The exponential consensus of linear multi-agent systems with binary-valued measurements and Markovian switching topologies

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Abstract

This paper studies consensus of linear multi-agent systems with binary-valued measurements and switching topologies. Unlike the existing consensus of multi-agent systems with binary-valued measurements, Markovian switching topology is introduced in this paper. A new algorithm is proposed to improve the consensus speed of multi-agent systems, with constant gains in both estimation and control, instead of time-varying gains. By analyzing the estimation error and the consensus error simultaneously, we prove that the proposed algorithm can make agents achieve consensus in a bounded range, and the consensus speed is negative exponential under certain conditions, which is faster than that in existing literature. Finally, simulation results are given to demonstrate the theoretical results.
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Summary

This paper studies consensus of linear multi-agent systems with binary-valued measurements and switching topologies. Unlike the existing consensus of multi-agent systems with binary-valued measurements, Markovian switching topology is introduced in this paper. A new algorithm is proposed to improve the consensus speed of multi-agent systems, with constant gains in both estimation and control, instead of time-varying gains. By analyzing the estimation error and the consensus error simultaneously, we prove that the proposed algorithm can make agents achieve consensus in a bounded range, and the consensus speed is negative exponential under certain conditions, which is faster than that in existing literature. Finally, simulation results are given to demonstrate the theoretical results.

KEYWORDS:
Binary-valued measurements, linear multi-agent systems, Markovian switching topologies, consensus, consensus speed

1 INTRODUCTION

In recent years, multi-agent systems have attracted more and more researchers’ attention due to their wide applications. Multi-agent systems are mainly applied in multi-sensor cooperative information processing, multi-robot assistance, unmanned aerial vehicle and underwater vehicle formation. Most existing literatures consider consensus problems with accurate neighbors’ states. However, the transferred information between two agents is usually quantized due to limited bandwidth. And there inevitably exists noise in information transmission. Besides, the topologies of multi-agent systems are usually changing due to many external factors, such as communication cutoff, transmission delay, and lack of input. Therefore, the study of consensus with quantization, noise and switching topology is more practical.

Consensus of multi-agent systems with quantization has been studied recently with the development of digital signal. Refs. consider consensus problem for continuous-time systems with logarithmic quantizers. An explicit upper bound of the consensus error are given in terms of the initial states, the quantization density and the parameters of the network graph. Refs. consider consensus problems with finite-level quantizers. It is shown that the asymptotic convergence rate is related to the scale of the network, the number of quantization levels and the connectivity of communication graph. Refs. consider consensus problems with binary-valued measurements. Ref. proposes a two-time-scale consensus algorithm to achieve average consensus under undirected topologies, in which all agents keep unchanged when agents estimate their neighbours’ states and control is designed at some skipping times when estimates error is in a bounded range. In Ref., each agent estimates its neighbors’ states by an online algorithm, and then the control is designed at every time based on the estimates, which leads to a faster convergence rate than that in Ref.
In Refs. [89,103,104] time-varying gains are used in the control algorithm to deal with random communication noises. The convergence rate in [89] is $O(1/t)$ for the best case. Such designs of time-varying gains are also used in Refs. [105,112] to reduce the impact of the communication noises, but they all cannot achieve exponential convergence rate. It is noticed that a constant gain is designed in Ref. [113], which can achieve exponential convergence rate for the case that neighbors’ states are obtained accurately. In [114], a constant gain is designed for the consensus control with binary-valued measurements and the convergence rate is faster than that in [89] with time-varying gains. But, the control in [114] is designed only at some skipping times, which does not good for consensus speed. To improve the consensus speed of multi-agent systems with binary-valued measurements, an online algorithm alternating estimation and control is proposed in this paper, in which, constant gains are designed in both estimation and control.

Not only bandwidth limitation and noises affect the consensus properties of multi-agent systems, but also the topology of multi-agent systems. Refs. [115,116,117] study multi-agent systems with switching topologies and precise states. Ref. [117] shows that consensus under changing topologies can be achieved if the union of the directed interaction graphs have a spanning tree frequently enough. Ref. [118] consider the consensus control of multi-agent systems with finite communication data rate and switching topology flows. A class of quantized-observer-based communication schemes and a class of certainty-equivalence-principle-based cooperative control laws are proposed with adaptive encoders and decoders. In Ref. [119], consensus algorithm is proposed for binary-valued measurements under switching topologies, each of which is required to emerge with a constant probability. In this paper, Markovian switching topologies are introduced into multi-agent systems with binary-valued measurements, where the probability of each topology is time-varying by the rule of Markov chain.

This paper studies exponential consensus for linear multi-agent systems with binary-valued measurements under Markovian switching topologies. A recursive consensus algorithm alternating estimation and control with constant gains is proposed. Firstly, each agent estimates its neighbors’ states by a recursive projection algorithm with a constant gain. Then, each agent designs consensus control with a constant gain based on the estimates. The contributions of this paper can be summarized as follows. (a) Markovian switching topologies are introduced into consensus with binary-valued measurements. The probability of each topology of multi-agent systems is time-varying, which is different from that in Ref. [106]. It brings difficulties to analyze Laplacian matrix. (b) An online algorithm alternating estimation and control with constant gains is proposed, which is different from those in Refs. [107,108]. The estimation and control are strongly coupling, which are analyzed at the same time. Then, the coefficient matrix in the updating of estimation and control become constant. (c) By choosing appropriate parameter in estimation, we can make sure the norm of coefficient matrix is less than 1. It is proved that the agents can achieve consensus in a bounded range. And, the consensus speed can be exponential, which is faster than those in Refs. [89,114].

The remaining content of this paper is structured as follows: In Section 2, problem formulation for consensus of multi-agent systems with binary-valued measurements is given. In Section 3, the consensus algorithm is proposed. In Section 4, we give the properties of the algorithm. In Sectoin 5, simulations are given to verify the correctness of theoretical results. In Section 6, we draw conclusions and discuss some relevant future research directions.

The following notations will be used in this paper. $\mathbb{R}^{m \times n}$ denotes $m \times n$ dimensional real matrix space. $I_n$ represents $n \times n$ identity matrix. For vector or matrix $X$, $X^T$ represents the transpose of $X$. $\|X\|$ represents the norm of $X$, which is defined as $\|X\| = \sqrt{\mathbb{E}(X^TX)}$. $J_m$ is an $m$-dimensional square matrix with all elements being $1/m$.

## 2 Problem Formulation

Consider a multi-agent linear system on binary-valued measurement with $n$ agents:

$$x_i(t + 1) = Ax_i(t) + Bu_i(t), \quad i = 1, \ldots, n.$$  \hspace{1cm} (1)

where $x_i(t) \in \mathbb{R}^m$, $u_i(t) \in \mathbb{R}^k$, $A \in \mathbb{R}^{m \times m}$ and $B \in \mathbb{R}^{m \times k}$, $x_i(t)$ and $u_i(t)$ are the state and the control of agent $i$ at time $t$ respectively.

Each agent $i$ gets the state of its neighbors through binary-valued observations:

$$\begin{align*}
y_{ij}(t) &= x_j(t) + \omega_{ij}(t), \\
s_{ij}(t) &= I\{y_{ij}(t) \leq C_{ij}\}, \quad j \in N_i(t), \quad i = 1, \ldots, n, \end{align*}$$  \hspace{1cm} (2)

where $y_{ij}(t) \in \mathbb{R}^m$ is the unmeasurable output, $\omega_{ij}(t)$ is the noise when agent $i$ gets information from agent $j$ at time $t$, $x_j(t) \in \mathbb{R}^m$ is the state of agent $j$ at time $t$, $s_{ij}(t) \in \mathbb{R}^m$ is the binary-valued measurements, $C_{ij} \in \mathbb{R}^m$ is a given vector, $N_i(t)$ is set of all adjacent agents of $i$ at time $t$. 

The agents are distributed by Markov switching topologies. Denote the switching topologies as \( G_{m(t)} = \{N_{m(t)}, E_{m(t)}\} \), where \( G_{m(t)} \) is the topology of the multi-agent system at time \( t \), \( N_{m(t)} \) is the set of agents of the multi-agent system and \( E_{m(t)} \) is the set of edges at time \( t \). The number of topological structures of multi-agent systems is limited, \( G_{m(t)} \in S = \{G_1, G_2, ..., G_s\} \), where \( s \) is the finite state space. The topology of the multi-agent system is switched by the Markov chain. The transition probability from topology \( l \) to topology \( k \) at time \( t \),

\[
t_{lk} = Pr \left\{ G_{m(t+1)} = k \mid G_{m(t)} = l \right\},
\]

where \( l, k \in S \). The Markov transition probability matrix \( T \) is given by

\[
T = \begin{bmatrix}
  t_{11} & t_{12} & \cdots & t_{1s} \\
  t_{21} & t_{22} & \cdots & t_{2s} \\
  \vdots & \vdots & \ddots & \vdots \\
  t_{s1} & t_{s2} & \cdots & t_{ss}
\end{bmatrix}.
\]

The goal of this paper is to design control \( \{u_i(t), i = 1, \ldots, n\} \) by using binary-valued measurements from switching topologies such that all the agents’ states achieve consensus.

**Assumption 1.** The union of the Markovian communication topology \( G_{\text{un}} = \{G_1, G_2, ..., G_s\} \) is connected.

**Assumption 2.** The norm of matrix \( A \) satisfies \( \|A\| \leq 1 \), the matrix \( B \) is an orthogonal matrix.

**Assumption 3.** The noises \( \{\omega_{ij}(t), (i, j) \in G_{m(t)}, t = 1, 2, \ldots\} \) in system (2) are identically distributed random variables with mean 0. The noises are independent with respect to \( i, j, t \). The marginal distribution function is \( [F_1(\cdot), ..., F_n(\cdot)]^T \). And the associated marginal density function is \( [f_1(\cdot), ..., f_2(\cdot)]^T \), where \( f_i(x) = \frac{dF_i(x)}{dx} \neq 0 \), for \( i = 1, 2, ..., n \).

**Assumption 4.** All the elements in the state transition matrix are positive. That is to say, \( t_{lm} > 0 \), for all \( i, m = 1, ..., s \).

## 3 CONTROL ALGORITHM

In this paper, we use the estimate of neighbor’s state since the neighbors’ states can not be obtained precisely. We use the recursive projection algorithm with constant gain to estimate the states of neighbors. Then, the controller is designed with a constant gain. The consensus algorithm consists of the following five steps:

1. **Initialization:** The initial state value of each node and the initial state estimation of each node to adjacent nodes are given respectively as following:

\[
x_i(0) = x_i^0, \quad \hat{x}_{ij}(0) = \hat{x}_{ij}^0,
\]

for \( j \in N_i(s), i = 1, 2, ..., n, s \in S \), where \( \|x_i(0)\| \leq M, \|\hat{x}_{ij}(0)\| \leq M \). \( M \) is a given constant.

2. **Observation:** Each agent gets the binary-valued measurements by equation (2).

3. **Estimation:** Based on the binary-valued measurements, the adjacent agents’ states are estimated as follows.

\[
\hat{x}_{ij}(t) = \prod_{M} \left\{ A\hat{x}_{ij}(t-1) + \beta \left[ F( C_{ij} - A\hat{x}_{ij}(t-1)) - s_{ij}(t) \right] \right\}, \quad j \in N_i(t),
\]

where \( \beta \) is estimated step size, \( F(\cdot) \) is the vector of marginal distributed function and \( F(z) = [F_1(z_1), ..., F_n(z_n)]^T \) is satisfied for any \( z = [z_1, z_2, z_3, ..., z_n]^T \). \( \Omega \) is a bounded set \( \Omega = \{\omega \in \mathbb{R}^n : \|\omega\| \leq M\} \). \( \prod_{M}(\cdot) \) is a projection mapping from \( \mathbb{R}^n \) to \( \Omega \) defined as

\[
\prod_{\Omega}(\phi) = \arg\min_{\psi \in \Omega} \|\phi - \psi\|, \forall \phi \in \mathbb{R}^n.
\]

4. **State updating:** According to the above state estimation, the control of each agent can be designed as:

\[
u_i(t) = -\frac{B^TA}{N} \sum_{j \in N_i(t)} (x_i(t) - \hat{x}_{ij}(t)), \quad i = 1, ..., n,
\]

where \( N \) is a given constant. By the control, the updating equation for each agent is as follows:

\[
x_i(t + 1) = Ax_i(t) - \frac{BB^TA}{N} \sum_{j \in N_i(t)} (x_i(t) - \hat{x}_{ij}(t))
\]

\[
= Ax_i(t) - \frac{A}{N} \sum_{j \in N_i(t)} (x_i(t) - \hat{x}_{ij}(t)).
\]
5. When \( t = t + 1 \), repeat the second step.

**Remark 1.** \((\text{E1}, \text{proposition 6})\). The projection function (5) satisfies
\[
\left\| \prod_{\Omega} (x_1) - \prod_{\Omega} (x_2) \right\| \leq \left\| x_1 - x_2 \right\|,
\]
for any \( x_1, x_2 \in \mathbb{R}^n \).

**Remark 2.** Due to the definition of projection operator function in the state estimation update, we can get that the estimate is bounded by \( M \), i.e.,
\[
\left\| \hat{x}_{ij} (t) \right\| \leq M, \; j \in N_i, \; i = 1, \ldots, n.
\]

**Proposition 1.** The state of agent \( i \) satisfies
\[
\left\| x_i (t) \right\| \leq M,
\]
if \( N \geq d_i (t) \), where \( d_i (t) \) is the degree of agent \( i \) at time \( t \). All agent states satisfy the following inequality
\[
\left\| x_i (t) \right\| \leq M, \; i = 1, \ldots, n
\]
if \( N \geq d_* \), where \( d_* \) is the maximum degree of the topologies in \( S \).

**Proof.**
1) When agent \( i \) has no neighbors, the state of agent \( i \) is the state at the previous moment, that is \( x_i (t + 1) = x_i (t) \).
2) By the state updating (6), we can get
\[
x_i (t + 1) = Ax_i (t) + \frac{1}{N} A \sum_{j \in N_i (t)} \left( \hat{x}_{ij} (t) - x_j (t) \right)
\]
\[
= \left( 1 - \frac{d_i (t)}{N} \right) Ax_i (t) + \frac{1}{N} \sum_{j \in N_i (t)} A \hat{x}_{ij} (t),
\]
if \( N = d_i (t) \), then
\[
x_i (t + 1) = \frac{1}{d_i (t)} \sum_{j \in N_i (t)} A \hat{x}_{ij} (t),
\]
we can get
\[
\left\| x_i (t + 1) \right\| \leq \frac{1}{d_i (t)} \sum_{j \in N_i (t)} \left\| A \right\| \left\| \hat{x}_{ij} (t) \right\| \leq M.
\]
Assume \( \left\| x_i (t) \right\| \leq M \) holds when \( N \geq d_i (t) \). Then, we can get
\[
\left\| x_i (t + 1) \right\| \leq \left( 1 - \frac{d_i (t)}{N} \right) \left\| A \right\| \left\| x_i (t) \right\| + \frac{1}{N} \sum_{j \in N_i (t)} \left\| A \right\| \left\| \hat{x}_{ij} (t) \right\|
\]
\[
\leq \left( 1 - \frac{d_i (t)}{N} \right) M + \frac{d_i (t)}{N} M = M.
\]
By mathematical induction and Assumption 3, we can get
\[
\left\| x_i (t) \right\| \leq M, \; \text{if} \; N > d_i (t).
\]
If \( N \geq d_* \), we have
\[
\left\| x_i (t) \right\| \leq M, \; \; i = 1, \ldots, n.
\]
\( \square \)

The state updating (6) can also be written as follows:
\[
x_i (t + 1) = Ax_i (t) - \frac{A}{N} \sum_{j \in N_i (t)} (x_i (t) - x_j (t)) + \frac{A}{N} \sum_{j \in N_i (t)} (\hat{x}_{ij} (t) - x_j (t)).
\]
Putting all the link \((i, j), j \in N_i \) of the unit graph \( G_{un} \) in a given order as follows
\[
l_1 = (1, r_1), l_2 = (1, r_2), \ldots, l_{d_i} = (1, r_{d_i}), l_{d_i+1} = (2, r_{d_i+1}), \ldots, l_{d_i+d_2} = (2, r(d_1 + d_2)), \ldots, l_{d_i+\ldots+d_*} = (n, r_{d_i+\ldots+d_*}).
\]
To label the link at time $t$, also the start point and the end point of each link, we define the a constant $w_{ij}(t)$, and $n$-dimensional vectors $p_{ij}(t)$ and $q_{ij}(t)$ as follows:

$$w_{ij}(t) = 1, \quad p_{ij}(t) = \begin{pmatrix} 0, ..., 0, 1, 0, ..., 0 \end{pmatrix}_T^T, \quad q_{ij}(t) = \begin{pmatrix} 0, ..., 0, 1, 0, ..., 0 \end{pmatrix}_T^T, \quad \text{if } j \in N_i(t);$$

$$w_{ij}(t) = 0, \quad p_{ij}(t) = 0_{1 \times n}, \quad q_{ij}(t) = 0_{1 \times n}, \quad \text{if } j \notin N_i(t).$$

for all $j \in N_i, i = 1, ..., n$. Putting \{ $w_{ij}(t), j \in N_i, i = 1, ..., n, \}$, \{ $p_{ij}(t), j \in N_i, i = 1, ..., n \}$ and \{ $q_{ij}(t), j \in N_i, i = 1, ..., n \}$ in the order as $l_1, l_2, ..., l_{d_i + +d_s}$, we can get matrixes $W_{m(t)}$, $P_{m(t)}$ and $Q_{m(t)}$ as follows:

$$W_{m(t)} = \text{diag}(w_{1r_1}, w_{2r_1}, ..., w_{2r_i}, ..., w_{2r_{d_i+1}}, ..., w_{nr_i+1 + d_s+1}, ..., w_{nr_i+1 + d_s});$$

$$P_{m(t)} = \begin{bmatrix} p_{1r_1}, ..., p_{2r_1}, p_{2r_1}, ..., p_{2r_1}, ..., p_{nr_i+1 + d_s}, ..., p_{nr_i+1 + d_s} \end{bmatrix};$$

$$Q_{m(t)} = \begin{bmatrix} q_{1r_1}^T, ..., q_{1r_1}^T, q_{2r_1+1}^T, ..., q_{nr_i+1 + d_s}^T \end{bmatrix}_{(d_i + +d_s) \times n.}$$

Let $\epsilon_{ij}(t) = \hat{x}_{ij}(t) - x_{ij}(t)$. Putting $\epsilon_{ij}(t)$ int the same order as matrix $P_{m(t)}^T$, we can get vector $\epsilon(t)$ as follows.

$$\epsilon(t) = \begin{bmatrix} \epsilon_{1r_1}(t), ..., \epsilon_{1r_1}(t), ..., \epsilon_{nr_i+1 + d_s} \end{bmatrix}.$$
then
\[
\lambda_i (C + D) \geq \begin{cases} 
\lambda_i (C) + \lambda_1 (D), \\
\lambda_{i-1} (C) + \lambda_2 (D), \\
\vdots \\
\lambda_1 (C) + \lambda_i (D).
\end{cases}
\] (16)

**Lemma 2.** The matrix \( \sum_{i=1}^n \phi_i (t) L_i \) is nonnegative definite with rank \( n - 1 \), where \( \phi_i (t) \) is the possibility of topology \( G_i \) occurs at time \( t \), \( L_i \) is the Laplacian Matrix of topology \( G_i \).

The proof of Lemma 2 is put in the appendix A.

Let \( \delta (t) = x (t) - J_n x (t) \). Then, define the mean square error of estimation \( R(t) \) and the consensus error \( V(t) \) as follows.

\[
R(t) = E \left( \varepsilon^T (t) \varepsilon (t) \right), \quad V(t) = E \left( \delta^T (t) \delta (t) \right).
\]

We can get two lemmas about \( R(t) \) and \( V(t) \).

**Lemma 3.** The consensus error \( V(t) \) satisfies
\[
V(t + 1) \leq 2 \left( 1 - \frac{\lambda_2}{N} \right)^2 V(t) + \frac{2\lambda_P}{N^2} R(t),
\] (17)

where \( \lambda_2 \) is the smallest eigenvalues of the matrix \( L_{m(t)} \) in \( \{ G_1, G_2, ..., G_s \} \), \( \lambda_P \) is the maximum norm of the matrix \( P_{m(t)}^T P_{m(t)} \) in \( \{ G_1, G_2, ..., G_s \} \), respectively.

The proof of Lemma 3 is put in the appendix B.

**Lemma 4.** The estimation error \( R(t) \) satisfies
\[
R(t + 1) \leq 2 \left( 1 - \beta f_{\min} \lambda_w \right)^2 \frac{\lambda_Q \lambda_P}{N^2} V(t) + 2 \left( 1 - \beta f_{\min} \lambda_w \right)^2 \left( 1 - \frac{\lambda_Q \lambda_P}{N^2} \right)^2 R(t) + \frac{\beta^2 \lambda_w^2}{4},
\] (18)

where \( \lambda_Q \) is the maximum eigenvalue of matrix \( Q_{m(t)}^T Q_{m(t)} \) in \( \{ G_1, G_2, ..., G_s \} \), \( \lambda_n \) is the maximum eigenvalue of the matrix \( L_{m(t)} \) in \( \{ G_1, G_2, ..., G_s \} \), \( \lambda_w \) is the maximum eigenvalue of the matrix \( W_{m(t)} \) in \( \{ G_1, G_2, ..., G_s \} \), \( f_{\min} = \min_{k=1,...,n} f_k (\max_{j \in N_i} c_{ij} (k) + M) \), and \( c_{ij} = [c_{ij} (1), ..., c_{ij} (n)]^T \) is the threshold vector, \( f_1 (\cdot), ..., f_n (\cdot) \) are the marginal desity functions of the noise.

The proof of Lemma 4 is put in the appendix C.

**Theorem 1.** Under the assumptions 1-4, choosing \( N \geq \lambda_Q \lambda_n \), the consensus error satisfies
\[
E \left( \delta^T (t) \delta (t) \right) \leq \frac{\beta^2 \lambda_w^2}{4 - 4a} + O \left( a' \right),
\]

where \( a \) is a constant. And, states of agents in linear system will reach consensus in a bounded range.

\[
\lim_{t \to \infty} E \left( \delta^T (t) \delta (t) \right) \leq \frac{\beta^2 \lambda_w^2}{4 - 4a},
\] (19)

if the estimated step size \( \beta \) is selected by
\[
\left( 1 - \beta f_{\min} \lambda_w \right)^4 > \frac{\left( 1 - \frac{\lambda_2}{N} \right)^4 + \frac{\lambda_P^2}{N^2} - \frac{1}{4}}{\left( 2 \left( 1 - \frac{\lambda_2}{N} \right)^2 \left( 1 - \frac{\lambda_Q \lambda_P}{N^2} \right)^2 - 2 \frac{\lambda_P^2 \lambda_Q \lambda_n}{N^4} \right)^2} - \left( 1 - \frac{\lambda_Q \lambda_P}{N^2} \right)^4 \frac{\lambda_Q^2}{N^4}.
\] (20)

**Proof.** By Lemma 3 and Lemma 4, we have
\[
\begin{align*}
V(t + 1) & \leq 2 \left( 1 - \frac{\lambda_2}{N} \right)^2 V(t) + \frac{2\lambda_P}{N^2} R(t), \\
R(t + 1) & \leq 2 \left( 1 - \beta f_{\min} \lambda_w \right)^2 \frac{\lambda_Q \lambda_P}{N^2} V(t) + 2 \left( 1 - \beta f_{\min} \lambda_w \right)^2 \left( 1 - \frac{\lambda_Q \lambda_P}{N^2} \right)^2 R(t) + \frac{\beta^2 \lambda_w^2}{4},
\end{align*}
\] (21)
where \( N \geq \lambda_O \lambda_n \). Let

\[
Z(t) = \begin{pmatrix} V(t) \\ R(t) \end{pmatrix}, \quad D = \begin{pmatrix} 0 \\ \frac{p^2 \zeta_c^2}{4} \end{pmatrix}.
\]

Let \( \zeta = (1 - \beta \lambda_{\text{min}})^2 \), and

\[
H = \begin{pmatrix} 2 \left( 1 - \frac{\lambda_n}{N} \right)^2, & 2 \frac{2 \lambda \lambda_{\text{max}}}{N^2} \\ 2 \frac{\lambda \lambda_{\text{max}}}{N^2}, & 2 \zeta \left( 1 - \frac{\alpha \lambda_{\text{max}}}{N} \right)^2 \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix},
\]

where \( e = 2 \left( 1 - \frac{\lambda_n}{N} \right)^2, f = \frac{2 \lambda \lambda_{\text{max}}}{N^2}, g = 2 \zeta \left( 1 - \frac{\alpha \lambda_{\text{max}}}{N} \right)^2, h = 2 \zeta \left( 1 - \frac{\alpha \lambda_{\text{max}}}{N} \right)^2 \).

By inequality (21), we have

\[
Z(t) \leq H Z(t-1) + D.
\]

Let \( a = \| H \| \), we can get by (22)

\[
V(t) \leq \| Z(t) \| \leq \| H Z(t-1) + D \|
\leq \| H \| \| Z(t-1) \| + \frac{\beta^2 \lambda_{\text{max}}^2}{4}
\leq a^{d-1} \| Z(1) \| + \frac{\beta^2 \lambda_{\text{max}}^2}{4} (1 - a^{-1})
= O(a^d) + \frac{\beta^2 \lambda_{\text{max}}^2}{4} (1 - a^{-1}).
\]

Let’s find the eigenvalues of coefficient matrix \( H^T H \).

\[
\begin{pmatrix} \lambda - (e^2 + g^2), & -(ef + gh) \\ -(ef + gh), & \lambda - (f^2 + h^2) \end{pmatrix},
\]

\[
\lambda^2 - (e^2 + f^2 + h^2 + g^2) \lambda + (eh - gf)^2 = 0,
\]

we need \( a = \lambda_{\text{max}}(H^T H) < 1 \). Let

\[
\frac{(e^2 + f^2 + h^2 + g^2) + \sqrt{(e^2 + f^2 + h^2 + g^2)^2 - 4 (eh - gf)^2}}{2} < 1.
\]

we have range of estimated step \( \beta \),

\[
(1 - \beta \lambda_{\text{min}})^4 > \frac{\left( 1 - \frac{\lambda_n}{N} \right)^4 + \frac{\lambda_{\text{max}}^2}{N^4} - \frac{1}{4}}{2 \left( 1 - \frac{\lambda_n}{N} \right)^2 \left( 1 - \frac{\alpha \lambda_{\text{max}}}{N} \right)^2 - 2 \frac{\lambda \lambda_{\text{max}}}{N^2} \left( 1 - \frac{\alpha \lambda_{\text{max}}}{N} \right)^2 - \left( 1 - \frac{\alpha \lambda_{\text{max}}}{N} \right)^4 - \frac{\lambda_{\text{max}}^2}{N^4}}.
\]

\[\square\]

**Theorem 2.** Under the assumptions 1-4, choosing \( N \geq \lambda_O \lambda_n \), the mean square error of estimation satisfies

\[
E \left( \epsilon(t)^T \epsilon(t) \right) \leq \frac{\beta^2 \lambda_{\text{max}}^2}{4 - 4d} + O \left( a^d \right).
\]

And, the estimation error of the neighbor state will decrease to a bounded range.

\[
\lim_{t \to \infty} E \left( \epsilon(t)^T \epsilon(t) \right) \leq \frac{\beta^2 \lambda_{\text{max}}^2}{4 - 4d},
\]

if the estimated step size \( \beta \) is selected by

\[
(1 - \beta \lambda_{\text{min}})^4 > \frac{\left( 1 - \frac{\lambda_n}{N} \right)^4 + \frac{\lambda_{\text{max}}^2}{N^4} - \frac{1}{4}}{2 \left( 1 - \frac{\lambda_n}{N} \right)^2 \left( 1 - \frac{\alpha \lambda_{\text{max}}}{N} \right)^2 - 2 \frac{\lambda \lambda_{\text{max}}}{N^2} \left( 1 - \frac{\alpha \lambda_{\text{max}}}{N} \right)^2 - \left( 1 - \frac{\alpha \lambda_{\text{max}}}{N} \right)^4 - \frac{\lambda_{\text{max}}^2}{N^4}}.
\]

The proof is omitted since it is similar to that of Theorem 1.
FIGURE 1 The topology $G_1$ of five agents

FIGURE 2 The topology $G_2$ of five agents

**Remark 3.** Let $\beta_G = \frac{1 - \lambda_2 N^4 + \lambda_2 P N^4 - 1}{2(1 - \lambda_2 N^2)^2 - 2 \lambda_2 Q \lambda_2 N^4 - 1}$, which is determined by the topology. If $0 < \beta_G < 1$, the step size $\beta$ should be smaller than a constant or bigger than a constant by condition (20), which is to ensure the norm of $H$ be less than 1. But, too big $\beta$ leads to a big consensus error by (19). So, the step size $\beta$ should be chosen smaller than a constant.

### 5 SIMULATION

Consider a multi-agent system with five nodes, and its topology is shown in Figure 1 and Figure 2. The corresponding Laplacian matrices are given as $L_1$ and $L_2$ in the following.

\[
L_1 = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 \\
0 & 0 & -1 & 2 & 1 \\
-1 & 0 & 0 & -1 & 2
\end{bmatrix}.
\]

Then, we can get the Laplacian matrix of the united graph $G_{\text{un}}$ as follows

\[
L = \begin{bmatrix}
2 & -1 & 0 & 0 & -1 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 \\
0 & 0 & -1 & 2 & 1 \\
-1 & 0 & 0 & -1 & 2
\end{bmatrix},
\]

which shows that the union of the Markovian communication topologies $\{G_1, G_2\}$ is connected.

In this multi-agent system, each agents’ state is composed of three-dimensional vectors, which are updated by

\[
x_i(t+1) = Ax_i(t) + Bu_i(t), \quad i = 1, ..., 5,
\]

where $x_i(t) \in R^3, u_i(t) \in R^3, A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Each agent can only obtain binary information from neighbors.

\[
\begin{align*}
\{ y_{ij}(t) &= x_j(t) + \omega_{ij}(t), \\
\hat{s}_{ij}(t) &= I \{ y_{ij}(t) \leq C_{ij}\}, \quad j \in N_i, i = 1, ..., n,
\end{align*}
\]

where $\omega_{ij}(t) \in R^3$ is the noise generated by joint normal distribution, the threshold $C_{ij}$ equals 0, and $s_{ij} \in R^3$ is the binary-valued observation.

Let the initial value of the state be $x(1) = [(-5, -2, -2)^T, (5, 3, 3)^T, (7, -4, 4)^T, (-4, 5, 5)^T, (2, 6, 6)^T]^T$ and the initial estimations be $\hat{x}(1) = [(0, 1, 1)^T, (0, 1, 1)^T, (0, 1, 1)^T, (0, 1, 1)^T, (0, 1, 1)^T]^T$. 

Take $G_{1}$ for example, $P_{1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $Q_{1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, $W_{1} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$, where $W_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, $W_{12} = W_{21} = W_{22} = 0_{5 \times 5}$. $\lambda_{P} = \max \{ \lambda_{\max} (P_{1}^{T} P_{1}), \lambda_{\max} (P_{2}^{T} P_{2}) \} = 2$, $\lambda_{Q} = \max \{ \lambda_{\max} (Q_{1}^{T} Q_{1}), \lambda_{\max} (Q_{2}^{T} Q_{2}) \} = 2$, $\lambda_{\mu} = 3.4142$, $\lambda_{\sigma} = 1$. Let $N = 20$, $M = 10$, $\mu = 0$ and $\sigma = 10$, we can get $f_{\min} = 0.0242$. By the condition (20) in Theorem 1, we can get $\beta < 4.7933$ or $\beta > 77.85$.

Let $\beta = 0.2 < 4.7933$, the trajectories of the states’ component are given in Figures 3-5. From Figures 3-5, we can see that the states of agents keep unchange sometimes. Because the topologies are switching and the agent may not be the neighbors at some times. By choosing $\beta = 0.2$, we can see that all the components of the states reach consensus in a bounded reigon, which is consistent with Theorem 1. Let $y_{1}(t) = 0.95 V(0)$, $y_{2}(t) = 0.8 V(0)$. Figure 6 shows that the consensus error $V(t)$ is between the $y_{1}(t)$ and $y_{2}(t)$, which implies that the consensus error converges with an exponential rate $O(\alpha t)$. The result is consistent with Theorem 1.
Figure 7 gives the consensus error $V(t)$ with different step size $\beta$. From Figure 7, we can see that larger $\beta$ leads to larger consensus error, which is consistent with Remark 3.

Let $r_1(t) = 0.95' R(0)$, $r_2(t) = 0.8' R(0)$. Figure 8 shows that the estimate error $R(t)$ is between the $r_1(t)$ and $r_2(t)$, which implies that the consensus converges in the rate of $O(a^t)$, $a < 1$. The result is consistent with Theorem 2.

6 CONCLUSION

In this paper, a consensus algorithm with binary-valued observations and Markov switching topology is proposed. The topology of multi-agent system changes according to Markov process. When the agent is not connected to the neighbor, the agent estimates the neighbor as the value of the previous moment. When the agent is connected the neighbor, estimation are updated based on binary-valued measurement. The states of systems under consensus control with a constant gain can achieve consensus. It is proved that the consensus speed can reach $O(a^t)$, where $a < 1$ is a constant.

In this paper, we assume that the norm of coefficient matrix of the linear multi-agent system is less than or equal to 1, but in real life, most of the coefficient matrices we encounter are more general, which may not satisfy the condition. Besides, we take the projection method to update the estimates in order to ensure the boundedness of estimates. If there is no projection function for estimation, could consensus be achieved? In the future, we will study a general coefficient matrix multi-agent systems on binary-valued measurement without projection method.

References


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APPENDIX

A PROOF OF LEMMA 2

Let \( \rho(1) = [\varrho_1(1), \varrho_2(1), \ldots, \varrho_s(1)] \) be the initial probability of different topologies. According to the Markov transition probability matrix, we can get

\[
\rho(2) = \rho(1)T = [\varrho_1(1), \varrho_2(1), \ldots, \varrho_s(1)] \\
\begin{bmatrix}
    t_{11} & t_{12} & \cdots & t_{1s} \\
    t_{21} & t_{22} & \cdots & t_{2s} \\
    \vdots & \vdots & \ddots & \vdots \\
    t_{s1} & t_{s2} & \cdots & t_{ss}
\end{bmatrix}
\]  

(A1)

Under Assumption 4, we know that there are non-zero values in \( \rho(1) \) and \( t_{im} > 0 \), for \( i, m = 1, \ldots, s \). So we can get

\[
\sum_{i=1}^{s} \varrho_i(1)t_{im} > 0, \quad m = 1, 2, \ldots, s
\]  

(A2)

From equation (A2), we know that when \( t = 2 \), every topology has a non-zero probability. Suppose \( \varrho_i(t-1) > 0 \), by Markov basic formula. we can get \( \rho(t) = \rho(t-1)T = \sum_{i=1}^{s} \varrho_i(t-1)t_{i1}empt; \sum_{i=1}^{s} \varrho_i(t-1)t_{i2}, \ldots, \sum_{i=1}^{s} \varrho_i(t-1)t_{is} \). So \( \varrho_i(t) > 0 \), for \( t > 1, i = 1, \ldots, s \). Prove in a similar way to (Lemma 2), we can get \( \sum_{i=1}^{s} \varrho_i(t)L_m \geq 0 \) and eigenvalues of \( \sum_{i=1}^{s} \varrho_i(t)L_m \) only one is 0.

B PROOF OF LEMMA 3

Denote \( \Xi_n = (I_n \otimes I_n - J_n \otimes I_n), \zeta(t) = \Xi_n(I_n \otimes A)(P_m(t) \otimes I_n) \varepsilon(t) \). By the state updating (14), we can get

\[
\delta(t+1) = \Xi_nx(t+1) \\
= \Xi_n \left( \left( I_n \otimes A - L_{mt} \otimes \frac{A}{N} \right)x(t) \right) - \frac{1}{N} \zeta(t) \\
= \left( I_n \otimes A - J_n \otimes A - L_{mt} \otimes \frac{A}{N} \right)x(t) - \frac{1}{N} \zeta(t) \\
= \left( I_n \otimes A - L_{mt} \otimes \frac{A}{N} \right)x(t) - \frac{1}{N} \zeta(t) \\
= \left( I_n \otimes A - \frac{L_{mt} \otimes A}{N} \right) \delta(t) - \frac{1}{N} \zeta(t) .
\]  

(B3)

we can get

\[
V(t+1) = E(\delta(t+1)^T \delta(t+1)) = E(\left( x(t+1)^T \Xi_n^T \Xi_n x(t+1) \right)) \\
= E \left\{ \left( I_n \otimes A - \frac{L_{mt} \otimes A}{N} \right)x(t) + \frac{1}{N} \zeta(t) \right\}^T \Xi_n^T \Xi_n \left[ \left( I_n \otimes A - \frac{L_{mt} \otimes A}{N} \right)x(t) + \frac{1}{N} \zeta(t) \right] \\
= E \left[ x(t)^T \left( I_n \otimes A - \frac{L_{mt} \otimes A}{N} \right)^T \Xi_n^T \Xi_n \left( I_n \otimes A - \frac{L_{mt} \otimes A}{N} \right)x(t) \right] \\
+ 2E \left[ x(t)^T \left( I_n \otimes A - \frac{L_{mt} \otimes A}{N} \right)^T \Xi_n^T \Xi_n \left( I_n \otimes A \right) \left( \frac{P_m(t)}{N} \otimes I_n \right) \varepsilon(t) \right] \\
+ E \left[ \varepsilon(t)^T \left( \frac{P_m(t)}{N} \otimes I_n \right)^T (I_n \otimes A)^T (I_n \otimes A) \left( \frac{P_m(t)}{N} \otimes I_n \right) \varepsilon(t) \right].
\]  

(B4)
The first part of the equation (B4), we have
\[
E \left[ x(t)^T \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right)^T \Xi_n^T \Xi_n \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right) x(t) \right]
\]
\[
= E \left[ x(t)^T \Xi_n^T \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right)^T \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right) \Xi_n x(t) \right]
\]
\[
= E \left[ \delta(t) \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right)^T \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right) \delta(t) \right]
\]
\[
= E \left[ \delta(t) \left( I_n \otimes A - \frac{\sum_{i=1}^t \theta_i(t) L_i \otimes A}{N} \right)^T \left( I_n \otimes A - \frac{\sum_{i=1}^t \theta_i(t) L_i \otimes A}{N} \right) \delta(t) \right].
\]

Let
\[
\delta(t) = U \tilde{\delta}(t) = [\tilde{\delta}_1(t), \tilde{\delta}_2(t)],
\]
where \(U\) is orthogonal matrix, \(U^T \sum_{i=1}^t \theta_i(t) L_i U = diag(1, 1 - \frac{\lambda_2}{N}, ..., 1 - \frac{\lambda_n}{N})\), since \(e_i = (1, 1, ...1)^T\), we have
\[
\tilde{\delta}_1(t) = e_1 \delta(t) = \sum_{i=1}^n \delta_i(t) = 0.
\]
we can have
\[
E \left\{ \tilde{\delta}_2(t)^T \left[ \text{diag} \left( 1 - \frac{\lambda_2}{N}, ..., 1 - \frac{\lambda_n}{N} \right) \otimes A \right] \left[ \text{diag} \left( 1 - \frac{\lambda_2}{N}, ..., 1 - \frac{\lambda_n}{N} \right) \otimes A \right] \tilde{\delta}_2(t) \right\} \leq \left( 1 - \frac{\lambda_2}{N} \right)^2 E \left( \tilde{\delta}_2(t)^T \tilde{\delta}_2(t) \right).
\]

Then, we have
\[
E \left[ x(t)^T \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right)^T \Xi_n^T \Xi_n \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right) x(t) \right] \leq \left( 1 - \frac{\lambda_2}{N} \right)^2 V(t),
\]
where \(\lambda_2\) is the smallest eigenvalue of matrix \(L_m(t)\) in \(\{G_1, G_2, ..., G_s\}\).
The second part of the equation (B4), we have
\[
2E \left[ x(t)^T \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right)^T \Xi_n^T \Xi_n \left( I \otimes A \right) \left( \frac{P_m(t) \otimes I_n}{N} \right) \varepsilon(t) \right]
\]
\[
\leq E \left[ x(t)^T \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right)^T \Xi_n^T \Xi_n \left( I_n \otimes A - \frac{L_m(t) \otimes A}{N} \right) x(t) \right]
\]
\[
+ E \left[ \varepsilon(t)^T \left( \frac{P_m(t)}{N} \otimes I_n \right)^T \left( I \otimes A \right)^T \Xi_n^T \Xi_n \left( I \otimes A \right) \left( \frac{P_m(t)}{N} \otimes I_n \right) \varepsilon(t) \right]
\]
\[
\leq \left( 1 - \frac{\lambda_2}{N} \right)^2 V(t) + \frac{\lambda_p}{N^2} R(t).
\]
The third part of the equation (B4), we have
\[
E \left[ \varepsilon(t)^T \left( \frac{P_m(t)}{N} \otimes I_n \right)^T \left( I \otimes A \right)^T \left( \frac{P_m(t)}{N} \otimes I_n \right) \varepsilon(t) \right] \leq \frac{\lambda_p}{N^2} R(t).
\]
From the (B7), (B10), (B11) inequality, we have
\[ E \left( \delta (t+1)^T \delta (t+1) \right) \leq 2 \left( 1 - \frac{\lambda_2}{N} \right)^2 V(t) + \frac{2\lambda_P}{N^2} R(t). \]  

**C PROOF OF LEMMA 4**

Let the estimation vector be as follows
\[ \hat{x}(t) = \left[ \hat{x}_{1r}(t), \ldots, \hat{x}_{1d_1}(t), \ldots, \hat{x}_{nr_{d_1+d_2+\ldots+d_i}}(t), \ldots, \hat{x}_{nr_{d_1+d_2+\ldots+d_i+1}}(t), \ldots \right]^T, \]

where \( r_{d_1+\ldots+d_i+1}, \ldots, r_{d_1+\ldots+d_i+1} \in N, i = 1, 2, \ldots, n \). By estimation (3), the vector form of estimation updating can be given:
\[ \hat{x}(t) = \prod_{m=1}^t \left\{ (A \otimes I_n) \hat{x}(t-1) + \beta (W_m(\otimes I_n) \left( F(C - (A \otimes I_n) \hat{x}(t-1) - s(t) \right) \right\}, \]

where \( W_{m(t)} \) is defined in equation (13). By Assumption (1), we can get \( \sum_{m(t)} W_{m(t)} \geq I \).

By estimation (1) and estimate updating (C14), we can get
\[ R(t+1) = E \left( \epsilon(t+1)^T \epsilon(t+1) \right) \]

\[ \leq E \left\{ \left[ (I_n \otimes A) \hat{x}(t) + \beta (W_m(\otimes I_n) \left( F(C - (A \otimes I_n) \hat{x}(t) - s(t+1) \right) - (Q_m(\otimes I_n) x(t+1) \right)^T \left[ (I_n \otimes A) \hat{x}(t) + \beta (W_m(\otimes I_n) \left( F(C - (A \otimes I_n) \hat{x}(t) - s(t+1) \right) - (Q_m(\otimes I_n) x(t+1) \right) \right] \right\} + \]

\[ 2\beta E \left\{ \left( W_m(\otimes I_n) \left( F(C - (A \otimes I_n) \hat{x}(t) - s(t+1) \right) \right)^T \left( W_m(\otimes I_n) \left( F(C - (A \otimes I_n) \hat{x}(t) - s(t+1) \right) \right) \right\} \]

We have
\[ (I_n \otimes A) \hat{x}(t) = (I_n \otimes A) \hat{x}(t) - (Q_m(\otimes I_n) x(t+1) \]

\[ = (I_n \otimes A) \hat{x}(t) - (Q_m(\otimes I_n) \left( I_n - \frac{1}{N} (Q_m P_m(\otimes I_n) \right) x(t) + (Q_m(\otimes I_n) \left( \frac{L_m(\otimes A)}{N} \right) x(t) \)

Since (C10), the first part of the equation (C15), we have
\[ E \left\{ \left( I_n \otimes A \right) \hat{x}(t) - (Q_m(\otimes I_n) x(t+1) \right\} ^T \left( I_n \otimes A \right) \hat{x}(t) - (Q_m(\otimes I_n) x(t+1) \right\} \]

\[ = E \left\{ \left( I_n \otimes A \right) \left( I_n - \frac{1}{N} (Q_m P_m(\otimes I_n) \right) x(t) + (Q_m(\otimes I_n) \left( \frac{L_m(\otimes A)}{N} \right) x(t) \right\} \right\} \]

\[ = E \left\{ \left( I_n \otimes A \right) \left( I_n - \frac{1}{N} (Q_m P_m(\otimes I_n) \right) x(t) + (Q_m(\otimes I_n) \left( \frac{L_m(\otimes A)}{N} \right) x(t) \right\} \right\} \]

\[ + 2E \left\{ \left( I_n \otimes A \right) \left( I_n - \frac{1}{N} (Q_m P_m(\otimes I_n) \right) x(t) + (Q_m(\otimes I_n) \left( \frac{L_m(\otimes A)}{N} \right) x(t) \right\} \right\} \]

\[ + E \left\{ \left( I_n \otimes A \right) \left( I_n - \frac{1}{N} (Q_m P_m(\otimes I_n) \right) x(t) + (Q_m(\otimes I_n) \left( \frac{L_m(\otimes A)}{N} \right) x(t) \right\} \right\} \]

The first part of the equation (C17), we have
\[ E \left\{ \left( I_n \otimes A \right) \left( I_n - \frac{1}{N} (Q_m P_m(\otimes I_n) \right) x(t) + (Q_m(\otimes I_n) \left( \frac{L_m(\otimes A)}{N} \right) x(t) \right\} \right\} \]

\[ \leq \left( 1 - \frac{\lambda Q \lambda_P}{N} \right)^2 R(t). \]
The second part of the equation (C17), we have
\[
2E \left\{ \left[ (I_n \otimes A) \left( I_n - \frac{1}{N} \left( Q_{m(t)} P_{m(t)} \otimes I_n \right) \right) \varepsilon (t) \right]^T \left[ \left( Q_{m(t)} \otimes I_n \right) \left( \frac{L_{m(t)} \otimes A}{N} \right) \right] x (t) \right\} \\
\leq E \left\{ \left[ (I_n \otimes A) \left( I_n - \frac{1}{N} \left( Q_{m(t)} P_{m(t)} \otimes I_n \right) \right) \varepsilon (t) \right]^T \left[ \left( I_n \otimes A \right) \left( I_n - \frac{1}{N} \left( Q_{m(t)} P_{m(t)} \otimes I_n \right) \right) \varepsilon (t) \right] \right\} \\
+ E \left\{ \left[ \left( Q_{m(t)} \otimes I_n \right) \left( \frac{L_{m(t)} \otimes A}{N} \right) x (t) \right]^T \left[ \left( Q_{m(t)} \otimes I_n \right) \left( \frac{L_{m(t)} \otimes A}{N} \right) x (t) \right] \right\} \\
\leq \left( 1 - \frac{\lambda_p A_2}{N^2} \right)^2 R(t) + \frac{\lambda_p A_2}{N^2} V(t).
\]

The third part of the equation (C17), we have
\[
E \left\{ \left[ \left( Q_{m(t)} \otimes I_n \right) \left( \frac{L_{m(t)} \otimes A}{N} \right) x (t) \right]^T \left[ \left( Q_{m(t)} \otimes I_n \right) \left( \frac{L_{m(t)} \otimes A}{N} \right) x (t) \right] \right\} \leq \frac{\lambda_n A_2}{N^2} V(t).
\]

Then from (C18)-(C20) inequality, we have
\[
E \left\{ \left[ (I_n \otimes A) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right] \right\}^T \left[ \left[ (I_n \otimes A) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right] \right\} \\
\leq 2 \left( 1 - \frac{\lambda_n A_2}{N^2} \right)^2 R(t) + 2 \frac{\lambda_n A_2}{N^2} V(t),
\]
where \( \lambda_n \) is the maximum eigenvalue of the matrix \( L_{m(t)} \) in \( \{G_1, G_2, ..., G_s\} \).

According to the mean value theorem, we can get
\[
F \left( C - (A \otimes I_n) \hat{x}(t) \right) - F \left( C - (Q_{m(t+1)} \otimes I_n) x(t+1) \right) \\
= -f \left( \phi \right) \left[ \left( I_n \otimes A \right) \hat{x}(t) - (Q_{m(t+1)} \otimes I_n) x(t+1) \right].
\]

By equation (C22), we have
\[
E \left\{ \left[ W_{m(t)} \otimes I_n \right] \left( F \left( C - (A \otimes I_n) \hat{x}(t) \right) - s(t+1) \right) \right\} \\
\leq -f \min \lambda_w E \left\{ \left[ I_n \otimes A \right] \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right\},
\]
where \( \lambda_w \) is the maximum eigenvalue of the matrix \( W_{m(t)} \) in \( \{G_1, G_2, ..., G_s\} \). By the equation (C23), we have the third part of the equation (C15)
\[
\beta^2 E \left\{ \left[ (W_{m(t)} \otimes I_n) \left( F \left( C - (A \otimes I_n) \hat{x}(t) \right) - s(t+1) \right) \right]^T \left[ \left( W_{m(t)} \otimes I_n \right) \left( F \left( C - (A \otimes I_n) \hat{x}(t) \right) - s(t+1) \right) \right] \right\} \\
\leq \beta^2 f_{\min}^2 \lambda_w E \left\{ \left[ \left( I_n \otimes A \right) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right]^T \left[ \left( I_n \otimes A \right) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right] \right\} \\
+ \beta^2 E \left\{ \left[ \left( I_n \otimes A \right) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right]^T \left[ \left( I_n \otimes A \right) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right] \right\} \\
\leq \beta^2 f_{\min}^2 \lambda_w E \left\{ \left[ \left( I_n \otimes A \right) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right]^T \left[ \left( I_n \otimes A \right) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right] \right\} + \frac{\beta^2 \lambda_w^2}{4}
\]

Be similar to equation (C24), we have the second part of the equation (C15)
\[
2\beta E \left\{ \left[ (I_n \otimes A) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right]^T \left[ (W_{m(t)} \otimes I_n) \left( F \left( C - (A \otimes I_n) \hat{x}(t) \right) - s(t+1) \right) \right] \right\} \\
\leq -2\beta f_{\min} \lambda_w E \left\{ \left[ \left( I_n \otimes A \right) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right]^T \left[ \left( I_n \otimes A \right) \hat{x}(t) - (Q_{m(t)} \otimes I_n) x(t+1) \right] \right\}
\]

By (C15), (C21), (C24) and (C25), we can get
\[
R(t+1) = E \left( \varepsilon (t+1)^T \varepsilon (t+1) \right) \\
\leq 2 \left( 1 - \beta f_{\min} \lambda_w \right)^2 \left( 1 - \frac{\lambda_p A_2}{N^2} \right)^2 E \left( \varepsilon (t)^T \varepsilon (t) \right) + 2 \left( 1 - \beta f_{\min} \lambda_w \right)^2 \frac{\lambda_n A_2}{N^2} E \left( \delta(t)^T \delta(t) \right) + \frac{\beta^2 \lambda_w^2}{4}.
\]