A Predefined-Time Stability Control Method Based on Fuzzy Compensation for Euler-Lagrange Systems and Its Applications to Manipulators

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Abstract—This paper presents an approach to predefined-time terminal sliding mode control (PT-TSMC) for Euler–Lagrange systems (ELSs) with actuator control input saturation, leveraging fuzzy compensation techniques. Initially, the paper focuses on tunable predefined-time stability (PTS), providing adjustable parameters to fine-tune the system’s stability time and thereby enhancing the adaptability of controller design. With this approach, the paper formulates a control input system that ensures PTS for such ELSs. Additionally, the paper proposes a strategy to enhance system performance, including robustness improvement, chattering reduction, and singularity elimination, through the design of an adaptive fuzzy logic system (AFLS). This AFLS estimates unstructured model uncertainty and compounded disturbances, seamlessly integrating them into the control system. Notably, this approach adeptly addresses the challenge of handling unknown model data. Finally, through comprehensive comparative simulations, the paper demonstrates the effectiveness of the proposed method, showcasing its commendable control performance.


I. INTRODUCTION

The dynamic model of ELSs has garnered significant interest in engineering applications, particularly for the study and modeling of nonlinear systems [1], [2]. This versatile model finds application in various practical scenarios, ranging from mobile robot platforms to spacecraft, due to its ability to capture complex dynamics. The potential applications of these technologies span diverse fields such as military operations, automation, surveillance, and space exploration. However, the nonlinear nature and mechanical instability inherent in ELSs necessitate the development of precise control and stability mechanisms.

Efforts to tackle uncertain ELSs affected by environmental disturbances have led to the development of robust adaptive control mechanisms. These mechanisms aim to enhance system robustness by dynamically updating control parameters in response to prevailing conditions, thereby ensuring an appropriate level of stability [3].

In practical production and daily life, disturbances, errors, human factors, and environmental conditions often hinder the accurate linear description of ELSs. As a result, researchers have shifted their focus towards studying nonlinear systems, with considerable attention devoted to controlling ELSs using various methods such as optimal control [4], adaptive control [5], and sliding mode control (SMC) [6]. Among these, SMC stands out for its robustness against uncertainties and disturbances. However, conventional SMC is limited to achieving asymptotic stability, which may not be sufficient for tasks requiring stability within a finite time, such as spacecraft attitude adjustments.

To address this limitation, finite-time control methods have been introduced, offering the advantage of estimating convergence time boundaries. One notable method is terminal sliding mode control (TSMC), which aims to improve effectiveness by ensuring stability within a finite time [7], [8]. However, accurate estimation of convergence time remains a challenge, particularly due to the dependence on initial states.

Fortunately, the fixed-time control method provides a solution by ensuring that the upper bound of convergence time remains constant and independent of the system’s initial states [9]. This approach has seen successful application in spacecraft attitude tracking control [10], uncertain magnetic levitation systems [11], and so on, offering stability even with varying initial conditions.

Despite its advantages, fixed-time control methods may still face challenges related to conservative convergence time estimation. To address this, researchers have introduced predefined-time control methods, allowing users to specify convergence time boundaries based on their requirements [12], [13]. Integration of predefined-time control with SMC ensures stability during the arrival stage while preserving characteristics of SMC [14], [15].

Additionally, discrepancies between actual convergence time and user-defined upper bounds can arise, prompting the introduction of adjustable parameters to fine-tune convergence time. This approach offers flexibility and adaptability in achieving desired stability outcomes. However, the singularity problem in SMC-based predefined-time controllers has not been completely resolved; therefore, it remains an open problem that needs to be addressed.

Actuator input saturation is a major obstacle in ELSs, stemming from constraints in hardware and software capabilities. The referenced study [16] addresses this by assuming the actuator saturation is unknown and approximating this unknown element using a NN. Subsequently, a saturation compensator is integrated into the feedforward path to mitigate the impact of saturation.

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One approach, as proposed by [17], [18], involves utilizing the Nussbaum function approach. This method employs a smoothing nonlinear function to approximate the saturated actuator input. An alternative method was introduced by [19]–[21], which involves designing an auxiliary dynamic system (ADS) to manage state error and address the input saturation problem. Several comparative analyses were conducted to evaluate the effectiveness of using a smoothing nonlinear function versus error transformation using an ADS. The findings favored the error transformation approach of the ADS, demonstrating superior performance in addressing saturation input. However, it’s important to note that traditional ADSs do not address fixed-time/predefined time convergence, highlighting an unresolved issue that warrants further investigation.

To enhance tracking accuracy and decrease computational complexity in controlled systems with unknown model dynamics and uncertain components, the prevailing approach involves approximating unstructured model uncertainty and compounded disturbances. Subsequently, these approximations are compensated for in the control design. Existing methods include neural networks (NN) [9], [22], [23], observers [24]–[28], filters [29], and fuzzy logic systems (FLS) [2], [30]–[32]. Among them, FLS integrate human knowledge and experiences into the system to enhance approximation performance. In particular, the AFLS provides robust approximation versus error transformation using an ADS. The findings illustrate superior performance in addressing saturation input. Several comparative analyses were conducted to evaluate the effectiveness of using a smoothing nonlinear function, which accurately estimates unstructured model uncertainty.

The paper comprises five primary sections. Initially, the introduction offers a comprehensive overview of the paper’s objectives and scope. Following this, the second section delineates the problem formulation and introduces essential preliminaries to establish a foundational understanding. In the subsequent section, the proposed approach is detailed, providing insights into the methodology employed. Section 4 is dedicated to the discussion of the method’s simulated performance, specifically focusing on its application to a 3-DOF robot manipulator. Finally, Section 5 encapsulates the study findings, serving as a conclusion to the paper while also outlining potential directions for future studies.

II. PRELIMINARIES AND PROBLEM FORMULATION

For $i = 1, 2, \ldots, n$ and vectors $x = [x_i, \ldots, x_n]^T \in \mathbb{R}^n$ and $a = [a_i, \ldots, a_n]^T \in \mathbb{R}^n$ with $a_i > 0$, then $|x|^a = [|x_1|^{a_1}\text{sign}(x_1), \ldots, |x_n|^{a_n}\text{sign}(x_n)]^T$, $|x_1|^{a_1} = |x_1|^{a_1}\text{sign}(x_1)$, $|x_1|^{a_1} = [|x_1|^{a_1}\text{sign}(x_1), \ldots, |x_n|^{a_n}\text{sign}(x_n)]^T$, $\phi = \text{diag}(\phi_1, \ldots, \phi_n)$ denotes the diagonal matrix, $\| \cdot \|$ represents the Euclidean norm, while $\lambda_{\text{max}}(\cdot)$ and $\lambda_{\text{min}}(\cdot)$ are the maximum and minimum eigenvalues, respectively.

A. Definitions and Lemmas

Consider the below system:

$$\dot{x} = f(x), \quad x(0) = x_0 \tag{1}$$

where $x \in \mathbb{R}$ represents the system states. Function $f: \mathbb{R}^n \to \mathbb{R}^n$ is nonlinear and continuous, and $f(0) = 0$.

**Definition 1**: [15] Assuming the equilibrium point of system (1) stabilizes within a finite time, and the continuous-time function $T(x_0, \mu): \mathbb{R}^n \to \mathbb{R}$ satisfies $T(x_0, \mu) \leq T_c$ for all $x_0 \in \mathbb{R}^n$, then the equilibrium point of system (1) is considered tunably predefined-time stable, with $T_c$ representing a predefined time. In this context, $\mu$ serves as a tunable parameter.

**Lemma 1**: Let us consider a positive definite function $V(x)$ for the system described by (1), defined within the set $U$, where $U_0 \subseteq U$, and $x \in U_0 \setminus \{0\}$. This function $V(x)$ follows the subsequent formula:

$$\dot{V} \leq -\frac{2\pi^3}{T_c} \left( \beta_1 V + \beta_2 V^{\frac{2}{1-\alpha}} + \beta_3 V^{\frac{1}{1-\alpha}} \right) \tag{2}$$

Here, $0 < \alpha < 1$, $T_c > 0$, and $\beta_1, \beta_2, \beta_3 > 0$, with the condition $\beta_1^2 \leq 4\beta_2\beta_3$. Additionally, $\beta_1 = \frac{1}{\sqrt{3}}\beta_2\beta_3$. Consequently, (1) exhibits tunably predefined-time stability, with $T_c$ indicating the predefined time.

**Proof 1**: (2) can be express as:

$$dV \leq -\frac{2\pi^3}{T_c} V^{\frac{\alpha+1}{1-\alpha}} \left( \beta_1 V^{\frac{1-\alpha}{1-\alpha}} + \beta_2 V^{1-\alpha} + \beta_3 \right) dt \tag{3}$$

Rewrite (3) as follows:

$$-\frac{\pi^3}{T_c} dt \geq \frac{dV}{2V^{\frac{\alpha+1}{1-\alpha}} \left( \beta_1 V^{\frac{1-\alpha}{1-\alpha}} + \beta_2 V^{1-\alpha} + \beta_3 \right)} \geq \frac{1}{1-\alpha} \frac{dV^{\frac{1}{1-\alpha}}}{(\beta_1 V^{\frac{1-\alpha}{1-\alpha}} + \beta_2 V^{1-\alpha} + \beta_3)} \tag{4}$$

Suppose $T^*$ represents the stable time of the system, indicating that $V(T^*) = 0$ and $V(0) = V_0 > 0$. Upon integration from 0 to $T^*$, the following expression is obtained:

$$-\frac{\pi^3}{T_c} \int_0^{T^*} dt \geq \frac{1}{1-\alpha} \int_{V(0)}^{V(T^*)} \frac{1}{(\beta_1 V^{\frac{1-\alpha}{1-\alpha}} + \beta_2 V^{1-\alpha} + \beta_3)} \tag{5}$$
\[ \frac{\pi}{\mathcal{E}} \int_0^{T^*} dt \leq \frac{1}{1 - \alpha} \int_0^1 \left( \beta_1 V^{1+\alpha} + \beta_2 V^{1-\alpha} + \beta_3 \right) \]

The solution of (6) leads to the following inequality:

\[ \frac{\pi}{\mathcal{E}} T^* \leq \frac{2}{(1 - \alpha) \sqrt{4 \beta_3 \beta_4 - \beta_4^2}} \left( \text{atan} \left( \frac{2 \beta_2 V_0 + \beta_1}{\sqrt{4 \beta_3 \beta_4 - \beta_4^2}} \right) - \text{atan} \left( \frac{\beta_3}{\sqrt{4 \beta_3 \beta_4 - \beta_4^2}} \right) \right) \]

\[ \leq 2 \mathcal{E} \left( \frac{\pi}{2} - \text{atan} \left( \frac{\beta_3}{\sqrt{4 \beta_3 \beta_4 - \beta_4^2}} \right) \right) \quad (7) \]

As \( \text{atan} \left( \frac{\beta_3}{\sqrt{4 \beta_3 \beta_4 - \beta_4^2}} \right) \in (0, \frac{\pi}{2}) \), it follows that:

\[ T^* \leq T_c \frac{2}{\pi} \left( \frac{\pi}{2} - \left( \frac{\beta_3}{\sqrt{4 \beta_3 \beta_4 - \beta_4^2}} \right) \right) \leq T_c \quad (8) \]

Thus, according to Definition 1 and (8), (1) is considered tunably predefined-time stable, where \( T_c \) represents the predefined time.

**Lemma 2:** Suppose there exists a positive definite function \( V(x) \) for (1), defined over set \( U \), where \( U_0 \subseteq U \) and \( x \in U_0 \{ 0 \} \), satisfying the following inequality:

\[ \dot{V} \leq -\frac{2 \pi}{\mathcal{E}} \left( \beta_1 V + \beta_2 V^{1+\alpha} + \beta_3 V^{1-\alpha} \right) + \eta \quad (9) \]

Here, \( 0 < \alpha < 1 \), \( T_c > 0 \), \( 0 < \eta < \infty \), and \( \beta_1, \beta_2, \beta_3 > 0 \), with the condition \( \beta_4^2 \leq 4 \beta_3 \beta_4 - \beta_4^2 \).

Consequently, (1) converges to a small domain \( \Omega \) close to the origin within a predefined time \( \sqrt{\mathcal{E}} T_c \). The residual set of the system solution is characterized as follows: \( \Omega = \left\{ x \mid V(x) \leq \min \left\{ \left[ \frac{T_{\pi/\mathcal{E}}} {\beta_3}, \left[ \frac{T_{\pi/\mathcal{E}}}{\beta_3 \beta_4 - \beta_4^2} \right]^{\pi/2}, \left[ \frac{T_{\pi/\mathcal{E}}}{\beta_3 \beta_4 - \beta_4^2} \right]^{2\pi/3} \right\} \right\} \).

**Proof 2:** Deriving insights from (9), we deduce the following outcomes:

\[ \dot{V} \leq -\frac{2 \pi}{\mathcal{E}} \left( \beta_1 V + \beta_2 V^{1+\alpha} + \beta_3 V^{1-\alpha} \right) + \left( \eta - \frac{\pi}{\mathcal{E}} \beta_1 V \right) \quad (10a) \]

\[ \dot{V} \leq -\frac{2 \pi}{\mathcal{E}} \left( \beta_1 V + \frac{1}{2} \beta_2 V^{1+\alpha} + \beta_3 V^{1-\alpha} \right) \quad (10b) \]

\[ \dot{V} \leq -\frac{2 \pi}{\mathcal{E}} \left( \beta_1 V + \beta_2 V^{1+\alpha} + \frac{1}{2} \beta_3 V^{1-\alpha} \right) \quad (10c) \]

Breaking down each equation:

For (10a), if \( V > \frac{\mathcal{E}}{\pi \beta_3} \), then \( \dot{V} \leq -\frac{2 \pi}{\mathcal{E}} \left( \frac{1}{2} \beta_1 V + \beta_2 V^{1+\alpha} + \beta_3 V^{1-\alpha} \right) \). Hence, the system trajectory enters the region \( V > \frac{\mathcal{E}}{\pi \beta_3} \) within the predefined time \( \sqrt{\mathcal{E}} T_c \).

For (10b), if \( V^{1+\alpha} > \frac{\mathcal{E}}{\pi \beta_3} \), then \( \dot{V} \leq -\frac{2 \pi}{\mathcal{E}} \left( \beta_1 V + \frac{1}{2} \beta_2 V^{1+\alpha} + \beta_3 V^{1-\alpha} \right) \). Therefore, the system trajectory enters the region \( V^{1+\alpha} > \frac{\mathcal{E}}{\pi \beta_3} \) within the predefined time \( \sqrt{\mathcal{E}} T_c \).

For (10c), if \( V^{1+\alpha} > \frac{\mathcal{E}}{\pi \beta_3} \), then \( \dot{V} \leq -\frac{2 \pi}{\mathcal{E}} \left( \beta_1 V + \beta_2 V^{1+\alpha} + \frac{1}{2} \beta_3 V^{1-\alpha} \right) \). Thus, the system trajectory enters the region \( V^{1+\alpha} > \frac{\mathcal{E}}{\pi \beta_3} \) within the predefined time \( \sqrt{\mathcal{E}} T_c \).

**B. Fuzzy Logic System**

A FLS [2] employs a collection of rules known as IF-THEN rules to establish a relationship between a set of input variables, denoted by \( u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n \), and an output variable \( z \in \mathbb{R} \). Each rule, indexed by \( j \), takes the form:

\[ \text{Rule } j: \text{ If } R^j_1 \text{ and } \ldots \text{ and } R^j_n \text{ then } z = Q^j \],

where \( R^j_1, R^j_2, \ldots, R^j_n \) and \( Q^j \) represent fuzzy sets. The fuzzy output, when utilizing a singleton fuzzifier, is determined as follows:

\[ z = \frac{\sum_{j=1}^k w_j \left( \Pi_{i=1}^n \phi_{R_i}^j(u_i) \right)}{\sum_{j=1}^k \left( \Pi_{i=1}^n \phi_{R_i}^j(u_i) \right)} = \mathbf{W}^T \Psi(u), \quad (11) \]

where \( k \) is the total number of rules employed. Here, \( \phi_{R_i}^j(u) \) represents the membership function of the \( i \)-th input variable \( u_i \). For \( j = 1, \ldots, k \), the vector \( \mathbf{W} = [w_1, \ldots, w_k]^T \) comprises weight components that require adjustment during training. Additionally, \( \Psi(u) = [\psi_1(u), \ldots, \psi_k(u)]^T \) denotes a fuzzy basis vector, where each element \( \psi_j(u) \) is expressed as:

\[ \psi_j(u) = \frac{\Pi_{i=1}^n \phi_{R_i}^j(u_i)}{\sum_{j=1}^k \left( \Pi_{i=1}^n \phi_{R_i}^j(u_i) \right)}. \quad (12) \]

**Lemma 3:** [35] Suppose there’s a continuous function \( f(u) \) defined on a compact set \( \mathcal{Y} \subset \mathbb{R}^{n \times 1} \). It’s possible to approximate this function \( f(u) \) using a FLS, denoted as \( \mathbf{W}^T \Psi(u) \), such that:

\[ \sup_{u \in \mathcal{Y}} |f(u) - \mathbf{W}^T \Psi(u)| \leq \varphi \quad (13) \]

Here, \( \varphi \) is a positive constant. The basis function vector \( \Psi(u) \) is defined as:

\[ \Psi(u) = [\psi_1(u), \ldots, \psi_k(u)]^T \frac{1}{\sum_{j=1}^k \psi_j(u)}, \]

where \( \mathbf{W} = [w_1, \ldots, w_k]^T \) is the ideal constant weight vector, with \( k > 1 \). Each \( \psi_j(u) \) is selected as Gaussian functions with the form:

\[ \psi_j(u) = \exp \left( \frac{-|u - \nu_j|^2}{\sigma_j^2} \right). \]

Here, \( \nu_j \) denotes the width of the Gaussian function, and \( \nu_j = [\nu_{j1}, \ldots, \nu_{jn}]^T \) represents the center vector.

According to the mentioned FLS’s properties, we can infer that \( \|\Psi(u)\| \leq \sigma \), where \( \sigma \) is an unknown positive constant.

**C. Problem Formulation**

The dynamic behavior of ELSs is typically described by the following equation [5]:

\[ M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau - \tau_d. \quad (14) \]
In this equation, the angular acceleration vector \( \ddot{q} \), the angular velocity vector \( \dot{q} \), and the position vector \( q \in \mathbb{R}^{n \times 1} \) are represented. The matrices \( M(q) = M_0(q) + \delta M(q) \in \mathbb{R}^{n \times n} \), \( C(q, \dot{q}) = C_0(q, \dot{q}) + \delta C(q, \dot{q}) \in \mathbb{R}^{n \times n} \), and the vector \( G(q) = G_0(q) + \delta G(q) \in \mathbb{R}^{n \times 1} \) correspond to the inertia, centripetal-Coriolis, and gravitational force, respectively. The parameters \( M_0, C_0, G_0 \) and \( \delta M, \delta C, \delta G \) are normal and uncertain, respectively. Additionally, \( \tau_d \in \mathbb{R}^{n \times 1} \) represents external disturbances and friction, while \( \tau \in \mathbb{R}^{n \times 1} \) denotes the control input torque.

Rewriting (14) more concisely yields:

\[
\ddot{q} = M_0^{-1}(q)\tau - M_0^{-1}(q) [C(q, \dot{q})\dot{q} + G(q)] - M_0^{-1}(q) [\tau_d + \delta M\ddot{q} + \delta C(q, \dot{q})\dot{q} + \delta G(q)],
\]

where \( \tau_d \in \mathbb{R}^{n \times 1} \) represents external disturbances and friction, while \( \tau \in \mathbb{R}^{n \times 1} \) denotes the control input torque.

Now, defining \( x_1 = q \) and \( x_2 = \dot{q} = \ddot{q} \in \mathbb{R}^{n \times 1} \) as \( x = [x_1 \ x_2]^T \). Therefore, (15) can be expressed as:

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= M_0^{-1}(q)\tau - \dot{H}(x) + L(x, \dot{x}, \tau_d)
\end{align*}
\]

Here, \( H(x) = -M_0^{-1}(q)[C(q, \dot{q})\dot{q} + G(q)] \) is the smooth function of nominal model and \( L(x, \dot{x}, \tau_d) = -M_0^{-1}(q)[\tau_d + \delta M\ddot{q} + \delta C(q, \dot{q})\dot{q} + \delta G(q)] \) accounts for uncertainties and external disturbances in the system.

Let us define the position and velocity tracking errors as \( e_1 = x_1 - x_{1d} \) and \( e_2 = x_2 - x_{2d} \), where \( x_{1d} \) and \( \dot{x}_{1d} \) denote the reference position and velocity, respectively. The vector of tracking errors is defined as \( e = [e_1 \ e_2]^T \). Thus, Eq. (16) can be reconstructed as follows:

\[
\begin{align*}
\dot{e}_1 &= e_2 \\
\dot{e}_2 &= M_0^{-1}(q)\tau - \dot{H} + L
\end{align*}
\]

Here, \( \dot{H} = H(x) - \ddot{x}_{1d}, L = L(x, \dot{x}, \tau_d) \), and the saturation function \( \tau_{sat} \) is defined as the actuator \( \tau \) with saturation and has the following expression:

\[
\tau_{sat} = \begin{cases} 
\tau_{max} \quad \text{if} \quad \tau_{i} \geq \tau_{max} \\
\tau_{i} \quad \quad \quad \quad \quad \text{if} \quad \tau_{min} < \tau_{i} < \tau_{max} \\
\tau_{min} \quad \text{if} \quad \tau_{i} < \tau_{min}
\end{cases}
\]

where \( \tau_{max} > 0 \) and \( \tau_{min} < 0 \) are the upper and lower bounds of the actuator, respectively.

### III. SYNTHESIS OF CONTROL DESIGN

#### A. Predefined-Time Auxiliary Dynamic System (PTADS)

The proposed PTADS, inspired by studies [19]-[21], effectively mitigates saturation’s impact, as follows:

\[
\dot{p} = -M_0^{-1}\dot{\tau}_{sat}(p) + M_0^{-1}\delta \tau - \frac{\pi^3}{T_{cp}} \left( \beta_{11}p + \beta_{21}[p]^{2-\alpha} + \beta_{31}[p]^\alpha \right)
\]

where \( \beta_{11} = \beta_1, \beta_{21} = \frac{\beta_2}{\pi \cdot \tau_{cp}}, \text{and } \beta_{31} = \beta_3, T_{cp} > 0, \delta \tau = \tau_{sat} - \tau, \text{and} \| \delta \tau \| \leq \bar{\tau}, \quad \bar{\tau} = \text{diag}(\bar{\tau}_i, \ldots, \bar{\tau}_n), \bar{\tau}_i > 0 \). Then, (17) can be transformed into

\[
\begin{align*}
\dot{v}_1 &= v_2 + p \\
\dot{v}_2 &= M_0^{-1}\tau_{sat} + H + L + M_0^{-1}\dot{\tau}_{sat}(p) - M_0^{-1}\delta \tau + \frac{\pi^3}{T_{cp}} \left( \beta_{11}p + \beta_{21}[p]^{2-\alpha} + \beta_{31}[p]^\alpha \right)
\end{align*}
\]

where \( v_1 = e_1 \) and \( v_2 = e_2 - p \) with \( v_2 = [v_{21}, \ldots, v_{2n}]^T \in \mathbb{R}^{n \times 1} \).

**Theorem 1:** Referring to (19), the origin, denoted as \( p_i = 0 \), acts as a globally finite-time stable equilibrium point, with the state of (19) converging to zero within a predefined time.

#### Proof 3: Considering the Lyapunov function \( V_1 = p_i^T p_i \), its time derivative, while utilizing Lemma 3 of the study [36], is given by:

\[
\begin{align*}
\dot{V}_1 &= -2[p_i^T M_0^{-1} \dot{\tau} + 2p_i^T M_0^{-1}\delta \tau] \\
&- \frac{2\pi^3}{T_{cp}} p_i^T (\beta_{11}p + \beta_{21}[p]^{2-\alpha} + \beta_{31}[p]^\alpha) \\
&\leq 2[p_i^T M_0^{-1} (\delta \tau - \bar{\tau}) - \frac{2\pi^3}{T_{cp}} \left( \beta_{31} \sum_{i=1}^{n} |p_i|^{1+\alpha} \right)] \\
&- \frac{2\pi^3}{T_{cp}} \beta_1 \left( \sum_{i=1}^{n} p_i \right)^2 + \frac{2\pi^3}{T_{cp}} \beta_2 \left( \sum_{i=1}^{n} p_i \right)^{2-\alpha} \\
&\leq \frac{2\pi^3}{T_{cp}} \beta_1 V_1 + \frac{2\pi^3}{T_{cp}} \beta_2 V_1^{\frac{2-\alpha}{2}} + \frac{2\pi^3}{T_{cp}} \beta_3 V_1^{\frac{1+\alpha}{2}}
\end{align*}
\]

Referring to Lemma 1, it is ensured that \( p_i = 0 \) within a predetermined time interval denoted as \( T_{cp} \).

#### B. Formulation of Sliding Mode Surface

The SM surface is formulated as follows:

\[
s = v_2 + \frac{\pi^3}{T_{cs}} (\beta_{12}e_1 + \beta_{22}[e_1]^{2-\alpha} + \beta_{32}[e_1]^\alpha)
\]

where \( \beta_{12} = \beta_1, \beta_{22} = \frac{\beta_2}{\pi \cdot \tau_{cs}}, \text{and } \beta_{32} = \beta_3. T_{cs} > 0 \) and \( s = [s_1, \ldots, s_n]^T \in \mathbb{R}^{n \times 1} \).

Upon reaching the SM surface (22), \( s_i = 0 \), the system’s ideal sliding-mode motion is governed by the following nonlinear differential equation:

\[
v_2 = -\frac{\pi^3}{T_{cs}} (\beta_{12}e_1 + \beta_{22}[e_1]^{2-\alpha} + \beta_{32}[e_1]^\alpha)
\]

Noting \( v_2 = e_2 - p \) and based on the proof result of Theorem 1, \( p_i \) is fixed-time stable, meaning \( p_i = 0 \) will be achieved within a predefined time. Therefore, (23) can be expressed as:

\[
\dot{e}_{1i} = \frac{\pi^3}{T_{cs}} (\beta_{12}e_1 + \beta_{22}[e_1]^{2-\alpha} + \beta_{32}[e_1]^\alpha)
\]

**Theorem 2:** The designed sliding-mode surface (22) ensures that the tracking errors converge to the origin within a predefined time \( T_{cs} \).

#### Proof 4: Constructing a positive Lyapunov function as:

\[
V_2 = e_1^T e_1
\]
Taking the derivative of (25) while using (24) and Lemma 3 of the study [36], we obtain:

\[
\dot{V}_2 = 2\pi \frac{3}{T_{cs}} \left( \beta_1 \sum_{i=1}^{n} e_{1i}^2 + \beta_2 \sum_{i=1}^{n} |e_{1i}|^{2-\alpha} \right) \\
- 2\pi \frac{3}{T_{cp}} \beta_2 \sum_{i=1}^{n} |e_{1i}|^{2+\alpha} \leq -2\pi \frac{3}{T_{cs}} \left( \beta_1 \sum_{i=1}^{n} e_{1i}^2 + \beta_2 \left( \sum_{i=1}^{n} e_{1i}^2 \right)^{\frac{3-\alpha}{2}} \right) \\
- 2\pi \frac{3}{T_{cp}} \beta_3 \left( \sum_{i=1}^{n} e_{1i}^2 \right)^{\frac{2+\alpha}{2}} \leq -2\pi \frac{3}{T_{cs}} \left( \beta_1 V_2 + \beta_2 V_2^{\frac{3-\alpha}{2}} + \beta_3 V_2^{\frac{1+\alpha}{2}} \right)
\]  

(26)

Referring to Lemma 1, it is guaranteed that the tracking error \(e_{1i} = 0\) will be achieved within a predetermined time interval denoted as \(T_{cs}\).

### C. Formulation of Controller

Taking the derivative of (22) with respect to time yields:

\[
\dot{s} = \dot{v}_2 + P(e_1, \dot{e}_1) + Q(e_1, \dot{e}_1)
\]

(27)

where \(P(e_1, \dot{e}_1) = \frac{2\pi}{T_{cs}} (\beta_1 \dot{e}_1 + \beta_2 (2-\alpha) |e_{1}|^{1-\alpha} \dot{e}_1)\) and \(Q(e_1, \dot{e}_1) = \frac{2\pi}{T_{cp}} \beta_3 (e_{1})^{2+\alpha}\).

While incorporating (20), (27) yields:

\[
\dot{s} = M_o^{-1} \tau + H + L + Q(e_1, \dot{e}_1) + M_o^{-1} \tau \text{sign}(p)
\]

\[
+ \frac{\pi \frac{3}{T_{cs}}}{(\beta_1 |p| + \beta_2 |p|^{2-\alpha} + \beta_3 |p|^\alpha) + P(e_1, \dot{e}_1)}
\]

(28)

Upon examination of (28), we note the presence of a negative power term \(|e_1|^{\alpha-1} \dot{e}_1\) in the function \(Q(e_1, \dot{e}_1)\). In cases where \(e_{1i} = 0\) and \(\dot{e}_{1i} \neq 0\), this term becomes infinite, leading to a singularity problem in the control input derived from (28). Such singularity renders it impractical for real-world applications. Furthermore, prior knowledge of the term \(L\) also poses a challenge in control design. Therefore, the sum of \(L\) and \(Q(e_1, \dot{e}_1)\) can be expressed in an AFSL, \(L + Q(e_1, \dot{e}_1) = \widetilde{W}^T \Psi (\mathbf{u}) + \varphi\) with \(\widetilde{W} = [\widetilde{W}_1, \ldots, \widetilde{W}_n] \in \mathbb{R}^{n \times n}\) and \(\varphi = [\varphi_1, \ldots, \varphi_n] \in \mathbb{R}^{n \times 1}\). Therefore, (28) becomes:

\[
\dot{s} = M_o^{-1} \tau + \widetilde{W}^T \Psi (\mathbf{u}) + \varphi + M_o^{-1} \tau \text{sign}(p)
\]

\[
+ \frac{\pi \frac{3}{T_{cs}}}{(\beta_1 |p| + \beta_2 |p|^{2-\alpha} + \beta_3 |p|^\alpha) + P(e_1, \dot{e}_1)}
\]

(29)

Drawing upon (29), we formulate the controller as follows:

\[
\tau = -M_0 (\tau_{eq} + \tau_r),
\]

(30a)

\[
\tau_{eq} = H + \widetilde{W}^T \Psi (\mathbf{u}) + P(e_1, \dot{e}_1) + M_o^{-1} \tau \text{sign}(p)
\]

\[
+ \frac{\pi \frac{3}{T_{cs}}}{(\beta_1 |p| + \beta_2 |p|^{2-\alpha} + \beta_3 |p|^\alpha)},
\]

(30b)

\[
\tau_r = \frac{\pi \frac{3}{T_{cr}}}{(\beta_3 |s|^{\alpha} - \beta_3 |s|^\alpha + \beta_3 |s|^\alpha)}\]

(30c)

where \(\beta_{13} = \beta_1, \beta_{23} = \frac{\beta_2}{\beta_1}, \beta_{33} = \beta_3, \text{ and } \nu > 0\).

**Theorem 3:** The proposed approach (30) provides the desired performance within predefined time stability for ELSs with uncertain components.

The control design overview is briefly described in Fig. 1

**Proof 5:** The Lyapunov function \(V_3 = V_{31} + V_{32} = s^T s + \sum_{i=1}^{n} \widetilde{W}_i \gamma_i^{-1} \tilde{w}_i\) is utilized with \(\tilde{W}_i = \widetilde{W}_i - W_i\), where \(\tilde{W}_i\) is the \(i\)th estimation error matrix of the weight. The time derivative of the Lyapunov function is expressed as:

\[
\dot{V}_3 = 2s^T \dot{s} + \sum_{i=1}^{n} \widetilde{W}_i^T \gamma_i^{-1} \dot{\tilde{w}}_i
\]

(31)

where \(\dot{s} = -\widetilde{W}^T \Psi (\mathbf{u}) + \varphi - \tau_r\) and \(\widetilde{W} = \tilde{W} - W\). Therefore, \(V_3 = 2s^T \left(-\widetilde{W}^T \Psi (\mathbf{u}) + \varphi - \tau_r\right) + 2 \sum_{i=1}^{n} \widetilde{W}_i^T \gamma_i^{-1} \tilde{w}_i
\)

\[
= 2s^T \left(-\widetilde{W}^T \Psi (\mathbf{u}) + \varphi - v \text{sign}(s)\right)
\]

\[
- 2\pi \frac{3}{T_{cr}} \left(\beta_1 \sum_{i=1}^{n} s_i^2 + \beta_2 \sum_{i=1}^{n} |s_i|^{3-\alpha} \right)
\]

\[
- 2\pi \frac{3}{T_{cr}} \beta_3 \sum_{i=1}^{n} |s_i|^{4+\alpha} + 2 \sum_{i=1}^{n} \widetilde{W}_i^T \gamma_i^{-1} \tilde{w}_i
\]

\[
\leq 2(\|\varphi\| - \|v\|) \|s\| + 2 \sum_{i=1}^{n} \widetilde{W}_i^T \left(\frac{1}{\gamma_i} \tilde{w}_i - s_i \Psi (\mathbf{u})\right)
\]

\[
- 2\pi \frac{3}{T_{cr}} \left(\beta_1 \sum_{i=1}^{n} s_i^2 + \beta_2 \left(\sum_{i=1}^{n} s_i^2\right)^{\frac{3-\alpha}{2}} \right)
\]

\[
- 2\pi \frac{3}{T_{cr}} \beta_3 \left(\sum_{i=1}^{n} s_i^4 \right)^{\frac{1+\alpha}{2}}
\]

(32)

Let the adaptive rule be: \(\dot{\tilde{W}}_i = \gamma_i s_i \Psi (\mathbf{u})\), where \(\gamma_i > 0\).

With the designed adaptive rule and utilizing Lemma 3 from the study [36], (32) yields:

\[
\dot{V}_3 \leq -2\pi \frac{3}{T_{cr}} \left(\beta_1 V_{31} + \beta_2 V_{31}^{\frac{3-\alpha}{2}} + \beta_3 V_{31}^{\frac{1+\alpha}{2}}\right)
\]

\[
\leq -\frac{2\pi}{T_{cr}} \beta_1 (V_{31} + V_{32})
\]

\[
- 2\pi \frac{3}{T_{cr}} \beta_2 \left(V_{31}^{\frac{3-\alpha}{2}} + V_{32}^{\frac{3-\alpha}{2}}\right)
\]

\[
- 2\pi \frac{3}{T_{cr}} \beta_3 \left(V_{31}^{\frac{1+\alpha}{2}} + V_{32}^{\frac{1+\alpha}{2}}\right) + \Gamma
\]

(33)

\[
\leq -\frac{2\pi}{T_{cr}} \left(\beta_1 V_3 + \beta_2 V_3^{\frac{3-\alpha}{2}} + \beta_3 V_3^{\frac{1+\alpha}{2}}\right) + \Gamma
\]
where $\Gamma = \frac{2\pi^3}{T_{cr}} \left( \beta_{13}V_{32} + \beta_{23}V_{32}^{\frac{1}{2}} + \beta_{33}V_{32}^{\frac{1}{3}} \right) > 0$.

As it well known, the weight update rates of FLS are bounded. $\Gamma$ is a function related to adaptive law of FLS, therefore, it is bounded.

Referring to Lemma 2, it is guaranteed that $s_1 = 0$ within a predetermined time interval denoted as $T_{cr}$.

IV. SIMULATIONS

A. Configuration of the Testing System

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Link 1</th>
<th>Link 2</th>
<th>Link 3</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>$l_{11}, l_{22}, l_{33}$</td>
<td>0.25</td>
<td>0.7</td>
<td>0.6</td>
</tr>
<tr>
<td>Weight</td>
<td>$m_{11}, m_{22}, m_{33}$</td>
<td>33.429</td>
<td>34.129</td>
<td>15.612</td>
</tr>
<tr>
<td>Inertia</td>
<td>$I_{1xx}, I_{2xx}, I_{3xx}$</td>
<td>0.7486</td>
<td>0.3080</td>
<td>0.0446</td>
</tr>
<tr>
<td></td>
<td>$I_{1yy}, I_{2yy}, I_{3yy}$</td>
<td>0.5518</td>
<td>2.4655</td>
<td>0.7092</td>
</tr>
<tr>
<td>Center</td>
<td>$l_{11z}, l_{22z}, l_{33z}$</td>
<td>0.5570</td>
<td>2.3938</td>
<td>0.7207</td>
</tr>
<tr>
<td>of</td>
<td>$l_{11y}, l_{22y}, l_{33y}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Mass</td>
<td>$m_{11z}, m_{22z}, m_{33z}$</td>
<td>-0.7461</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The controller’s efficacy is rigorously evaluated through trajectory-tracking motion control simulations conducted in MATLAB/SIMULINK. The testbed for these simulations is a 3-DOF robotic manipulator, modeled after the PUMA560 robot, as depicted in Fig. 2. This robotic model is meticulously structured, following the rigorous principles of dynamics and kinematics as detailed in research [37]. The selection of system parameters is based on an extensive review of the existing literature, ensuring the model’s accuracy and reliability through simulations in MATLAB/SIMULINK and design in SOLIDWORKS. The geometric parameters are adopted from a 3-DOF robotic manipulator described in our prior work [38], including detailed specifications such as link dimensions, positions of the centers of mass, and inertia properties, as listed in Table I. Fig. 2 offers a visual representation of the robot model through a SOLIDWORKS illustration.

These simulations are pivotal in evaluating several critical performance metrics essential for assessing the controller’s efficiency. These metrics include the convergence rate and tracking accuracy. Moreover, the simulations provide a detailed comparison of tracking error accuracy, resolution of chattering phenomena, and robustness against uncertainties, contrasting the proposed approach (M4) with established methods such as SMC (M1), finite-time SMC (M2) [39] and fixed-time SMC (M3) [40]. The differential equations inherent in the simulations are solved using Euler’s method, ensuring precision with a meticulously set sampling time of $t_s = 10^{-3}$ [s]. This rigorous simulation process highlights the controller’s potential to significantly enhance trajectory tracking in robotic manipulators, offering insights into its comparatively advantageous over traditional control strategies.

<table>
<thead>
<tr>
<th>Method</th>
<th>Control Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1</td>
<td>$\eta_{11}, \eta_{12}$</td>
<td>5, 5</td>
</tr>
<tr>
<td>M2</td>
<td>$\eta_{21}, \eta_{22}, \eta_{23}, \eta_{24}$</td>
<td>5, 5, 5, 5</td>
</tr>
<tr>
<td>M3</td>
<td>$\eta_{31}, \eta_{32}, \eta_{33}, \eta_{34}$</td>
<td>5, 5, 5, 5</td>
</tr>
<tr>
<td>M4</td>
<td>$\beta_{1}, \beta_{2}, \beta_{3}, \alpha_{i}$</td>
<td>0.8, 1.2, 0.8, 1.2, 1.5</td>
</tr>
<tr>
<td></td>
<td>$T_{cr}, T_{ca}, T_{cr}, v$</td>
<td>0.5, 0.2, 0.1, 0.1</td>
</tr>
<tr>
<td></td>
<td>$\tau_{\max, \min}, \tau$</td>
<td>$(\pm 150, \pm 360, \pm 100)^T$, $(0.1, 0.1, 0.1)^T$</td>
</tr>
</tbody>
</table>

The control strategies for manipulator systems, M1, M2, and M3, are summarized with their respective control signal equations.

The M1 [41] controller has the control input given by: $\tau = -M_0 (\eta_{11} \dot{e}_1 + \mathbf{H} + \mathbf{L} \text{sign}(s) + \eta_{12} s)$, where $s = \dot{e}_1 + \eta_{11} e_1$. The parameters $\eta_{11}$ and $\eta_{12}$ are positive constants.

The M2 employs a control law: $\tau = -M_0 \left( \left( \eta_{21} + \eta_{22} \eta_{21} e_1 \right) e_2^{\frac{1}{3}} \right) \dot{e}_1 + \mathbf{H} + \mathbf{L} \text{sign}(s) + \eta_{32} s + \eta_{32} \left( s^{\frac{3}{2}} \right)^{\eta_{32}}$, where $s = \dot{e}_1 + \eta_{21} e_1 + \eta_{22} e_1^{\eta_{22}}$. The parameters $\eta_{21}, \eta_{22}, \eta_{32}, \eta_{34}$ as positive constants, and $\eta_{22} > 0, 0 < \eta_{24} < 1$.

The M3 is defined as: $\tau = -M_0 \left( \left( \eta_{31}, \eta_{32} \right) e_1^{\frac{1}{3}} + \eta_{32} e_2^{\frac{1}{2}} \right) \dot{e}_1 + \mathbf{H} + \mathbf{L} \text{sign}(s) + \eta_{33} \left( s^{\frac{3}{2}} \right) + \eta_{34} \left( s^{\frac{3}{2}} \right)^{\eta_{34}}$, with $s = \dot{e}_1 + \eta_{31} e_1 + \eta_{32} e_1^{\eta_{32}} + \eta_{32} e_1^{\eta_{32}}$. The parameters $\eta_{31}, \eta_{32}, \eta_{33}, \eta_{34}$ are positive constants, and $\eta_{31}, \eta_{33} > 1$, $0 < \eta_{32}, \eta_{34} < 1$.

The robot arm’s tracking task is to follow a predefined trajectory with its end effector, defined as follows: $X = 0.85 - 0.01 t$ [m]; $Y = 0.2 + 0.2 \sin(0.5 t)$ [m]; and $Z = 0.7 + 0.2 \cos(0.5 t)$ [m].

To evaluate the proposed solution’s effectiveness and robustness, simulations incorporate uncertain factors such as dynamic errors, disturbances, and friction. Dynamic errors are modeled as $\delta \mathbf{m}(q) = 0.3 M_0(q)$, $\delta \mathbf{C}(q, \dot{q}) = 0.3 C_0(q, \dot{q})$, and $\delta \mathbf{G}(q) = 0.3 G_0(q)$, representing 30% of $M_0(q)$, $C_0(q, \dot{q})$, and $G_0(q)$, respectively. For each joint, disturbances and friction forces are defined respectively as: $\tau_{d1} = 4 \sin(t) + 0.1 \sin(\dot{q}_1) + 2 \dot{q}_1$ [N.m], $\tau_{d2} = 5 \sin(t) + 0.1 \sin(\dot{q}_2) + 2 \dot{q}_2$ [N.m], and $\tau_{d3} = 6 \sin(t) + 0.1 \sin(\dot{q}_3) + 2 \dot{q}_3$ [N.m]. Identical initial conditions are established for all robot states.

The chosen control parameters are detailed in Table II. In the proposed strategy, the AFLS receives inputs reflecting position tracking errors and their derivatives, represented as $\mathbf{u} = \left( \dot{e}_1^T, \dot{e}_2^T \right)^T \in \mathbb{R}^{2n+1}$. This setup employs five symmetric Gaussian membership functions, numbered $j = 1$ through
Fig. 4. Uncertain Components Approximation Output of AFLS at Each Joint.

Fig. 3. State Response of Auxiliary Dynamic System.

\[ j = 5, \text{ each fine-tuned through a detailed trial-and-error methodology.} \]

These membership functions, denoted as \( \phi_{R_i} \), are structured as follows:

\[ \phi_{R_i} = \exp \left[ -\left( \frac{u_i + \frac{3}{200}}{\frac{200}{200}} \right)^2 \right], \quad \phi_{R_i}^2 = \exp \left[ -\left( \frac{u_i + \frac{3}{200}}{\frac{200}{200}} \right)^2 \right], \]

\[ \phi_{R_i}^3 = \exp \left[ -\left( \frac{u_i - \frac{3}{200}}{\frac{200}{200}} \right)^2 \right]. \]

The root-mean-square method (RMSM) quantifies tracking error, aiding in the comparative evaluation of control methodologies for precision and stability. The RMSM formula is:

\[ E_{1,2,3} = \sqrt{\frac{1}{J} \sum_{i=1}^{J} (q_{d1,i} - q_{1,i})^2}, \text{ where } J \text{ is the total number of samples.} \]

In assessing the accuracy of the compared control methods, we utilize the root-mean-square error (RMSE) to quantify tracking errors at each joint following convergence to equilibrium, spanning from 2 to 20 seconds. The detailed results can be found in Table III. Additionally, Fig. 5 depicts the end-effector trajectory for all four controllers. Upon examining these figures, it is evident that controllers \( M_1, M_2, M_3, \) and \( M_4 \) display commendable tracking performance, indicating their robustness in handling uncertainties.

In Fig. 6, a 3D diagram showcases a comparison of RMSE observed at the robot’s joint angles. Figs. 7, 8, and 9 provide further insights into the tracking errors at these joints resulting from the various controllers.

Upon examining the convergence rates of all methods from the zoomed-in subplots of Figs. 7 through 9, offering a detailed view of this phase from 0 to 1 second, it becomes readily apparent the convergence time of the tracking errors. Both \( M_2 \) and \( M_3 \) demonstrate quicker convergence rates compared to \( M_1 \), with \( M_4 \) emerging as the fastest among them. The convergence time of all methods at each joint, in order, is approximately \( M_1 (\approx 1 \text{ second}), M_2 (\approx 0.5 \text{ seconds}), M_3 (\approx 0.3 \text{ seconds}), \) and \( M_4 (\approx 0.2 \text{ seconds}) \). This indicates that the convergence time of \( M_4 \) falls within the predefined time.

Additionally, examining the data presented in Figs. 6 through 9, which include zoomed-in subplots spanning intervals from 1 to 20 seconds, alongside the results tabulated in Table III, reveals that \( M_2 \) and \( M_3 \) exhibit notably superior accuracy when compared to \( M_1 \), with \( M_3 \) showing a slightly higher degree of precision than \( M_2 \). This heightened accuracy is attributed to the utilization of advanced sliding mode surface designs in \( M_2 \) and \( M_3 \), which embrace nonlinearity, diverging from the linear approach of \( M_1 \). Moreover, \( M_4 \) achieves superior tracking accuracy among all methods, as evidenced by the data in Figs. 6 through 9 and Table III, when compared to the other methods.

The control torques of all four methods are depicted in Fig. 10. Upon observation, \( M_1, M_2, \) and \( M_3 \) exhibit apparent chattering behavior in their control signals. This phenomenon stems from the substantial gain values assigned in their reaching control law (RCL) to improve the robustness in handling total uncertainties. However, such chattering behavior can have detrimental effects on the mechanical components and operational lifespan of the robot manipulators. There are trade-offs to consider.

In contrast, the chattering phenomenon observed in \( M_4 \) is not serious. This smoother behavior is made possible by the AFLS designed in \( M_4 \), enabling the approximation of unstructured model uncertainty and compounded disturbances, seamlessly integrating them into the control design. As a result, only an insignificant gain value is utilized to address

**B. Discussion of Performance Results**

Fig. 3 indicates that the convergence time of the state response of the proposed PTADS at each joint falls within the predefined time.

As shown in Fig. 4, the sum of \( L \) and \( Q(e_1, \dot{e}_1) \), representing uncertain and singularity components, can be quickly obtained by an AFLS. This provides precise information to the control loop, thereby improving control performance.

In assessing the accuracy of the compared control methods, we utilize the root-mean-square error (RMSE) to quantify tracking errors at each joint following convergence to equilibrium, spanning from 2 to 20 seconds. The detailed results can be found in Table III. Additionally, Fig. 5 depicts the end-effector trajectory for all four controllers. Upon examining these figures, it is evident that controllers \( M_1, M_2, M_3, \) and \( M_4 \) display commendable tracking performance, indicating their robustness in handling uncertainties.

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---

**TABLE III**

**RMSEs via Three Control Algorithms.**

<table>
<thead>
<tr>
<th>Method</th>
<th>( E_1 )</th>
<th>( E_2 )</th>
<th>( E_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_1 )</td>
<td>( 1.774 \times 10^{-4} )</td>
<td>( 3.305 \times 10^{-4} )</td>
<td>( 3.439 \times 10^{-4} )</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>( 2.325 \times 10^{-5} )</td>
<td>( 4.433 \times 10^{-5} )</td>
<td>( 5.136 \times 10^{-5} )</td>
</tr>
<tr>
<td>( M_3 )</td>
<td>( 2.072 \times 10^{-5} )</td>
<td>( 3.625 \times 10^{-5} )</td>
<td>( 4.924 \times 10^{-5} )</td>
</tr>
<tr>
<td>( M_4 )</td>
<td>( 6.71 \times 10^{-7} )</td>
<td>( 2.42 \times 10^{-7} )</td>
<td>( 5.44 \times 10^{-6} )</td>
</tr>
</tbody>
</table>
the approximation error raised by the AFLS. This approach ensures that M4 exhibits smoother control torques among the four controllers, as evidenced in Fig. 10, while still maintaining robustness and high accuracy, as shown in Figs. 7 through 9.

Upon closer examination of the enlarged plots in Fig. 10b, it is apparent that the actual control torques \((\tau_1, \tau_2, \tau_3)\) consistently stay within the bounds of input saturation. This finding suggests that the PTADS effectively addresses the nonlinearity introduced by input saturation.

V. CONCLUSION

In summary, this paper introduced a novel approach to PT-TSMC, specifically tailored for ELSs facing actuator con-
control input saturation. By leveraging compensation techniques grounded in an AFLS, we have effectively enhanced control performance and robustness, while also addressing challenges such as chattering reduction and singularity elimination. Moreover, our method adeptly handles unknown model information through a fuzzy approximation mechanism. The incorporation of adjustable parameters further enhances the adaptability of the controller design, ensuring optimal stability time tuning for the system. Through rigorous comparative simulations, our proposed approach consistently demonstrates commendable control performance across various scenarios, validating its effectiveness in achieving the desired performance within predefined time stability for ELSs.

Future work may include validating the proposed approach in real-world experiments, optimizing controller parameters for specific ELSs, and extending the method to tackle other control challenges. Additionally, enhancing computational efficiency for real-time implementation and exploring applicability to various ELS types are potential research directions.

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REFERENCES


