Sparse Regression Codes for Non-Coherent SIMO channels

Sai Dinesh Kancharana, Madhusudan Kumar Sinha, and Arun Pachai

1Department of Electrical Engineering, Indian Institute of Technology Madras

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Sai Dinesh Kancharana †, Madhusudan Kumar Sinha ‡, Arun Pachai Kannu ‡.
Department of Electrical Engineering
Indian Institute of Technology Madras, Chennai - 600036, India
Email: ※ee20d401@smail.iitm.ac.in, ‡ee16d028@smail.iitm.ac.in, ‡arunpachai@ee.iitm.ac.in

Abstract—We study the sparse regression codes over flat-fading channels with multiple receive antennas. We consider a practical scenario where the channel state information is not available at the transmitter and the receiver. In this setting, we study the maximum likelihood (ML) detector for SPARC, which has a prohibitively high search complexity. We propose a novel practical decoding algorithm, named maximum likelihood matching pursuit (MLMP), which incorporates a greedy search mechanism along with the ML metric. Using simulation results, we show that MLMP has significant gains over the block-OMP based decoders considered in [7]. We also compare with a pilot-based scheme with polar codes, where the estimated channel is used to decode the codeword. With our proposed MLMP decoder, SPARC shows significant gains over pilot-aided polar codes.

Notation: Scalars are small case letters x, vectors and matrices are bold font lower x and upper-case X, respectively. x* is the hermitian of x and ||x|| is the ℓ2 norm. log and log2 represent the natural and base-2 logarithm respectively. For any positive integer N, [N] represents the set {1, 2, . . . , N}. (a, b) is the inner product between a and b. |X| is the size of the set X. CN(μ, σ2) is the complex Gaussian pdf with mean μ and variance σ2.

I. INTRODUCTION

Sparse regression codes, introduced in [1], have been shown to be capacity-achieving for the AWGN channel with an approximate message passing (AMP) decoder [2]. Spatial coupling techniques [3] and power allocation techniques [4] for SPARC have been studied in the literature. SPARC for small block lengths in AWGN has been studied in [5].

Motivated by ultra-reliable low latency communications (URLLC), we consider short block lengths and study SPARC in flat-fading, single-input multi-output (SIMO) channels. We focus on the non-coherent model where the channel state information (CSI) is unavailable at the transmitter and the receiver. In this setting, sparse vector coding (SVC) [6], which is closely related to SPARC, has been studied in [7], [8]. In [6], the authors have used the multi-path matching pursuit algorithm [9], which extends the greedy orthogonal matching pursuit (OMP) algorithm [10], [11] by incorporating a tree-searching strategy. In [7], a pilot-less SVC scheme with a block-OMP decoder has been studied for the unknown fading channels. A pilot-less one-shot transmission and a deep neural network-based decoder were proposed in [8]. SVC has been applied for multi-user multi-input multi-output in [12]. Later, SVC has also been extended to encode additional bits using constellation symbols in [13].

In this paper, we first present the maximum likelihood (ML) detector for SPARC in the non-coherent SIMO channel. Since the search complexity of ML is infeasible, we develop a novel algorithm, maximum likelihood matching pursuit (MLMP), which incorporates the greedy search mechanism with the ML metric. Similar to [9], we also study parallel search mechanisms with MLMP in order to mitigate the error propagation in the greedy algorithm. Using simulation results, we show that the MLMP algorithm gives significant gains over the block-OMP based decoders considered in [7]. We also compare with a pilot-based scheme with polar codes, where the estimated channel is used to decode the codeword. With our proposed MLMP decoder, SPARC shows significant gains over pilot-aided polar codes.

II. BACKGROUND ON SPARC

SPARCs are defined by a dictionary matrix A of size N × L where L ≥ N. Let A = {a1, a2, . . . , aL} denote the set of columns of A. This set is partitioned into K disjoint sections, with Ak denoting the set of columns in the kth section. We assume that Lk = |Ak| is a power of two for any k. Based on the information bits, the encoder selects one column from each section Ak ∀k ∈ [K], and the codeword is the sum of the chosen columns. Since each column in Ak can be indexed using log2 Lk bits, the number of bits conveyed by the SPARC codeword is N̄ = K k=1 log2 Lk. Let the support set S ⊂ [L] contain the indices of the chosen columns from A. Now, we can represent the codeword as

$$s = Ax = \sum_{m \in S} a_m,$$

where x is a K-sparse vector with exactly K non-zero entries (which are equal to one) corresponding to the locations of the chosen columns. Let C denote the complete set of all possible codewords that can be formed. Since each section contains Lk columns ∀k ∈ K, the total number of codewords in the set C are, |C| = \prod_{k=1}^{K} L_k. If all the sections are of equal size, then the total number of codewords are |C| = \left(\frac{N}{K}\right)^K. The code rate R for SPARC is measured in bits per real channel use.
An important property of the dictionary matrix is its mutual coherence,
\[ \mu(A) = \max_{i \neq j} \frac{\|a_i \cdot a_j\|}{\|a_i\| \cdot \|a_j\|}. \] (2)
This parameter, which significantly impacts the sparse recovery performance, represents a measure of the maximum correlation between any two columns of the matrix \( A \). In our study, we use mutually unbiased bases (MUB) [5], [14] from quantum information theory literature for the dictionary matrix, with \( L = N^2 \) unit norm columns and mutual coherence \( \mu = \frac{1}{\sqrt{N}} \).

III. SPARC FOR UNKNOWN FADING CHANNEL
We consider a non-coherent fading model where the receiver lacks channel state information. We assume a flat-fading channel, where the fading coefficients remain constant over the entire duration of the codeword and change independently over different codewords. This is particularly valid since we are working with very short block lengths. We aim to build computationally efficient non-coherent decoders for SPARC in flat-fading fading channels. With \( D \) receive antennas, the observation vector \( y_i \) of size \( (N \times 1) \) at each receive antenna is,
\[ y_i = h_i s + v_i, \quad \forall i \in [D], \] (3)
where \( h_i \sim CN(0, \sigma_h^2) \) is the unknown fading coefficient, and \( v_i \sim CN(0, \sigma_v^2) \) is the complex white Gaussian noise (CWGN) in the \( i^{th} \) receive antenna. Given that the receiver observes \( y_i, \forall i \in [D] \) and has the knowledge of \( A \), the decoder’s task is efficiently to identify the locations of the non-zero columns of \( A \) that formed the codeword.

In sparse signal recovery, a significant body of literature frequently highlights the computational superiority of greedy algorithms compared to the conventional convex optimization-based \( \ell_1 \) minimization techniques. Hence, we propose a greedy algorithm based on ML detection for the model in (3) discussed in the next subsection.

A. ML Detector for single antenna model
The model for SPARC through an unknown flat-fading channel with a single receive antenna can be represented as,
\[ y = h s + v, \]
\[ = h(\sum_{m \in S} a_m) + v. \] (4)
Now, the ML detector for \( s \) from the observation \( y \) can be written as,
\[ \hat{s} = \arg\max_{s \in \mathcal{C}} p(y|s). \] (5)
From (4), \( y|s \) is distributed as,
\[ y|s \sim CN(0, \sigma_h^2 s^* s + \sigma_v^2 I_N), \] (6)
therefore,
\[ p(y|s) = \frac{1}{\pi^N|\mathcal{C}|} \exp(-y^* C^{-1} y). \] (7)
We can write \(|\mathcal{C}|\) as a product of its eigenvalues,
\[ |\mathcal{C}| = \sigma_v^{2(N-1)}(\sigma_h^2 s^* s + \sigma_v^2), \] (8)
and \( C^{-1} = (\sigma_h^2 s^* s + \sigma_v^2 I)^{-1} \) can be expanded using the Woodbury identity as,
\[ C^{-1} = \frac{1}{\sigma_v^2} (I - \frac{\sigma_h^2}{\sigma_v^2 + \sigma_h^2} s^* s). \] (9)
Now, plugging back the equations (8) and (9) back into (7) and simplifying it, we get the ML detector as
\[ \hat{s} = \arg\max_{s \in \mathcal{C}} \beta_s |\langle y, s \rangle|^2 - \gamma_s. \] (10)
Where,
\[ \beta_s = \frac{\sigma_h^2}{\sigma_v^2 + \sigma_h^2} \] (11)
As we see in (10), the ML detector needs to search over all the possible SPARC codewords, which can be prohibitively complex.

B. ML detector for multi-antenna model
The ML detector for the multi-antenna model in (3) can be derived as an extension of the ML estimate of the single-antenna model. The ML detector given the observations \( y_i, \forall i \in [D] \) at the receive antennas, is,
\[ \hat{s} = \arg\max_{s \in \mathcal{C}} \log p(y_1, y_2 \ldots y_D|s), \] (12)
\[ = \arg\max_{s \in \mathcal{C}} \sum_{i=1}^D \log(p(y_i|s)), \] (13)
\[ = \arg\max_{s \in \mathcal{C}} \beta_s \sum_{i=1}^D |\langle y_i, s \rangle|^2 - D\gamma_s. \] (14)
The \( \beta_s \) and \( \gamma_s \) are the same as in (11). Equation (13) is valid since we assume the channel fading coefficients to be i.i.d. across channels. By leveraging the spatial diversity, the BLER performance significantly improves by increasing the number of receive antennas.

C. Maximum Likelihood Matching Pursuit
To find the ML detector of \( s \), we need to compute the metric in (10) or (14) for all possible codewords in \( \mathcal{C} \). For example, the single antenna detector can be represented as,
\[ \hat{s} = \arg\max_{m_1, m_2 \ldots m_K} \beta_m |\{y^* a_{m_1} + \cdots + y^* a_{m_K}\}|^2 - \gamma_m, \] (15)
where, \( S = \{m_1, m_2, \ldots m_K\} \) is the support set that forms the codeword. This is infeasible even for small block lengths, as we see that \(|\mathcal{C}|\) is quite large. So, we introduce an intelligent way to approximate the ML detector into an iterative detector named the maximum likelihood matching pursuit (MLMP) algorithm. MLMP iteratively finds the columns that make up the codeword, one at a time.
In a single antenna system, the observed codeword can be represented as,
\[ y = hs + v, \]
\[ = ha_{m_1} + h \sum_{m \in S \setminus m_1} a_m + v. \]  
(16)  
(17)

In the first iteration, the aim is to detect only one column (we consider it as the first one in (16) w.l.o.g) that made up the codeword s. While estimating this column, the rest of the \(|S \setminus m_1| = K - 1\) undetected columns act as interference. We set the effective noise variance in the first iteration as,
\[ \hat{\sigma}^2_{v,1} = \sigma^2_v + \frac{\sigma^2_h}{N} \sum_{m \in S \setminus m_1} \|a_m\|^2, \]  
(18)

where,
\[ \| \sum_{m \in S \setminus m_1} a_m \|^2 = (\sum_{m \in S \setminus m_1} a_m, \sum_{m \in S \setminus m_1} a_m), \]
\[ = \sum_{p, q} \langle a_p, a_q \rangle, \]
\[ = \sum_{m \in S \setminus m_1} \|a_m\|^2 + \sum_{p, q \neq p} \langle a_p, a_q \rangle. \]  
(19)  
(20)

The cross terms in (20) can be bounded using the mutual coherence of the matrix \( \mu(A) \) (2). Hence, the sum in (20) lies in the interval
\[ [(K-1)-(K-1)(K-2)\mu(A), (K-1)+(K-1)(K-2)\mu(A)]. \]

So, we approximate the sum by taking the mid-point of the interval, which makes (18),
\[ \hat{\sigma}^2_{v,1} \approx \sigma^2_v + \frac{\sigma^2_h}{N}(K-1). \]  
(21)

In the first iteration of MLMP, we are trying to find a single active column from the codeword by searching over all \( a_m \in A \). Since all columns have unit norm, the terms \( \beta \) and \( \gamma \) in the metric (10) remain the same \( \forall a_m \in A \). Hence, the first iteration of MLMP selects an active column from the codeword as
\[ a_{\hat{m}_1} = \arg \max_{a_m \in A} |\langle y, a_m \rangle|^2. \]  
(22)

For the next iteration, the detected column in the first iteration is assumed to be correct, and we try to find the next column, which is part of the codeword. Hence, the observed signal can be written as,
\[ y = h(a_{\hat{m}_1} + a_{m_2}) + h \sum_{m \in S \setminus \{\hat{m}_1, m_2\}} a_m + v. \]  
(23)

Now, the remaining \( K - 2 \) columns act as interference, and hence the effective noise is, \( \hat{\sigma}^2_{v,2} = \sigma^2_v + \frac{\sigma^2_h}{N}(K-2) \). The second active column \( a_{\hat{m}_2} \) is detected as,
\[ a_{\hat{m}_2} = \arg \max_{a_m \in A \setminus \{\hat{m}_1\}} |\langle y, a_{\hat{m}_1} + a_m \rangle|^2 - \gamma_{2,m}, \]  
(24)

\[ \beta_{2,m} = \frac{\sigma^2_h}{\hat{\sigma}^2_{v,2} + \|a_{\hat{m}_1} + a_m\|^2}, \]
\[ \gamma_{2,m} = \log \left( \frac{\sigma^2_h}{\hat{\sigma}^2_{v,2}} \|a_{\hat{m}_1} + a_m\|^2 + 1 \right), \]  
(25)

where \( \hat{A}_1 \) is the section corresponding to the first detected column \( a_{\hat{m}_1} \).

In general, with multiple receive antennas, the selection of an active column in the \( k^{th} \) iteration is done as
\[ a_{\hat{m}_k} = \arg \max_{a_m \in A \setminus \{\hat{A}_1, \ldots, \hat{A}_{k-1}\}} \beta_{k,m} \sum_{i=1}^D (\langle y_i, \sum_{\ell=1}^{k-1} a_{\hat{m}_\ell} + a_m \rangle)^2 - \gamma_{k,m}, \]  
(26)

where,
\[ \beta_{k,m} = \frac{\sigma^2_h}{\hat{\sigma}^2_{v,k} + \|a_{\hat{m}_1} + a_m\|^2}, \]
\[ \gamma_{k,m} = \log \left( \frac{\sigma^2_h}{\hat{\sigma}^2_{v,k}} \|a_{\hat{m}_1} + a_m\|^2 + 1 \right). \]  
(27)

The MLMP algorithm stops after \( K \) iterations. Since the active columns from the codeword are obtained in a greedy manner, the search space of MLMP is linear in \( K \) while that of the optimal ML detector grows exponentially with \( K \).

D. Parallel-MLMP

In the MLMP algorithm, detecting the first column is the most susceptible to errors, as the remaining \( K - 1 \) undetected columns act as interference. Given the iterative nature of greedy algorithms like MLMP, any error occurring in the first iteration tends to propagate through subsequent iterations, leading to errors in detecting the subsequent columns [9].

To overcome this, we introduce Parallel-MLMP (P-MLMP), which chooses the top \( \cdot P \) columns with the highest metrics in the first iteration. We then run the MLMP algorithm \( P \) times in parallel, assuming one of the top \( P \) columns from the first iteration is an active column from the codeword. We get \( P \) different codewords from these \( P \) parallel MLMP decoders. Among these \( P \) candidates, we select the candidate with the largest ML metric as the final chosen codeword. Algorithm 1 shows the working for Parallel-MLMP.

**Algorithm 1 Parallel-MLMP**

1. **Input:** \( y, A, K, P \).
2. **Compute first metric:** \( q(i) = |\langle y, a_i \rangle|^2, \forall a_i \in A \).
3. **Let** \( J = \{j_1, \ldots, j_P\} \) **denote the set of indices corresponding to the top \( P \) values of** \( q(i) \) **such that** \( q(j_1) \geq q(j_2) \geq \cdots \geq q(i) \geq q(j) \) **for any** \( i \in [L] \setminus J \).
4. **Initialise parallel path index:** \( n = 1 \).
5. **Set the first active column (as one of the top \( P \) candidates obtained previously):** \( a_{\hat{m}_1} = a_{j_1} \).
6. **Run MLMP algorithm and find the remaining** \( K - 1 \) **active columns:** Denote the detected codeword as \( \hat{s}_n \).
7. **Update:** \( n = n + 1 \); if \( n \leq P \) go back to step 5.
8. **Select best among the** \( P \) **candiates based on ML metric from (7):** \( \hat{s} = \arg \max_{1 \leq n \leq P} p(y|\hat{s}_n) \).
E. Modified Block-OMP

SPARCs in unknown fading channels have been studied in [7], where Block-OMP (BOMP) [11] was used for the decoding. The BOMP algorithm estimates the unknown fading channel based on the detected columns and then removes their contribution from the observation to compute the residual. For the flat-fading model, we can improve this BOMP algorithm by using the modification described below. The residual \( r^{(k)}_i \) based on the columns detected so far \( \{a_{\hat{m}_1}, \cdots, a_{\hat{m}_k}\} \) is computed as,

\[
r^{(k)}_i = y_i - \hat{h}_{i,k} \sum_{\ell=1}^{k} a_{\hat{m}_\ell},
\]

\[
\hat{h}_{i,k} = \frac{\langle y_i, \sum_{\ell=1}^{k} a_{\hat{m}_\ell} \rangle}{\| \sum_{\ell=1}^{k} a_{\hat{m}_\ell} \|^2}.
\]

The next active column is chosen as \( a_{\hat{m}_{k+1}} = \arg \max_m \sum_{i=1}^{D} |\langle r^{(k)}_i, a_{\hat{m}} \rangle|^2 \). The algorithm stops after recovering all the \( K \) active columns. We can also introduce a parallel search mechanism in MBOMP, P-MBOMP, as done in MLMP.

IV. PILOT-AIDED TRANSMISSION

Typically, to communicate over unknown fading channels, pilot-aided transmissions (PAT) are employed. This section presents details of the PAT strategy for the fading model we have considered. Since there is only one unknown fading coefficient (per receive antenna), we send a single pilot symbol to facilitate the channel estimation. Subsequently, the estimated channel is used to detect the data (codeword) transmitted. Here, we can employ error control codes which are designed for channels with known CSI. In PAT, we have to allot a fraction of the available energy for pilots and the remaining fraction for the data codeword. Let \( \alpha \) be the fraction of the total energy \( E \) allocated to the pilots, i.e., \( E_p = \alpha E \) and \( E_d = (1-\alpha)E \) is the energy for the data sequence. Increasing the energy for the pilots improves the channel estimation, but the energy allocated for the data is reduced, thereby increasing the probability of error. Hence, an optimal power allocation should maximize the effective SNR, incorporating the channel estimation error. One complex pilot symbol per codeword will suffice for a flat-fading channel that remains constant over the entire codeword.

\[
y_{p,i} = h_i \sqrt{E_p} + v,
\]

\[
y_{d,i} = h_i \sqrt{E_d} \frac{N}{N} s_d + v,
\]

where \( y_{p,i} \) is the observation of the single pilot symbol, and \( y_{d,i} \) is the observation (of length \( N \)) corresponding to the (unit norm) codeword \( s_d \), at the \( i \)th receive antenna. The MMSE estimate of the channel is \( \hat{h}_i = \frac{\sqrt{E_p} \sigma_i^2}{E_p \sigma_i^2 + \sigma_v^2} y_{p,i} \) and the estimation error is \( \hat{h}_i = h_i - \hat{h}_i \). Now, (29) can be written as,

\[
y_{d,i} = \hat{h}_i \sqrt{E_d} \frac{N}{N} s_d + \hat{h}_i \sqrt{E_d} \frac{N}{N} s_d + v.
\]

The last two terms of the above equation act as noise, which includes the CWGN noise and the channel estimation error. Hence, the \( \alpha \), which maximizes this effective SINR satisfies

\[
\alpha_{opt} = \arg \max_{0<\alpha<1} \frac{E_d}{N} \sigma_h^2 \frac{\sigma_h^2}{\sigma_h^2 + \sigma_v^2},
\]

where \( \sigma_h^2 = \frac{\sigma_i^2}{E_p \sigma_i^2 + \sigma_v^2} \) and \( \sigma_v^2 = \sigma_h^2 - \sigma_i^2 \). Thus, the formulation in (31) gives us the optimal power allocation for pilots and data used during transmission. Once we have the channel estimate of the unknown fading channel, we can perform maximal ratio combining of observations \( z_d = \sum_{k=1}^{D} \hat{h}_i y_{d,i} \), and decode the codeword \( s_d \) using \( z_d \).

V. SIMULATION RESULTS

In this section, we study the BLER performance of MLMP compared to other widely used algorithms in the literature that apply to our system model. We use a complex MUB matrix [14] of size \( 64 \times 4096 \) with unit norm columns as the dictionary matrix to generate the codeword, thereby making the length of the codeword \( N = 64 \). This is equivalent to 128 real channel uses. The expected energy of the codeword is \( E_s = K \), with the dictionary columns being the unit norm. The energy per bit \( E_b = E_s/N_b \), where \( N_b \) is the number of information bits transmitted by the codeword. Finally, our simulations show the performance of BLER vs \( E_b/N_0 \), where \( N_0 = \sigma_i^2 \) for the complex codewords. The SPARC scheme transmitting \( N_b \) bits using \( 2N \) real dimensions is \( (2N, N_b) \), error control code with the rate being \( N_b/2N \) bits per real dimensions or bits per channel use (bpcu). With \( D \) receive antennas, we normalize the channel variance \( \sigma_i^2 = 1/D \), so that the array gain from the receive antennas is unity. Hence, the performance gain from the multiple antennas is only through the diversity gain.

Figure 1 shows the BLER of the MLMP algorithm for (128,40) SPARC with MUB64 dictionary matrix and \( K = 4 \). We also plot the lower bound on the BLER by computing the sphere packing bound for the known CSI coherent communication [15]. We observe that MLMP has significant gains over the MBOMP algorithm. We also note that parallel MLMP has significant gains over MLMP.

In Figure 2, we also include the performance of (128,40) polar code [16] used in 5G NR standards along with the pilot-aided transmission and list decoding [17]. We see that SPARC, with our proposed parallel MLMP decoder, has significant gains over polar codes with the PAT strategy. We also see that due to the diversity gains, a higher number of receive antennas gives better BLER.

In Figure 3, we study the performance of MLMP for an even shorter 32-length complex codeword SPARC generated using a MUB matrix of size \( 32 \times 1024 \). BLER performance slightly degrades owing to the higher mutual coherence of MUB32 compared to that of MUB64. We note from the figure that MLMP still maintains the performance gain over MBOMP, even at shorter block lengths. This figure also shows the marginal performance degradation by increasing the code rate.
VI. CONCLUSION

In this letter, we study the BLER performance of SPARCs over an unknown fading channel with multiple receive antennas. We presented MLMP, a novel non-coherent greedy algorithm inspired by approximating the search of the ML detector. Using MUB as the dictionary matrix, we study the BLER performance of SPARCs over a multi-antenna fading channel. We have shown that MLMP significantly outperforms the Block-OMP widely used in the literature. We also show that MLMP with SPARC performs better than Polar codes with the pilot-aided transmission.

REFERENCES