Constraint Concentration in Distribution Networks

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Abstract—The model complexity in grid-related applications can be reduced by the concentration of distribution network constraints (DNC), which removes the original need for power flow calculations. To conquer the "curse of dimensionality" inherent in existing computational concentration methods, this letter proposes an analytical concentration scheme that utilizes Gaussian elimination and theoretical inner approximation. Based on the 16-dimensional numerical studies, this new scheme proves effective, which eventually enhances the applicability of physical constraint awareness in concrete grid management practices.

Index Terms—Distribution network, grid feasibility, feasible region, constraint concentration.

I. INTRODUCTION

THE DNC are crucial in numerous grid management problems, for instance flexibility market clearing and grid integration planning [1], [2]. The critical variables in DNC, which represent injection active/reactive power at each bus, are sufficient to determine power flow and to detect any DNC violations. Meanwhile, the original DNC model contains additional variables that denote bus voltage, cable current, and cable power flow [3]. These auxiliary variables complicate the DNC application process by requiring complex power flow analysis. To omit this complexity, our goal is DNC concentration by excluding these auxiliary variables.

A brute-force concentration approach involves computing the multidimensional feasible region of critical variables, alias "hosting capacity region" (HCR), which advances the concept of “hosting capacity” [4], [5]. Leveraging computational geometry techniques, this region can be assessed by an expansive polytope. The concentrated DNC are then outlined by this polytope, a geometric shape formed by the intersection of multiple linear constraints. However, this method encounters the “curse of dimensionality” [6], [7]. The exponentially increased number of constraints, especially in high-dimensional settings, negates the advantage gained from variable number reduction. This trade-off represents a signification drawback of the computational ideology in DNC concentration.

To address the issue with the constraint quantity, this letter proposes an analytical concentration approach. The main contributions of this work are twofold:
1) Through Gaussian elimination and inner approximation, an analytical DNC concentration scheme is developed, applicable for both convex and exact distribution network models.
2) Both benchmark and proposed schemes undergo theoretical analysis. Moreover, numerical studies are conducted in a 16-dimensional case, eventually confirming that the proposed one effectively overcomes the "curse of dimensionality".

II. ORIGINAL DISTRIBUTION NETWORK CONSTRAINT

A. Distribution grid model

Radial network topology has been widely deployed in distribution grids. Therefore, the DistFlow model is adopted for distribution network modeling as given in (1), which contributes to the exact DNC that is denoted by $DNC-e$. The variable and parameter notations are provided in Table I, where $P_i, Q_i$ colored in grey are critical variables [3].

\[
P_{ij} = \sum_{k:(j,k) \in G} P_{jk} + r_{ij}l_{ij} - P_j (1a)
\]

\[
Q_{ij} = \sum_{k:(j,k) \in E} Q_{jk} + x_{ij}l_{ij} - Q_j (1b)
\]

\[
v_j = v_i - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (r^2_{ij} + x^2_{ij})l_{ij} (1c)
\]

\[
l_{ij} = \frac{P^2_{ij} + Q^2_{ij}}{v_i}, l_{ij} \leq l_{ij}^{\text{max}} (1d)
\]

\[
P_i^{\text{min}} \leq P_i \leq P_i^{\text{max}}, Q_i^{\text{min}} \leq Q_i \leq Q_i^{\text{max}} (1e)
\]

\[
v_i^{\text{min}} \leq v_i \leq v_i^{\text{max}} (1f)
\]

DNC-e: (1).

B. Convex DNC variants

The DNC-e is non-convex and difficult to process for Gaussian elimination. Since the accuracy of LinDistFlow model has been numerically verified acceptable as tested in [4], we use this convex variant as an approximated network model, thus contributing to DNC-c1 as written below.

\[
P_{ij} = \sum_{k:(j,k) \in E} P_{jk} - P_j, Q_{ij} = \sum_{k:(j,k) \in E} Q_{jk} - Q_j (2a)
\]

\[
v_j = v_i - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}), P^2_{ij} + Q^2_{ij} \leq v_i^{\text{max}} (2b)
\]

DNC-c1: (2), (1e), (1f).

Meanwhile, considering the conservativeness of grid operator, compared to immeasurable approximation in DNC-c1, an inner approximation of DNC-e is anticipated. As proven in the Appendix of [8] and depicted in Fig. 1, there exist two convex
various variants, numerated as DNC-c2 and DNC-c3, whose feasible region intersection must be a subset of that defined by DNC-e. Therefore, the concentration of DNC-e can be finalized using an union of concentrated DNC-c2 and DNC-c3. For further illustration, $E_j$ denotes the set of links in the shortest path from POC $j$ to the slack bus. Intermediate parameters $l_{ij}^{max}, \Delta P_{ij}, \Delta Q_{ij}$ can be calculated as (4).

$$P_{ij} = \sum_{k:(j,k)\in E} P_{kj} + r_{ij}l_{ij}^{max} - P_j$$

$$Q_{ij} = \sum_{k:(j,k)\in E} Q_{kj} + x_{ij}l_{ij}^{max} - Q_j$$

$$v_j = v_0 - 2(r_{ij}P_{ij} + x_{ij}Q_{ij}) + (2r_{ij}Q_{ij} + 2x_{ij}l_{ij}^{max})$$

$$P_j^2 + Q_j^2 \leq v_{i}l_{ij}^{max}, \quad l_{ij}^{max} = (\sqrt{l_{ij}} - \sqrt{\Delta l_{ij}})^2$$

$$\Delta l_{ij} = (\Delta P_{ij}^2 + (\Delta Q_{ij})^2$$

$$\Delta P_{ij} = \sum_{(a,b)\in E_j} r_{ab}l_{ab}^{max}, \quad \Delta Q_{ij} = \sum_{(a,b)\in E_j} x_{ab}l_{ab}^{max}$$

**DNC-c2:** (2a), (3c), (4), (1e), (1f).

**DNC-c3:** (3a), (3b), (2b), (1e), (1f).

### III. CONSTRAINT CONCENTRATION IN CHAIN

#### A. Concentration of DNC-c1

Since (1e) involves no auxiliary variables, we transfer (1e) from DNC-c1 to the concentration result. After this preprocessing, we start the variable elimination process from (2a). In reference with Proposition I in [8], by defining vectors $P_j$ and $P_{ij}$ in (5), we have (6), where $V_j$ denotes the vertex set describing the shortest path from POC $j$ to the slack bus.

$$P_j := [P_1, P_2, ..., P_j]^T, \quad P_{ij} := [P_{i1}, P_{i2}, ..., P_{ij}]^T$$

$$P_j = MP_j, \quad Q_{ij} = MQ_j, \quad M[i,j] := \begin{cases} 1 & i \in V_j \\ 0 & i \notin V_j \end{cases}$$

$$\mathbf{v} := [v_1, v_2, ..., v_j]^T, \quad \mathbf{R} := \text{diag}(R_{01}, R_{12}, ..., R_{ij})$$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{M}^T[2(\mathbf{RP}_{ij} + \mathbf{XQ}_{ij})]$$

Through defining voltage boundary vectors in (9), we substitute (7) and (8) into (1f). Continuing this Gaussian elimination process, we acquire (10), which merely involves critical variables. The operator $\leq$ compares the same-position elements of various vectors.

$$\mathbf{v}_{min} := [v_{1min}, v_{2min}, ..., v_{jmin}]^T$$

$$\mathbf{v}_{max} := [v_{1max}, v_{2max}, ..., v_{jmax}]^T$$

$$N_R := M^2 \mathbf{RM}, \quad N_X := M^2 X$$

$$\mathbf{v}_{min} \leq \mathbf{v}_0 + 2N_R \mathbf{P}_j + 2N_X \mathbf{Q}_j \leq \mathbf{v}_{max}$$

To this end, we define new vectors as given in (11), where $\text{par}(\cdot)$ and $s$ returns the parent and root bus in this tree-type network. With the assistance of mathematical induction, we finally derive (12) through combining (8), (11f), and the right part of (2b). $\odot$ denotes the operation of Hadamard product.

$$\mathbf{l}^m := [l_{01}^{max}, l_{12}^{max}, ..., l_{ij}^{max}]^T$$

$$\mathbf{v}_s[i] := \begin{cases} v_0 & s = \text{par}(i) \\ 0 & s \neq \text{par}(i) \end{cases}$$

$$\mathbf{T}[i, j] := \begin{cases} 1 & i = \text{par}(j) \\ 0 & i \neq \text{par}(j) \end{cases}$$

$$\mathbf{T}(\mathbf{v}_0 + 2N_R \mathbf{P}_j + 2N_X \mathbf{Q}_j) + \mathbf{v}_s \odot \mathbf{l}^m \geq (MP_j) \odot (MP_j) + (MQ_j) \odot (MQ_j)$$

In summary, the concentrated DNC-c1 can be represented by (10), (12), and (1e), where all auxiliary variables have been eliminated. Such concentration result is analytical, where the accuracy loss is fully imported by the original DNC-c1.

#### B. Concentration of DNC-e

As stated in Section II-B, the concentration of DNC-e can be acquired by processing DNC-c2 and DNC-c3. Following the same derivation principles at Section III-A, we concentrate DNC-c2 as (13) and (1e). In analogy, DNC-c3 is reformulated as (14) and (1e). Finally, the concentrated DNC-e can be represented by (13), (14), and (1e), with no presence of auxiliary variables in the original DNC-e.

$$\mathbf{v}_{min} \leq \mathbf{v}_0 + 2N_R \mathbf{P}_j + 2N_X \mathbf{Q}_j - N_{Z2} \mathbf{l}^m \leq \mathbf{v}_{max}$$

$$\mathbf{T}(\mathbf{v}_0 + 2N_R \mathbf{P}_j + 2N_X \mathbf{Q}_j - N_{Z2} \mathbf{l}^m) + \mathbf{v}_s \odot \mathbf{l}^m \geq (MP_j) \odot (MP_j) + (MQ_j) \odot (MQ_j)$$

$$\mathbf{Z}_2 := (\mathbf{R}^2 + \mathbf{X}^2), \quad N_{Z2} := M^2 \mathbf{Z}_2$$

$$\tilde{\mathbf{l}}^m := [l_{01}^{max}, l_{12}^{max}, ..., l_{ij}^{max}]^T$$

$$\mathbf{Z}_2 := (\mathbf{R}^2 + \mathbf{X}^2), \quad N_{Z2} := M^2 \mathbf{Z}_2$$

$$\mathbf{v}_{min} \leq \mathbf{v}_0 + 2N_R \mathbf{P}_j + 2N_X \mathbf{Q}_j - N_{Z3} \mathbf{l}^m \leq \mathbf{v}_{max}$$

$$\mathbf{T}(\mathbf{v}_0 + 2N_R \mathbf{P}_j + 2N_X \mathbf{Q}_j - N_{Z3} \mathbf{l}^m) + \mathbf{v}_s \odot \mathbf{l}^m \geq (MP_j) \odot (MP_j) + (MQ_j) \odot (MQ_j)$$

$$N_{Z3} := 2(N_R \mathbf{R} + N_X \mathbf{X})$$

#### C. Theoretical performance evaluation

The HCR computation method mentioned in Section I is taken as the benchmark concentration scheme, for comparison with the proposed analytical one. Table II concludes how various concentration schemes reduce the model complexity, where $J$ counts all lines and the “original” column refers to
the unconsecrated DNC. Notably, all matrix constraints will be converted into a set of meta inequalities, which is scalar and contains only one sign of inequality. Evidenced by the smaller constraint and variable numbers in Table II, no matter either DNC-c1 or DNC-e selected, the analytical method owns theoretically superior concentration performances.

### TABLE II: Theoretical concentration and variable numbers

<table>
<thead>
<tr>
<th>Model</th>
<th>Cardinality</th>
<th>Origin</th>
<th>Computation</th>
<th>Analytics</th>
</tr>
</thead>
<tbody>
<tr>
<td>DNC-c1</td>
<td>Constraint Variable</td>
<td>10.5</td>
<td>Unbounded</td>
<td>7.5</td>
</tr>
<tr>
<td>DNC-c1</td>
<td>Constraint Variable</td>
<td>6.5</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>DNC-e</td>
<td>Constraint Variable</td>
<td>11.5</td>
<td>Unbounded</td>
<td>10.5</td>
</tr>
<tr>
<td>DNC-e</td>
<td>Constraint Variable</td>
<td>10.5</td>
<td>2.5</td>
<td>2.5</td>
</tr>
</tbody>
</table>

### IV. NUMERICAL VALIDATION

To verify the numerical efficacy of the proposed scheme, a Dutch network model depicted in Fig. 2 is employed for testing, whose parameters are provided in [4]. Bus 9 is the slack bus. There exist 16 critical variables in this DNC to include active/reactive power from all non-slack 8 buses, for consideration of future energy unit integration.

Firstly, we set (1e) to be an orthotope with each dimensional between -4 MVar and 4 MVar. Within this orthotope, we generate 10000 random points as a testing set for feasibility assessment. By examining whether these points violate the original or concentrated DNC, alongside counting their examination time, we provide a comparative performance analysis in the 16-dimensional space as listed in Table III. All numerical experiments are implemented on the same platform.

### TABLE III: Time expenses for feasibility assessment

<table>
<thead>
<tr>
<th>Method</th>
<th>DNC-c1</th>
<th>DNC-e</th>
</tr>
</thead>
<tbody>
<tr>
<td>Origin (Power Flow Calculation)</td>
<td>232.73 s</td>
<td>2197.60 s</td>
</tr>
<tr>
<td>Computation (HCR-based Benchmark in [8])</td>
<td>Fail</td>
<td>Fail</td>
</tr>
<tr>
<td>Analytics (This Paper)</td>
<td>14.39 s</td>
<td>20.91 s</td>
</tr>
</tbody>
</table>

Evidenced by at least 93.82% assessment time reduction attributed to the analytical concentration, the proposed method successfully overcomes the “curse of dimensionality” in this 16-dimensional case. Meanwhile, the benchmark one fails under buffer overflow. In detail, this benchmark is rendered inoperative due to overwhelmed geometric computation for high-dimensional HCR expansion. Besides, from the accuracy perspective, after clearing out the accumulative numerical error in power flow calculation, the assessment accuracy loss of the analytical DNC-c1 and DNC-e concentration is as low as zero and 1.4%. Notably, the latter loss is imported by the inner approximation process discussed in Section II-B. To summarize, all phenomena above validate the effectiveness of the proposed DNC concentration method eventually.

Additionally, to provide a visualized comparison of computational and analytical concentrations, based on DNC-c1 with slack (1e) removed, we focus on a 3-dimensional domain that represents active power at Bus 1,2,3. The results are depicted in Fig. 3a. Consisting of 24 inequalities, the blue upper envelope and yellow lower envelope co-formulate the analytical feasible region. The green selvedge denotes no space between two envelopes. Comparatively, the red polytope composed of 48 linear constraints is computationally generated after figuring out feasible points among 10000 candidates. Besides region loss, this polytope already suffers more constraints than the analytical result. By incorporating more feasible points, this polytope expands to approach the analytical region, with the price of further more constraints. The same phenomena are observed in the other case of Bus 3,5,7 as illustrated by Fig. 3b. These 3-dimensional studies have twice confirmed the superiority of the proposed analytical method.

### V. CONCLUSION

This letter provides a closed-form expression of DNC, thus reducing the model complexity by omitting the need of power flow calculations. Relevant theoretical and numerical analyses have confirmed its efficacy. In future, the authors will apply this analytical concentration method to concrete grid-involved applications, for instance grid-aware flexibility market design.

### REFERENCES