy associated to the near fields of the GF. However, since previously it was chosen to retain only the visible portion of the outgoing fields one can perform the mutual coupling integrals in (B3) retaining only the visible portion of the mode, in (B4). To obtain it one can start by
representing the GF as the superposition of a visible and an invisible component as

$$\tilde{g}(\vec{r}) = \tilde{g}_{inv}(\vec{r}) + \tilde{g}_{vis}(\vec{r})$$  \hspace{1cm} (B8)

where

$$\tilde{g}_{inv}(\vec{r}) = \frac{\i k}{k(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{\tilde{B}(k\vec{k})}{k^2-k_r^2} \frac{k^2}{k_r^2} e^{-i\vec{k} \cdot \vec{r}} d\vec{k}$$  \hspace{1cm} (B9)

and

$$\tilde{g}_{vis}(\vec{r}) = \frac{\i k}{k(2\pi)^3} \iiint_{-\infty}^{\infty} \frac{\tilde{B}(k\vec{k})}{k^2-k_r^2} \frac{k^2}{k_r^2} e^{-i\vec{k} \cdot \vec{r}} d\vec{k}$$  \hspace{1cm} (B10)

where $\vec{k} = 1/k_r \vec{k}$. Introducing the 3D spectral representation of the visible GF's in B3 the mutual coupling is formally expressed as a nine folded integral

$$C_{mn}^{pq} = \left(\frac{\i}{k}\right) l^2 e^{\frac{1}{2} (2\pi)^3} \iiint_{-\infty}^{\infty} \frac{\tilde{D}(k\vec{k}) \cdot \tilde{B}(k\vec{k})}{k^2-k_r^2} \frac{k^2}{k_r^2} e^{-i\vec{k} \cdot \vec{r} - i\vec{r} \cdot \vec{m}} d\vec{k}$$  \hspace{1cm} (B11)

However, recognizing in (B11)

$$\iiint e^{-i\vec{k} \cdot \vec{r} - i\vec{r} \cdot \vec{m}} d\vec{r} = (2\pi)^3 \delta(\vec{k} - \vec{k'})$$  \hspace{1cm} (B12)

we can replace everywhere in (B11) $\vec{k'} = \vec{k}^*$ and eliminate the integration in $\vec{k'}$ reducing the mutual coupling to a 3 folded integral

$$C_{mn}^{pq} = \left(\frac{\i}{k}\right)^2 \delta_{et} e^{\frac{1}{2} (2\pi)^3} \iiint_{-\infty}^{\infty} \frac{\tilde{D}(k\vec{k}) \cdot \tilde{B}(k\vec{k})}{k^2-k_r^2} \frac{k^2}{k_r^2} e^{-i\vec{k} \cdot \vec{r} - i\vec{r} \cdot \vec{m}} \frac{k^2}{k_r^2} d\vec{k}$$  \hspace{1cm} (B13)

Indicating $\left[\tilde{D}(k\vec{k}) \cdot \tilde{B}(k\vec{k})\right] = Pol(\vec{k}; \vec{r}, \vec{q})$ and noticing that

$$\left[\frac{\i k}{k^2-k_r^2} \right]^2 \frac{k^2}{k_r^2} = \left[\frac{\i k}{k^2-k_r^2} \right]^2$$  \hspace{1cm} (B14)

and

$$k^2 k_r^2 = |k|^2 |k_r|^2$$  \hspace{1cm} (B15)

The expression in (13) is simplified into

$$C_{mn}^{pq} = \left(\frac{\i}{k}\right)^2 \delta_{et} e^{\frac{1}{2} (2\pi)^3} \iiint_{-\infty}^{\infty} Pol(\vec{k}; \vec{r}, \vec{q}) F(k_r) e^{-i\vec{k} \cdot \vec{d}} d\vec{k}$$  \hspace{1cm} (B16)

where

$$F(k_r) = \frac{1}{k^2-k_r^2} \frac{1}{k^2-k_r^2} \frac{1}{k_r^2} |k|^2$$  \hspace{1cm} (B17)

Adopting the standard change of variables

$$\begin{align*}
k_x &= k_r \sin \beta \cos \alpha \\
k_y &= k_r \sin \beta \sin \alpha \\
k_z &= k_r \cos \beta
\end{align*}$$  \hspace{1cm} (B18)

Eq. (B16) can be expressed as

$$C_{mn}^{pq} = \left(\frac{\i}{k}\right) \delta_{et} e^{\frac{1}{2} (2\pi)^3} \iiint_{-\infty}^{\infty} Pol(\vec{k}; \vec{r}, \vec{q}) F(k_r) k_r^2 e^{-i\vec{k} \cdot \vec{d}} d\vec{k}$$  \hspace{1cm} (B18)

One can then decide to focus the integration in the solid angle only the domains where $\vec{k} \cdot \vec{d} > 0$ using the procedure in Appendix C. Accordingly

$$C_{mn}^{pq} = \left(\frac{\i}{k}\right) \delta_{et} e^{\frac{1}{2} (2\pi)^3} \iiint_{-\infty}^{\infty} Pol(\vec{k}; \vec{r}, \vec{q}) F(k_r) k_r^2 e^{-i\vec{k} \cdot \vec{d}} d\vec{k}$$  \hspace{1cm} (B19)

The integration in $k_r$ can be performed capturing the residues in $k$ or $k^*$. The contributions from the poles in zero are neglected since they would be important for the evaluation of the self coupling, but in that case the reaction integral can be done much in a much simpler way directly in the space domain. The integration in $k_r$ can be explicitly expressed as

$$I = \int_{-\infty}^{\infty} \frac{1}{(k-k_r)(k+k_r)} \frac{e^{-j\vec{k} \cdot \vec{d}}}{k_r^2} |k_r|^2 d\vec{k}$$  \hspace{1cm} (B20)

Capturing the residues in $k$ and in $-k^*$, it results

$$I = 2\pi j \frac{1}{(2k)(k^*+k)} \left(\frac{e^{-jkk^*d}}{k^2} + \frac{e^{-jkk^*d}}{k^2} \right)$$  \hspace{1cm} (B20)

Rearranging the terms

$$I = \pi \frac{1}{(2\alpha)(2\beta)} \left(\frac{e^{-jkk^*d}}{k^*} - \frac{e^{-jkk^*d}}{k^*}\right)$$  \hspace{1cm} (B21)

By indicating $A = e^{-j\vec{k} \cdot \vec{d}}$, equation (21) can be simplified as

$$I = \frac{\pi}{4\alpha\beta} e^{-akd^2} A - A^*$$  \hspace{1cm} (B22)

Eventually, it results

$$C_{mn}^{pq} = -\left(\frac{\i}{k}\right)^2 \delta_{et} e^{\frac{1}{2} (2\pi)^3} \frac{\pi}{4\alpha\beta} 2\frac{j\alpha d}{(2\beta)}$$  \hspace{1cm} (B23)

The integral in $d\vec{k}$ can be performed asymptotically using a stationary phase technique. Since the stationary phase is $\vec{k} \cdot \vec{d} = 1$ it is apparent that that when $\beta d = (n+1)\pi$ the integral $I = 0$
The angular integrations can then be performed analytically in the stationary phase points. The result depends on the specific polarization considered. For instance in the case that the two original dipoles are parallel

\[ C_{\text{pp}}^{\text{pp}} = \frac{8\varepsilon_0 e^2}{\lambda_0^2} \int_0^{\pi} e^{-ad \sin (\beta d)} \text{Pol} [\hat{p}, \hat{q}] \]  

(B24)

where

\[ d = |\hat{r}_m - \hat{r}_n|, \hat{k} = (\hat{r}_m - \hat{r}_n) / d \] and \( \text{Pol} [\hat{p}, \hat{p}] = 1 \)  

(B25)

All other dyadic combinations \( \text{Pol} [\hat{p}, \hat{q}] \) will give rise to different results. However in the present context, is only important to observe that all mutual couplings go to zero for \( (\beta d) = (n + 1)\pi \).

In the case of a self coupling of the visible fields does not require such a spectral decomposition it results in

\[ C_{\text{pp}}^{\text{pp}} = \frac{8\varepsilon_0 e^2}{\lambda_0^2} \int_0^{\pi} e^{-ad \sin (\beta d)} \text{Pol} [\hat{p}] \]  

(B26)

APPENDIX C: REDUCTION OF INTEGRALS

In this appendix a procedure is presented to transform the 3D integrals of the type

\[ I = \int_{\text{4V}} \int_{0}^{\infty} \frac{1}{k^2 - k_r^2} e^{-j k_r \cdot \hat{k}} \, dk_r \]  

(C1)

in the equivalent form

\[ I = \int_{k_r > 0}^{\infty} \int_{-\infty}^{\infty} \frac{1}{k^2 - k_r^2} e^{-j k_r \cdot \hat{k}} \, dk_r \, d\hat{k} \]  

(C2)

Proof

One can first recognize that the angular, \( \hat{k} \), integration in (C1) is divided in parts for which \( \hat{k} \cdot \hat{r} > 0 \) and parts for which \( \hat{k} \cdot \hat{r} < 0 \). Accordingly the integration \( I \) can be expressed as the superposition of two integrals

\[ I = I_+ + I_- \]  

(C3)

where

\[ I_+ = \int_{k_r > 0}^{\infty} \int_{0}^{\infty} \frac{1}{k^2 - k_r^2} e^{-j k_r \cdot \hat{k}} \, dk_r \, d\hat{k} \]  

(C4)

and

\[ I_- = \int_{k_r < 0}^{\infty} \int_{0}^{\infty} \frac{1}{k^2 - k_r^2} e^{-j k_r \cdot \hat{k}} \, dk_r \, d\hat{k} \]  

(C5)

It is simple to realize that \( I_+ \) and \( I_- \) are exactly the same except for the sign of the phase term. Accordingly we can express \( I_- \) as

\[ I_- = \int_{k_r > 0}^{\infty} \int_{0}^{\infty} \frac{1}{k^2 - k_r^2} e^{j k_r \cdot \hat{k}} \, dk_r \, d\hat{k} \]  

(C6)

Substituting (C6) in (C3)

\[ I = \int_{k_r > 0}^{\infty} \int_{0}^{\infty} \frac{1}{k^2 - k_r^2} (e^{-j k_r \cdot \hat{k}} + e^{j k_r \cdot \hat{k}}) \, dk_r \, d\hat{k} \]  

(C7)

At this point one can directly focus on the integration in \( k_r \), indicated as \( I_k \), and recognize that

\[ I_k = \int_{0}^{\infty} \frac{1}{k^2 - k_r^2} e^{-j k_r \cdot \hat{k}} \, dk_r + \int_{0}^{\infty} \frac{1}{k^2 - k_r^2} e^{j k_r \cdot \hat{k}} \, dk_r \]  

(C8)

One can then focus on the second integration in (C8)

\[ \int_{0}^{\infty} \frac{1}{k^2 - k_r^2} e^{j k_r \cdot \hat{k}} \, dk_r = \int_{-\infty}^{0} \frac{1}{k^2 - k_r^2} e^{-j k_r \cdot \hat{k}} \, dk_r \]  

(C9)

So that

\[ I_k = \int_{-\infty}^{0} \frac{1}{k^2 - k_r^2} e^{-j k_r \cdot \hat{k}} \, dk_r \]  

(C10)

and

\[ I = \int_{k_r > 0}^{\infty} \int_{-\infty}^{0} \frac{1}{k^2 - k_r^2} e^{-j k_r \cdot \hat{k}} \, dk_r \, d\hat{k} \]  

(C11)

Proving the thesis of the appendix.

The integration in \( k_r \) can now for instance be performed using residue theory Since \( \hat{k} \cdot \hat{r} > 0 \) the integration path can be close at infinity on the Lower Half Space to capture the residue in \( k_r = k \),

\[ \int_{-\infty}^{0} \frac{1}{k^2 - k_r^2} e^{-j k_r \cdot \hat{k}} \, dk_r = \frac{\pi j}{k} e^{-j k \cdot \hat{k}} \]  

(C12)

so that

\[ I = \frac{\pi j}{k} \int_{k_r > 0}^{\infty} e^{-j k \cdot \hat{k}} \, d\hat{k} \]  

(C13)

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