Receiver Architecture Design and Analysis for NOMA-Based Multi-User Communication Systems

Shimaa Naser$^1$, Sami Muhaedat$^1$, and Zhiguo Ding$^1$

$^1$Affiliation not available

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Abstract — The sixth-generation (6G) wireless network promises unprecedented enhancements in terms of system throughput, energy efficiency, traffic capacity per area, spectral efficiency, and low latency. To meet these demands, the radio interface must demonstrate the adaptability and efficient utilization of scarce frequency resources, necessitating novel multiple access techniques and new waveforms. Furthermore, in large-bandwidth multi-user networks, intersymbol interference (ISI) and inter-user interference (IUI) represent major design challenges. In this regard, time reversal (TR) has emerged as a promising 6G waveform candidate, concentrating signal energy in both the time and space domains in multipath environments. On the other hand, non-orthogonal multiple access (NOMA) promises high spectral efficiency and enhanced connectivity, serving multiple users over the same time-frequency-code resources. In this paper, we investigate the integration of NOMA and TR to mitigate these challenges and propose, for the first time in the literature, a novel receiver architecture for downlink NOMA-based TR communications, which does not require precoding at the transmitter. In more detail, power-domain NOMA is employed by the transmitter, while TR filtering is carried out at each receiving end. We derive novel approximated expressions for the pairwise error probability (PEP), a fundamental component in establishing the union bound on the bit error rate (BER), to characterize users’ performance. We extensively employ Monte Carlo simulations and numerical analyses to verify the analytical expressions, providing significant insights into the error rate performance for each user. Also, we investigate the performance gain of the proposed NOMA-based TR receiver, over the orthogonal multiple access scheme, namely time-reversal multiple access (TRMA). Results demonstrate the superiority of our scheme, in terms of the bit error rate (BER), particularly in sparse multi-path environments compared to TRMA, with a percent of improvement in the average BER between 73.5%—98.31%. This improvement is also accompanied by a reduced overhead compared to traditional TRMA, which necessitates users’ channel state information feedback to the base station for TR precoding. Moreover, our findings indicate that at high signal-to-noise ratio values, the diversity gain for a particular user is proportional to the product of the user’s order, determined by its channel strength, and the number of its channel taps.

Index Terms — Error rate analysis, multi-user communications, non-orthogonal multiple access, pairwise error probability, and time reversal.

I. INTRODUCTION

The sixth generation (6G) of wireless communication is envisaged to support a plethora of data-hungry and delay-sensitive applications, which impose challenging limitations on the network’s performance in terms of reliability, latency, data rate, and energy consumption. This has motivated the research community to explore higher frequency ranges to meet the envisioned data rate requirements in 6G networks. Nevertheless, communication at higher frequencies requires energy-efficient radio frequency (RF) front-end design and computationally efficient signal processing techniques, which are the main challenges to be tackled to achieve the vision of 6G networks. Therefore, new waveforms must be efficiently designed to utilize the available spectrum to deliver higher system throughput, adapt to diverse and dynamic environments, and ensure energy efficiency in the face of evolving concerns about sustainability. Moreover, the waveforms of 6G networks should exhibit improved spectral efficiency and security features to meet the demands of emerging applications.

Within this context, time reversal (TR), a linear precoding technique, has been proposed as a promising 6G waveform candidate to preprocess the signals before transmission by using the time-reversed (and conjugate if complex) channel impulse response (CIR) [1]. Such a filter can focus the energy of the signals, in a rich scattering environment, in both space and time domains at the intended receiver. It is worth noting that the time-focusing property of TR can assist in reducing the inter-symbol interference (ISI) that severely degrades the system performance in multipath environments. Furthermore, it is noteworthy that, the spatial focusing property offers great potential for reducing inter-user interference (IUI) [2], [3]. Therefore, the intended user receives high energy from all harvested multipath components without the need for complicated multi-antenna RF front ends at the receiver which requires heavy signal processing techniques. Thus, it is anticipated that TR will play a crucial role in supporting low-complexity, energy-efficient, and secure communications in future 6G networks.

From the perspective of energy harvesting, TR enhances the capabilities of wirelessly powered devices by making full use of the multipath propagation to sustain the received power at the harvester input [4]. This is different from conventional harvesting techniques where the power density of the wireless signal received by the antenna is affected by multiple factors including multipath channels, shadowing effects, and large-scale path loss. As a consequence, only a fraction of the transmitted energy from the source can be successfully harvested by the receiver. Although multi-antenna techniques have been
employed to tackle the limited power issue, incorporating additional RF chains can increase the overall implementation cost. Moreover, in indoor scenarios with rich scattering, multi-antenna processing schemes may not function properly due to the blockage caused by potential obstacles.

Motivated by the promising capabilities of TR, several research works have investigated its performance in different contexts. The authors in [5] and [6] studied the performance of TR communications in an ultra-wideband (UWB) communication system assuming perfect channel state information (CSI) at the transmitter. However, in practical scenarios, channel estimation experiences uncertainties, mainly due to the uplink and downlink estimation errors (CEEs), which highly affect the focusing feature of TR and hence, the performance of the underlying TR scheme. Within the same context, the authors in [7] quantified the channel capacity and the bit error rate (BER) performance of TR-assisted multi-user UWB systems, while considering both spatial correlation and CEE. It was demonstrated that spatial correlation and CEE result in significant deterioration in the signal-to-interference-plus-noise ratio (SINR), hence, both must be fully considered during the system design to avoid the need for deploying complicated receivers or a large number of antennas. Nevertheless, broadband systems encounter less scattering in the propagation channels [8]. Hence, in scenarios where users are located close to each other, channels of different users experience high correlation. In such cases, the focusing gain of TR can be significantly degraded, which leads to a significant loss of system performance. By focusing the signal toward a legitimate receiver, TR is also expected to enhance the secrecy of the communication. Thus, different attempts have investigated the performance of TR in the context of physical layer security [9], [10]. Finally, the authors in [11] experimentally demonstrated the spatiotemporal-focusing capability of TR at different carrier frequencies in the sub-6-GHz, mmWave, and sub-THz bands.

Although the primary goal of the TR transmissions is to focus the received signal in time and space at a designated receiver, interference in both time and space domains is unavoidable. This interference manifests itself as ISI and IUI, particularly in multi-user scenarios [12]. This challenge becomes particularly severe in scenarios with spatially correlated signal propagation paths among different users. Furthermore, existing research works on TR-based multi-user communication typically assume that the number of channel taps is substantially greater than the number of users. However, in situations with reduced scattering, the correlation among the users’ channels significantly increases. Such correlation reduces the effectiveness of spatial-temporal focusing, resulting in a degraded performance of the TR-based multi-user communication. Consequently, it becomes critical to develop a new framework that mitigates the above-mentioned challenges for the successful implementation of TR-based multi-user communication systems. In this regard, non-orthogonal multiple access (NOMA) has emerged as a promising multiple access scheme for improved spectral efficiency and enhanced connectivity [13]–[15]. The fundamental principle of NOMA involves enabling multiple users to concurrently share the same time/frequency resources. In this approach, distinct power levels are assigned to different users at the transmitter, followed by the utilization of superposition coding (SC) to multiplex and transmit the signals. On the receiver side, successive interference cancellation (SIC) is applied to eliminate signals from users with higher power levels, facilitating the detection of the intended user’s signal [16]. While NOMA has been explored in various systems, its integration with TR systems has received limited attention. To the best of the authors’ knowledge, the only existing work in the literature that explores NOMA for TR systems was presented in [17], where the authors investigated the uplink sum rate performance. In [17], the base station (BS) was assumed to employ a TR filter for each user to suppress the IUI and employ SIC for the detection of each user’s signal. Unlike the work presented in [17], in this paper, we investigate the integration of NOMA into a multi-user downlink TR communication system. Compared to [17] which employs TR filters at the BS, our work considers employing TR filters at the receivers’ end; thereby reducing the feedback overhead required for CSI acquisition at the BS. The main contributions of this paper are summarized as follow:

- Propose a novel receiver architecture that integrates NOMA with TR in a downlink frequency selective channel. The proposed NOMA-based TR receiver mitigates IUI and realizes enhanced connectivity at a lower cost compared to conventional TR systems. This property is due to the fact that the proposed receiver does not require CSI knowledge or TR filtering at the transmitter.
- Investigate the pair-wise probability (PEP) performance of the proposed NOMA-based TR receiver, where novel closed-form PEP expressions are derived under the assumption of sufficient upsampling ratio.
- Demonstrate that the proposed NOMA-based TR receiver outperforms conventional TR transmissions and offers a low complexity solution, particularly in sparse multipath environments where the spatial/temporal focusing capability of TR deteriorates.
- Capitalizing on the derived PEP expressions, the BER union bound is deduced for the proposed NOMA-based TR receiver, where meaningful insights are obtained. Moreover, the obtained formulations for the PEP are utilized to derive the diversity gain of all users.
- Monte Carlo simulations are conducted to validate the derived expressions and to demonstrate the error rate performance and diversity gain of individual users compared to the conventional TR multiple access (TRMA). Results demonstrate the superiority of our receiver in sparse multi-path environments. This improvement is also accompanied by a reduced overhead compared to TRMA.
- Demonstrate that at high signal-to-noise ratio (SNR) val-
ues, the diversity gain for a particular user is proportional to the product of the user’s order, determined by its channel strength, and the number of channel taps.

The rest of the paper is organized as follows. In Sec. II, the adopted system and channel models are presented. The analysis of PEP for different users is investigated in Sec. III, where the diversity gain achieved by TR-NOMA is obtained. Numerical and simulation results are presented in Sec. IV and the paper is concluded in Sec. V.

Notation: The complex conjugate, absolute value, the $l_2$ norm operations, and the complex conjugate are denoted by $(\cdot)^*$ and $|\cdot|$, $||\cdot||$, $\circ$, respectively. Also, the ceiling operation is represented by $\lceil \cdot \rceil$. The real part of a complex number is denoted by $\Re\{\cdot\}$. Detected data symbols are represented as $\hat{x}$ while incorrectly detected symbols are denoted as $\tilde{x}$. The difference $x - \hat{x}$ is denoted by $\Delta$. Finally, the expectation is donated by $\mathbb{E}(\cdot)$.

II. SYSTEM AND CHANNEL MODELS

We consider a multi-user downlink system, which consists of $K$ single-antenna users and a BS, as depicted in Fig. 1. The distance between BS and the $k$th users is denoted by $d_k$. Furthermore, the channels from the BS to the users are assumed to be independent and identically distributed (IID) quasi-static fading channels and experience frequency selective fading with the same number of taps $L$. The number of channel taps $L$ is calculated as $\lceil (-\sigma_{T_s} \log(A)/T_s) \rceil + 1$, where $A$ is a ratio of non-negligible path power to the first path power in linear scale, $\sigma_{T_s}$ is the delay spread of the $k$th user channel, and $T_s$ is the sampling period, i.e., $T_s = 1/B$ where $B$ is the bandwidth [18]. More specifically, $h_k = [h_k[0], h_k[1], ..., h_k[L-1]]^T \in \mathbb{C}^{L \times 1}$ denotes the $L$-tap CIR of the link between the BS and the $k$th user. In this work, we assume that the $l$th tap channel coefficient $h_k[l]$ is a circular symmetric complex Gaussian random variable (RV) with zero mean and the following variance

$$\sigma_{k,l}^2 = \mathbb{E}[|h_k[l]|^2] = \frac{\eta_{h_k}}{\sigma_{T_s} \sum_{l=0}^{L-1} \sigma_{k,l}^2} \exp(-l T_s/\sigma_{T_s}), \forall l$$

where $\eta_{h_k}$ is the large-scale path loss. We assume that the BS employs NOMA to multiplex users’ messages in the power domain. Thus, without loss of generality, the $K$ users are sorted first in ascending order based on their channel gains as $||h_1||^2 \leq ||h_2||^2 \leq \ldots \leq ||h_K||^2$, hence, the power allocation coefficients are selected based on the ordered channel gains of the users, i.e., $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_K$. Then, the user’s sequences, $x_1, x_2, \ldots, x_K$, each of length $N$, are first upsampled at the BS with a rate back-off factor $D$ to reduce the ISI, i.e., $D-1$ zero samples are inserted between each two original consecutive symbols. Therefore, the $n_{th}$ sample of the $k$th user upsampled signal is given by

$$x_k^D[n] = \begin{cases} x_k[m], & n = mD \\ 0, & \text{else} \end{cases}, \quad (2)$$

where $x_k^D$ is the upsampled sequence of user $k$. Then, the BS transmits the superposition of all upsampled users’ messages as follows:

$$x = \sum_{k=1}^{K} \sqrt{\alpha_k} x_k^D,$$

where $\alpha_k$ is the power allocation of the $k$th user. Thus, the received signal at the $k$th user is the result of the convolution of its CIR and the transmitted signal as follows

$$\tilde{y}_k^D = h_k \circ x + \tilde{n}_k = h_k \circ \sum_{i=1}^{K} \sqrt{\alpha_i} x_i^D + \tilde{n}_k,$$

where each component of $\tilde{n}_k$ is an additive white Gaussian noise (AWGN) with zero mean and $\sigma^2$ variance. At the $k$th user, the received signal is filtered out using the time-reversed and conjugate of the corresponding CIR vector, $g_k$, as the following:

$$y_k^D = g_k \circ h_k \circ \sum_{i=1}^{K} \sqrt{\alpha_i} x_i^D + n_k^D,$$

where $\sigma_{T_s}$ is the delay spread of the $k$th user channel, and $T_s$ is the sampling period, i.e., $T_s = 1/B$ where $B$ is the bandwidth [18]. More specifically, $h_k = [h_k[0], h_k[1], ..., h_k[L-1]]^T \in \mathbb{C}^{L \times 1}$ denotes the $L$-tap CIR of the link between the BS and the $k$th user.
where $\mathbf{n}^D_k = \mathbf{g}_k \odot \tilde{n}_k$ and $\mathbf{g}_k$ represents the time-reversed and conjugate of the $k^{th}$ user CIR vector, which is given by:

$$\mathbf{g}_k[n] = \frac{\mathbf{h}^*_k[L-1-n]}{||\mathbf{h}_k||}.$$  

(6)

It is worth noting that the term $1/||\mathbf{h}_k||$ is added for normalization purposes to ensure that the transmitted power remains constant in every realization. Then, by downsampling $\mathbf{y}_k^D$ by the same rate back-off factor $D$, the $n^{th}$ sample of the received signal can be represented by the following

$$\mathbf{y}_k[n] = \sum_{i=1}^{K} \sum_{l=0}^{(2L-2)/D} \sqrt{\alpha_i} (\mathbf{h}_k \odot \mathbf{g}_k)[Dl]\mathbf{x}_k[n-l]+\mathbf{n}_k[n],$$  

(7)

where $n = 0, 1, ..., 2L-2$ and $\mathbf{n}_k[n] = \mathbf{n}^D_k[DN]$ is an AWGN with zero mean and variance $\sigma^2$. Also,

$$(\mathbf{h}_k \odot \mathbf{g}_k)[n] = \sum_{l=0}^{L-1} \mathbf{h}_k[l]^{\odot} \mathbf{g}_k[n-l] - \mathbf{h}_k[l_1]^{\odot} \mathbf{g}_k[n-l_1];$$  

(8)

It is important to highlight that the $k^{th}$ user selects the sample characterized by the maximum-power central peak, denoted by $n = (L - 1)/D + \nu$, $\nu = 0, ..., N - 1$, to decode the $\nu^{th}$ symbol. The signal received at this sample is expressed in (9), as shown on the top of the subsequent page. Notably, the initial term in (9) denotes the combination of the desired signal and interference from the other users, which is removed through SIC. Conversely, the second term includes both ISI and IUI. However, owing to the higher power of the first term attributed to TR focusing, the second term is treated as noise in the detection process. The SIC receiver leverages variations in power levels among users to eliminate interference caused by users’ signals with higher allocated power. For instance, the $k^{th}$ user receiver cancels the effect of the interfering signals of users $1, ..., k-1$ from the first term in (9). The rest of the users signals, i.e., $k+1, ..., K$, will be treated as noise. Since the first user is assigned the largest power coefficient, this user does not need to carry out SIC, where signals from all other users are treated as noise. Thus, utilizing the maximum likelihood detection (MLD), the first user decodes its sequence sequentially using the following rule

$$\hat{x}_1[n - \frac{L-1}{D}] = \arg \min_{\hat{x}_1 \in \mathcal{X}} \left( \left| y_1[n] - \sqrt{\alpha_1}||\mathbf{h}_1||\hat{x}_1[n - \frac{L-1}{D}] \right| \right)^2,$$  

(10)

where $\mathcal{X}$ is a set of an arbitrary signal constellation. On the other hand, for the $k^{th}$ user, $k > 1$, the output of the MLD process is given as

$$\hat{x}_k[n - \frac{L-1}{D}] = \arg \min_{\hat{x}_k \in \mathcal{X}} \left( \left| \tilde{y}_k[n] - \sqrt{\alpha_k}||\mathbf{h}_k||\hat{x}_k[n - \frac{L-1}{D}] \right| \right)^2,$$  

(11)

where $\tilde{y}_k[n]$ is the result of the $k-1$ SIC iterations given by (12) on the top of the next page.

### III. PEP Analysis

In this section, we derive the expressions for the PEP for the $k^{th}$ users of the proposed NOMA-based TR system. Leveraging the derived PEP expression, we obtain the BER union bound, a widely embraced method for obtaining a reliable assessment of error rate performance when exact BER derivation is mathematically challenging.

#### A. PEP analysis for the first user

Since the first user is allocated the highest power level, it decodes its own message directly, by assuming the interference from other users’ messages as a noise. Utilizing (10), the PEP of the first user, $\Pr(x_1[\nu], \hat{x}_1[\nu]|\mathbf{h}_1)$, is defined as the probability of detecting symbol $\hat{x}_1[n - (L-1)/D]$ given that symbol $x_1[n - (L-1)/D]$ was transmitted. Since $\nu = n - (L-1)/D$, the conditional PEP of the first user can be evaluated as follows

$$\Pr(\hat{x}_1[\nu], \tilde{x}_1[\nu]|\mathbf{h}_1) = \Pr \left( \left| y_1[n] - \sqrt{\alpha_1}||\mathbf{h}_1||\hat{x}_1[\nu] \right|^2 \leq \left| y_1[n] - \sqrt{\alpha_1}||\mathbf{h}_1||\tilde{x}_1[\nu] \right|^2 \right).$$  

(13)

where $\hat{x}_1[\nu] \neq x_1[\nu]$. By expanding (13), the conditional PEP of the first user can be represented as (14) at the bottom of the next page, where $\Delta_1 = x_1[\nu] - \hat{x}_1[\nu]$ and $n = (L-1)/D + \nu$, $\nu \in [0, N - 1]$. It is worth mentioning that the real-valued part of the Gaussian noise $\tilde{n}_1[n]$ in (14) is normally distributed with zero mean and $\sigma^2/2$ variance, and hence, the decision variable is Gaussian, i.e.,

$$2 \Re \left( \sqrt{\alpha_1}||\mathbf{h}_1||\Delta_1\tilde{n}_1[n] \right) \sim \mathcal{N} \left( 0, 2\alpha_1\sigma^2||\mathbf{h}_1||^2||\Delta_1||^2 \right).$$

Recalling that, for $X \sim \mathcal{N} \left( \mu, \sigma^2 \right)$ [19]

$$\Pr(\hat{x}_1[\nu], \tilde{x}_1[\nu]|\mathbf{h}_1) = \mathcal{Q} \left( \frac{\zeta_1}{\sqrt{2}\sigma||\Delta_1||} \right),$$  

(17)

where $\zeta_1$ is given in (15) at the bottom of the next page with $n = (L-1)/D + \nu$, $\nu \in [0, N - 1]$. It is important to point out that deriving a closed-form expression for the PEP is challenging, mainly due to the ISI term in the $Q$-function in (17). However, when $D$ (the up-sampling factor) is assumed to be sufficiently large, the ISI power becomes insignificant since increasing the up-sampling factor further helps in reducing the ISI power. Thus, the PEP for the first user in (17) can be approximated as follows

$$\Pr(\hat{x}_1[\nu], \tilde{x}_1[\nu]|\mathbf{h}_1) = \mathcal{Q} \left( \frac{||\mathbf{h}_1||B_1}{A_1} \right),$$  

(18)

where $B_1 = \sqrt{\alpha_1}||\Delta_1||^2 + 2\Re \left( \Delta_1\sum_{i=2}^{K} \sqrt{\alpha_i}\hat{x}_i[\nu] \right)$ and $A_1 = \sqrt{2}\sigma||\Delta_1||$. Then, to get the unconditional PEP, we average over the probability density function (PDF) of $||\mathbf{h}_1||$. 

...
where the PDF and CDF of the norm of the channel vector are given as

\[ f_{\|h\|} = \sum_{i=1}^{K} \sqrt{\alpha_i} |h_i| |x_i[n - L/D]| + \sum_{i=1}^{K} \sum_{l=0, l \neq (L-1)/D}^{(2L-2)/D} \sqrt{\alpha_i} \frac{\sum_{j=0}^{L-1} h_k[j] h_k^*[L - DL + j]}{\|h_k\|} x_i[n - l] + n_k[n], \]

\[ n = \frac{L-1}{D} + \nu, \nu = 0, \ldots, N-1. \]

The Proof follows:

\[ \bar{y}_k[n] = \sum_{i=k}^{K} \sqrt{\alpha_i} |h_i| |x_i[n - L/D]| + \sum_{i=1}^{K} \sum_{l=0, l \neq (L-1)/D}^{(2L-2)/D} \sqrt{\alpha_i} \frac{\sum_{j=0}^{L-1} h_k[j] h_k^*[L - DL + j]}{\|h_k\|} x_i[n - l] + n_k[n]. \]

Since the users are ordered based on their channel gains, ordered statistics should be considered when evaluating the PDF of the norm of the channels. Therefore, the ordered PDF of the norm of the \( k \)th user channel, \( \|h_k\| \), is given by

\[ f_{\|h\|}(\omega_k) = C_k f_x(\omega_k) [F_x(\omega_k)]^{k-1} (1 - F_x(\omega_k))^{K-k}, \]

where \( \omega_k = \|h_k\| \) and \( C_k = K!/(k-1)! (K-k)! \). Also, \( f_x(\omega_k) \) and \( F_x(\omega_k) \) are the PDF and cumulative distribution function (CDF) of \( \|h_k\| \), respectively.

**Lemma 1.** The PDF and CDF of the norm of the \( k \)th user channel vector are given as

\[ f_x(\omega_k) = 2\omega_k \sum_{l=0}^{L-1} W_{k,l} \exp(-\frac{\omega_k^2}{\sigma_{k,l}^2}) \]

and

\[ F_x(\omega_k) = \sum_{l=0}^{L-1} W_{k,l} (1 - \exp(-\frac{\omega_k^2}{\sigma_{k,l}^2})), \]

respectively.

**Proof.** Let \( S_k = \sum_{l=0}^{L-1} |h_k[l]|^2 \). From [20], the PDF of \( S_k \) is given by

\[ f_{S_k}(s_k) = \sum_{l=0}^{L-1} W_{k,l} \exp(-\frac{s_k}{\sigma_{k,l}^2}), \]

where \( W_{k,l} = \prod_{i=0, i \neq l}^{L-1} \sigma_{k,l}^2/(\sigma_{k,l}^2 - \sigma_{k,i}^2) \). Assuming that \( \omega_k = \sqrt{\frac{S_k}{\lambda_k}} \), then the PDF of \( \omega_k \) can be written as \( f_x(\omega_k) = 2\omega_k f_{S_k}(\omega_k^2) \). Alternatively, it can be written as

\[ f_x(\omega_k) = 2\omega_k \sum_{l=0}^{L-1} W_{k,l} \exp(-\frac{\omega_k^2}{\sigma_{k,l}^2}). \]

To evaluate the CDF of \( \omega_k \), we utilize the results for the CDF of \( S_k \) from [20]. The CDF of \( S_k \) is given as

\[ F_{S_k}(s_k) = \sum_{l=0}^{L-1} W_{k,l} (1 - \exp(-\frac{s_k}{\sigma_{k,l}^2})). \]

Thus, the CDF of \( \omega_k \) is obtained by transformation of variables, i.e., \( F_x(\omega_k) = F_{S_k}(\omega_k^2) \). Hence, \( F_x(\omega_k) \) can be written as

\[ F_x(\omega_k) = \sum_{l=0}^{L-1} W_{k,l} (1 - \exp(-\frac{\omega_k^2}{\sigma_{k,l}^2})). \]

\[ \Pr \left( x_1[n], x_1[n] | h_1 \right) = \Pr \left( 2 \sqrt{\alpha_1} |h_1| |\Delta_1 n_1[n]| \leq \frac{\alpha_1 |h_1| |\Delta_1| |\alpha_1 h_1|^2 |\Delta_1|^2}{\sum_{i=2}^{K} \sqrt{\alpha_i} x_i[n]} + \sum_{i=1}^{K} \sqrt{\alpha_i} x_i[n - l] \sum_{j=0}^{L-1} h_1[L - DL + j] h_i^*[j] \right). \]

\[ \zeta_1 = \sqrt{\alpha_1} |h_1| |\Delta_1|^2 + \sum_{i=2}^{K} \sqrt{\alpha_i} x_i[n] + \sum_{i=1}^{K} \sqrt{\alpha_i} x_i[n - l] \sum_{j=0}^{L-1} h_1[L - DL + j] h_i^*[j]. \]
By first utilizing (19) and Lemma 1 to obtain the ordered PDF of the first user, i.e., \( k = 1 \), and then averaging (18) over the evaluated PDF, the expression of the first user’s PEP can be derived.

**Proposition 1.** The exact PEP for the first user is given by (20), at the top of the next page, where \( \text{PEP}_1 := \Pr(x_1[v], \bar{x}_1[v]) \) and \( \mu_{1,l} = \frac{1}{\sigma_{l,j}} + \sum_{t=0}^{L-1} \frac{m_t}{\sigma_{l,t}}. \)

**Proof.** See Appendix I.

**B. PEP analysis for the \( k \)th user**

The \( k \)th user needs to detect and subtract the signals of all other users with lower detection order, i.e. \( x_1[v], \ldots, x_{k-1}[v] \) before decoding its own message, as indicated in (11) and (12). Consequently, the residual interference from \( x_{k+1}[v], \ldots, x_N[v] \) diminishes and can be treated as an additive noise. Therefore, the PEP of the \( k \)th user can be evaluated as shown in (21) on the top of the next page, where \( n = (L-1)/D + \nu_1, \nu_1 \in [0, N-1] \). Then, by utilizing (16), the conditional PEP of the \( k \)th user can be evaluated as follows

\[
\Pr\left(x_k[v], \bar{x}_k[v] \mid h_k\right) = Q\left(\frac{\zeta_k}{\sqrt{2}\sigma|\Delta_k|}\right),
\]

where \( \zeta_k \) is given in (23) on the top of the next page with \( n = (L-1)/D + \nu_1, \nu_1 \in [0, N-1] \). It is worth noting that since the \( K \)th user decodes all the interference caused by the \( K-1 \) users’ messages. Utilizing the same approximation as the one for the first user’s PEP, i.e., \( D \) is assumed to be sufficiently large, the ISI power becomes negligible compared to the useful signal part. Thus, the conditional PEP for user \( k \) in (22) can be rewritten as follows

\[
\Pr\left(x_k[v], \bar{x}_k[v] \mid h_k\right) \approx Q\left(\frac{||h_k||B_k}{A_k}\right),
\]

where \( B_k = \left[ \sqrt{\alpha_k} |\Delta_k|^2 + 2\Re\left\{\Delta_k \sum_{i=k+1}^{K} \sqrt{\alpha_i}X_i[v]\right\} \right] \) and \( A_k = \sqrt{2}\sigma|\Delta_k| \). Therefore, to get the unconditional PEP, we first evaluate the ordered PDF of the \( k \)th user channel gain using (19) and Lemma 1. Then, we average (24) over the evaluated PDF of \( ||h_k|| \).

**Proposition 2.** The exact PEP for the \( k \)th user, where \( k > 1 \), is given by (25), at the top of the next page, where \( \text{PEP}_k := \Pr(x_k[v], \bar{x}_k[v]) \) and \( \mu_{k,l} = \frac{1}{\sigma_{l,j}} + \sum_{t=0}^{L-1} \frac{m_t}{\sigma_{l,t}}. \)

**Proof.** See Appendix II.

**C. BER Union Bound**

To obtain an accurate indication of the error rate performance of the proposed NOMA-based TR system, we obtain the union bound on the error rate by utilizing the PEP expressions. Thus, the BER union bound can be evaluated using [21]

\[
P_e \leq \frac{b}{b} \sum_{x_k[v]} \sum_{x_{k} \neq x_k[v]} e_{x_k, x_k} \Pr\left(x_k[v], \bar{x}_k[v] \mid x_k[v]\right), \forall k \neq l.
\]

where \( b \) is the number of information bits in symbol \( x_k[v] \), \( P_{x_k[v]} \) denotes the probability of transmitting the symbol \( x_k[v] \) and \( e_{x_k, x_k} \) is the number of bit errors when \( x_k[v] \) is transmitted and \( x_k[v] \) is detected. Additionally, \( \Pr(x_k[v], \bar{x}_k[v]|x_k[v]) \) represents the conditional PEP on the other users messages. Since the conditional PEP depends on the transmitted and detected symbols, in order to get the average BER union bound, it is necessary to evaluate (26) by considering all possible scenarios of \( x_k[v], x_l[v], l \neq k, \) and \( x_k[v] \).

IV. NUMERICAL AND SIMULATION RESULTS

In this section, we present numerical and Monte Carlo simulations to evaluate the performance of the proposed downlink NOMA-based TR system and to confirm the validity of the derived analytical results. To this end, two or three user scenarios are adopted, where each user is equipped with a single antenna, the link between each user and the BS is modeled as a frequency-selective channel, and each channel tap is a complex Gaussian RV with zero mean and variances calculated according to (1). The large-scale path loss coefficients \( \eta_k \) are calculated according to the following:

\[
\eta_k = \eta_0(d_k/d_0)^{-a}, \quad \text{where} \quad d_k = \text{the distance between the BS and the} \ k \text{th user which is set to 50 m,} \quad a = \text{the path loss exponent set to 4, and} \quad \eta_0 = \text{the path loss at the reference distance chosen as } \approx -20 dB \ [22].
\]

Furthermore, the bandwidth \( B \) is set to 200 MHz, and the number of taps of each user link is assumed to be the same and is calculated based on the RMS delay spread, i.e., \( L = \left[ -\sigma l \log(A)/T_s \right] + 1, \) where \( A \), the ratio of non-negligible path power to the first path power, is set to \( -20 dB \). Finally, the noise variance \( \sigma^2 \) is set to \(-120 dB\). For convenience, a summary of the used parameters is depicted in Table I.

In the simulation, a total of \( 10^6 \) channel realizations are randomly generated to evaluate the average performance of the proposed NOMA-based TR receiver. The channels are generated randomly and ordered based on their norm values and then assigned to users based on their orders. Therefore, without loss of generality, the first user is allocated the lowest power coefficient while the second/third user is allocated the largest power coefficient. Unless stated otherwise, transmitted and detected signals are chosen randomly from the binary phase shift keying (BPSK) constellation.

Fig. 2 depicts the PEP performance for a two-user scenario with \( L = 3, 11, \) and 25. The power allocation coefficients
are chosen as $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$. It can be observed from the figure that the derived PEP expressions given in (20) and (25) perfectly match the simulation results over the entire transmit power range. Moreover, it can be noted that due to order statistics, the first user has worse performance compared to the second user. Furthermore, the performance for both users is improved with the number of channel taps due to the improved spatial/temporal focusing.

![Fig. 2: Analytical and simulated PEP for different users, $L = 25, 11, 3, \alpha_1 = 0.6, \alpha_2 = 0.4$.](image-url)
Average Bit Error Rate (ABER)

\[ 10 \]

10

\[ 10 \]

\[ 10 \]

\[ l \]

\[ l \]

i.e., to account only for the interference between two channel taps, \( D \) figure that the union bound converges to the exact simulated constellation. In specific, we compare the derived union bound with the simulation for different upsampling ratios to verify the approximations shown in (18) and (24). It is shown in the figure that the union bound converges to the exact simulated BER for a higher upsampling ratio \( D \). This is justified by the reduced power of the ISI term in the MLD equations in (10)-(11). For instance, at \( D = L - 1 \), the ISI term will be reduced to account only for the interference between two channel taps, i.e., \( l = 0 \) and \( l = 2 \), as follows

\[
ISI = \sum_{i=1}^{K} \sum_{l=0}^{2} \sqrt{\alpha_l} \sum_{j=0}^{L-1} h[k][j] h[k][L - 1 - Dl + j] ||h[k]|| x[i][n - l] \quad (27)
\]

Also, it can be observed that the derived bound for large \( D \) values is tight for both users. As \( D \) increases, both ISI and IUI decrease, consequently leading to a reduction in the BER.

A. Sparse Frequency Selective Channels

In this subsection, we provide a comparative analysis between the performance of the proposed framework and the conventional TRMA in sparse frequency-selective channels. First, we present an overview of the conventional TRMA systems, followed by a comparison for two distinct sparse frequency-selective channels.

1) Conventional TRMA: The conventional TRMA systems involve two phases, namely the recording and transmission phases. During the initial phase, each intended user transmits an impulse signal to the BS. Subsequently, the BS records the channel response for each link and preserves the time-reversed and conjugated version of each response for the transmission phase. Following the channel recording phase, the system initiates its transmission phase. At the BS, all users’ sequences are first up-sampled by a factor of \( D \). Then, the up-sampled sequences are fed into the bank of TR filters, where the output of the \( k \)th TR filter is the convolution of the \( k \)th up-sampled sequence and the corresponding TR filter. After that, all the outputs of the TR filters are added together, and then the combined signal is transmitted into wireless channels. Due to the temporal focusing effect, the signal energy is concentrated in a single time sample at each user. Therefore, the \( k \)th user applies a one-tap gain adjustment to the received signal to restore the signal, followed by downsampling it with the same factor \( D \).

2) Performance Comparison: In Figs 4-5, we compare the BER performance of the proposed NOMA-based TR system with the baseline TRMA, wherein the TR structure is employed in multi-user downlink systems over multi-path channels, such that signals of different users are separated solely by TRMA. Specifically, we considered two-user scenarios within distinct sparse multipath environments. In the first environment, characterized by a delay spread of \( \sigma_r = 2 \text{ ns} \), there are only three channel taps. Conversely, in the second environment with a delay spread of \( \sigma_r = 10 \text{ ns} \), the channel exhibits 11 distinct taps. It can be observed that for both scenarios, the proposed NOMA-based TR scheme achieves superior BER performance compared to the TRMA counterpart, and the gap is larger for smaller delay spread values i.e., the environment has sparse multipath components. For instance, the percent of improvement in the average BER of the two users for \( \sigma_r = 2 \text{ ns} \) and at \( P_t = 20 \text{ dBm} \) is 98.31%, while for \( \sigma_r = 10 \text{ ns} \) and \( D = 5 \), the percent of improvement is around 73.5%.

Subsequently, we consider in Fig. 6 the average BER for a 3-users scenario and we compare the performance of the proposed NOMA-based TR and TRMA, where the power allocation factors for the three users have been selected as \( \alpha_1 = 0.8 \), \( \alpha_2 = 0.15 \), and \( \alpha_3 = 0.05 \) and the delay spread of the channel is \( \sigma_r = 10 \text{ ns} \). Notably, 31.82% improvement on the BER is achieved for the 3-users NOMA-based TR systems compared to the TRMA for \( D = 5 \). On the other hand, for \( D = 10 \) the percentage of improvement compared to TRMA is 99.9%. This reveals the effectiveness of the proposed scheme in mitigating the IUI in multi-user TR systems when users experience sparse multipath channels. Moreover, as \( D \) increases, the proposed scheme efficiently mitigates both ISI and IUI compared to the conventional TRMA scheme. These benefits are accompanied by a reduced overhead compared to traditional TRMA, which necessitates user feedback to the BS for channel estimation to execute TR precoding.

Finally, we quantify the diversity gain, which is defined as the slope of the PEP when the SNR approaches infinity i.e.,...
where $\bar{\gamma}$ is the average SNR. The diversity gain for two users NOMA-based TR system is presented in Fig. 7, where the number of the taps has been chosen as either 25 or 3 by setting the delay spread to $\sigma_\tau = 25$ ns and $\sigma_\tau = 2$ ns, respectively. It is observed from the figure that the achieved diversity gain of the $k^{th}$ user converges to almost $kL$ when $\bar{\gamma}$ increases substantially. This is due to both the effect of the order statistics and the TR focusing. Given that $L = 25$, it can be observed that the diversity gains converge to almost 25 and 50 for the first and second users, respectively. On the other hand, when $L = 3$, it can be observed that the diversity gains converge to almost 3 and 6 for the first and second users, respectively. Nevertheless, it is observed from the figure that as the number of channel taps increases, the convergence rate becomes slower.
V. Conclusion

This paper addressed the challenges associated with TR that arise with in-time ISI and in-space IUI, especially in scenarios featuring sparse multipath components or correlated signal propagation paths among users. To mitigate the IUI and realize enhanced connectivity in frequency-selective channels, we proposed the NOMA-based TR scheme. Our investigation has focused on the performance analysis via the gain of the proposed NOMA-based TR in comparison to the benchmark TDMA technique. Novel closed-form expressions for the PEP for the proposed NOMA-based TR system have been derived for different users over frequency selective fading channels. Then, capitalizing on the derived PEP, the BER union bound was obtained, where simulation and analytical results show that the union bound provides a tight upper bound on the exact BER at a high upsampling ratio. Furthermore, results demonstrate the superiority of our scheme, in terms of the BER, in sparse multi-path environments compared to TRMA, which realize enhanced connectivity in frequency-selective channels, that arise with in-time ISI and in-space IUI, especially in the top of the next page.

Therefore, the integral in (30) can be represented as

\[ PEP_1 \approx \int_0^\infty \frac{1}{2} \text{erfc} \left( \frac{B_1 \omega_1}{\sqrt{2} A_1} \right) \times \left( 2K \omega_1 \sum_{l=0}^{L-1} W_{1,l} \exp \left( -\frac{\omega^2_1}{\sigma^2_{1,l}} \right) \right) \times \sum_{j=0}^{K-1} \left( \frac{K-1}{j} \right) (-1)^j \left( \sum_{l'=0}^{L-1} W_{1,l'} \left( 1 - \exp \left( -\frac{\omega^2_1}{\sigma^2_{1,l'}} \right) \right) \right)^j d\omega_1. \] (32)

Then, by utilizing the multinomial expansion to expand the following term

\[ \left( \sum_{l'=0}^{L-1} W_{1,l'} \left( 1 - \exp \left( -\frac{\omega^2_1}{\sigma^2_{1,l'}} \right) \right) \right)^j, \] (33)

the integral in (32) can be further expressed as (34) on the top of the next page.

Similarly, the binomial expansion is employed to the following term in (34),

\[ \left( 1 - \exp \left( -\frac{\omega^2_1}{\sigma^2_{1,l'}} \right) \right)^j. \] (37)

Consequently, the integral in (34) can be rewritten as (35) on the top of the next page. Finally, we rearrange the terms in (35) to obtain (36), and then, we utilize [24, Eq. 6.287.2] to solve the integral to yield the expression presented in (20).

APPENDIX II: PROOF OF PROPOSITION 2

To evaluate the unconditional PEP for the \( k^{th} \) ordered user, we write the PDF of the norm of the channel gain of user \( k \) as follows

\[ f_{\Omega}(\omega_k) = C_k \omega_k \sum_{l=0}^{L-1} W_{k,l} \exp \left( -\frac{\omega^2_k}{\sigma^2_{k,l}} \right) \times \left( \sum_{l'=0}^{L-1} W_{k,l'} \left( 1 - \exp \left( -\frac{\omega^2_k}{\sigma^2_{k,l'}} \right) \right) \right)^{k-1} \times \left( 1 - \sum_{l''=0}^{L-1} W_{k,l''} \left( 1 - \exp \left( -\frac{\omega^2_k}{\sigma^2_{k,l''}} \right) \right) \right)^{K-k}. \] (38)

Then, the integral of the conditional PEP in (24) over the PDF given in (38) is obtained as follows

\[ PEP_k \approx \int_0^\infty \frac{1}{2} \text{erfc} \left( \frac{B_k \omega_k}{\sqrt{2} A_k} \right) \times \left( 2C_k \omega_k \sum_{l=0}^{L-1} W_{k,l} \exp \left( -\frac{\omega^2_k}{\sigma^2_{k,l}} \right) \right) \times \left( \sum_{l'=0}^{L-1} W_{k,l'} \left( 1 - \exp \left( -\frac{\omega^2_k}{\sigma^2_{k,l'}} \right) \right) \right)^{k-1} \times \left( 1 - \sum_{l''=0}^{L-1} W_{k,l''} \left( 1 - \exp \left( -\frac{\omega^2_k}{\sigma^2_{k,l''}} \right) \right) \right)^{K-k} d\omega_k. \] (39)

Using the binomial expansion for the last term in (39) and rearranging the terms, the integral can be rewritten as (40) on the top of the next page.
\[ PEP_k \approx \int_0^\infty \frac{1}{2} \text{erfc} \left( \frac{B_k \omega_k}{\sqrt{2} A_k} \right) \times 2 C_k \omega_k \sum_{l=0}^{L-1} W_{k,l} \exp(-\frac{\omega_k^2}{2 \sigma_{k,l}}) \sum_{j=0}^{K-k} \left( K - k - j \right) (-1)^j \times \sum_{i_0, \ldots, i_{k-1}, i_{L-1}}^{v_k \rightarrow j} \left( i_0, \ldots, i_{L-1} \right) \times \prod_{t'=0}^{L-1} W_{l',l'}^{i_t'} \left( 1 - \exp(-\frac{\omega_k^2}{2 \sigma_{k,l,t'}}) \right) \, d\omega_k \] (41)


