Improved Design-Oriented Analytical Model for Switched Reluctance Machines Unsaturated Inductances

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Abstract—Design-oriented modelling of switched reluctance
machines is gaining more and more attention as it provides
an attractive solution to the considerable computational burden
associated to preliminary design stages. This work proposes a
novel analytical model to determine the unsaturated inductance
values, particularly in non-overlap conditions, based on a twofold
improvement of the permeance method: the use of elliptic flux
tubes, and the ability to self-tailor the flux tube shapes to any
machine geometry. Finally, the accuracy of the proposed model
is proven against finite element analysis applied to four designs,
along with experimental results of a physical prototype. Self-
tailoring is also assessed and its implementation in an automated
design routine is analysed.

Index Terms—Analytical Model, Elliptic, Flux Tube, In-
ductance Profile, Machine Design, Permeance, Unsaturated,
Switched Reluctance Machine.

I. INTRODUCTION

SWITCHED reluctance machines (SRMs) are nowadays
considered an attractive solution for many applications,
such as automotive, aerospace, or energy storage, where an
ample set of requirements is to be met, e.g., high efficiency,
high torque and power density, etc [1]–[4]. When designing for
these applications, thousands of candidates need to be assessed
before a satisfactory optimised solution is singled out. In
this context, analytical tools are gaining remarkable attention
as a means to avoid the considerable computational burden
required by Finite Element Analysis (FEA) [5]–[7]. In order
to provide a robust alternative to FEA, analytical models should
meet three main criteria, namely: high accuracy for different
geometries and operating conditions, ease of implementation,
and low computational burden.

The foundation of the SRM analytical modelling lies in
the unsaturated inductance vs. rotor position profile \( L(\theta) \), as
it is used to determine performance at high speeds, [8], as
well as to find the saturated flux linkage loci, [9], which,
in turn, is used to determine performance at medium and
low speeds [10]. An example of \( L(\theta) \) is shown in Fig. 1.
As it can be seen, the profile features two almost constant
inductance regions, under full- and non-overlap conditions,
and an almost linearly varying region under partial overlap.
\( L(\theta) \) is determined with a two-step process. The first is the
selection of the analytical expression for \( L(\theta) \); the second is
the calculation of the inductance values at various key rotor
positions, such as \( \theta_s \), \( \theta_1 \), \( \theta_2 \), and \( \theta_u \) (see Fig. 1). Concerning
the first step, \( L(\theta) \) can be expressed analytically either through
a continuous curve, [11], or by a piece-wise function [8], [9],
[12], [13]. With regards of the second, inductances can be
calculated analytically by either field-solution-based (FSB), or
flux-tube-based (FTB) models.

FSB models solve Maxwell’s equations either in the actual
geometry, [14], [15], or through the Schwarz-Christoffel trans-
formations [16], [17]. Their accuracy is generally comparable
to FEA, although at the price of an onerous implementation
and a significant computational burden.

In FTB models, a set of flux tubes is defined and used to
calculate reluctances, [12], [18]–[20], or permeances [21]–
[24]. Reluctances are commonly used in lumped-parameter
equivalent magnetic networks, whereas permeances are more
suitable for integral formulations. In accordance with the
three criteria identified for a robust FEA alternative, this
work considers FTB modelling based on permeances. Indeed,
thanks to their simple mathematical formulation, FTB models
are simple to implement and are computationally light to
run, while permeances ensure a simple correlation between

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Fig. 1. Unsaturated inductance vs rotor position profile.
machine geometry and mathematical expressions. However, in the way FTB models using permeances are commonly implemented, accuracy is limited for two main reasons.

The first is the limited adaptability to different SRM geometries. FTB models usually rely on flux tube geometries based on straight lines and circular arcs [12], [21], [23], [24]. When it comes to tailoring flux tubes to an arbitrary SRM geometry, i.e., defining angles of arcs and lengths of lines, two options are available. The first is to follow FEA results of a benchmark case [12]. The second is to adhere to simple rules, such as 90° arcs or straight lines [21], [23], [24]. In the same manner, points where flux tube shapes change are defined either via FEA or by very general rules, e.g., the middle of teeth edges [21], [24]. Additionally, once selected, flux tube shapes and locations are kept fixed for all geometries [12], [25].

The second reason limiting accuracy is the use of straight lines and circles to model flux tubes, especially in non-overlap conditions, which are critical for determining SRM performance at high speeds. By bearing in mind that torque is proportional to the phase current squared (which depends on \(L(\theta)\) and \(dL/d\theta\)), it is observed that at high speeds current reaches its peak in the proximity of the non-overlap region [8]. An alternative to model flux tubes more precisely has been proposed based on elliptical arcs [26]–[28]. However, there remains plenty of margin to improve this approach further. In [26], the model considers only partial-overlap conditions and a linearised geometry, whereas [27] proposes a mix of elliptical and circular shapes. Finally, [28] proposes a simplified mathematical formulation for fully elliptic arcs, although with an oversimplified expression of the ellipses eccentricity.

In a bid to overcome the two above issues in the determination of unsaturated, non-overlap inductances for arbitrary SRM geometries, this work proposes a novel approach to the permeance FTB method with a twofold improvement:

1) the use of elliptic arc flux tubes in non-overlap conditions and for semi-linearised geometries, based on a relatively simple single-variable formulation,
2) a simple yet effective strategy for automatically self-tailoring the flux tube shapes on an arbitrary SRM geometry with no need for a benchmark case.

The paper is structured as follows. The main modelling assumptions are given in Section II. Leakage and magnetising tubes are presented respectively in Sections III and IV. Section V wraps up the inductance calculation. Section VI describes the self-tailoring procedure. Finally, the proposed method is validated and discussed in Section VII.

### II. Modelling Assumptions

Fig. 2 shows the geometry of an arbitrary SRM considered in this paper for modelling purposes. Symbols are given in Table I. The main modelling assumptions are as follows:

1) iron core has infinite permeability,
2) conductors are uniformly distributed within each slot, with uniform current density,
3) curvature is taken into account by replacing circular arcs by the corresponding chords,
4) the model is 2D with axial-symmetry, i.e., end effects are neglected.

In particular, assumption 4 confines the flux tube curvature to the plane orthogonal to the machine’s axis of rotation. An example is sketched in Fig. 3. Furthermore, an infinitesimal flux tube carrying an infinitesimal flux \(d\Phi\) through an infinitesimal area \(dA\) is actually equivalent to a single flux line through \(dA\). For this reason, infinitesimal flux tubes are represented with dashed flux lines in the following figures, unless it is specified otherwise.

### III. Leakage Flux Tubes

Leakage flux tubes originate and terminate in the stator. Here they are modelled in two ways, \(l_1\) and \(l_2\), as shown in Fig. 4. \(l_1\) tubes model leakage flux between tooth side and back iron, while \(l_2\) tubes model leakage flux between adjacent teeth. The vertical abscissa \(y\) is used to derive the formulas of their corresponding permeances \(P_{l_1}\) and \(P_{l_2}\). \(l_1\) and \(l_2\) share a breakpoint at \(y_1\), meaning that leakage flux is

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**Table I. Geometric Parameters of an Arbitrary SRM**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Parameter</th>
<th>Symbol</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\zeta_{st})</td>
<td>Stator Pitch Angle</td>
<td>(M)</td>
<td>Stator Teeth Pairs per Phase</td>
</tr>
<tr>
<td>(\zeta_{rt})</td>
<td>Rotor Pitch Angle</td>
<td>(h_{st})</td>
<td>Stator Tooth Height</td>
</tr>
<tr>
<td>(\beta_{st})</td>
<td>Stator Tooth Angle</td>
<td>(h_{rt})</td>
<td>Rotor Tooth Height</td>
</tr>
<tr>
<td>(\beta_{rt})</td>
<td>Rotor Tooth Angle</td>
<td>(l_{stk})</td>
<td>Axial Stack Length</td>
</tr>
<tr>
<td>(A_{ss})</td>
<td>Slot Area</td>
<td>(l_s)</td>
<td>Airgap Thickness</td>
</tr>
<tr>
<td>(D_s)</td>
<td>Stator Bore Diameter</td>
<td>(m)</td>
<td>Number of Phases</td>
</tr>
<tr>
<td>(D_r)</td>
<td>Rotor Outer Diameter</td>
<td>(b_{sy})</td>
<td>Stator Yoke Thickness</td>
</tr>
<tr>
<td>(b_{ry})</td>
<td>Rotor Yoke Thickness</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
modelled using $l_1$ for $y < y_1$ and $l_2$ for $y > y_1$. Similarly, $y_2$ is the breakpoint where $l_2$ terminates and magnetising flux tubes begin. In general terms, breakpoints define the points around the tooth contour where each flux tube turns to the next, and are determined automatically by the proposed self-tailoring process described in section VI.

A. Leakage Flux Tubes $l_1$

Fig. 4(a) shows an $l_1$ leakage flux tube. This is modelled by means of circular arcs, which subend an angle of $\pi/2 - \delta_{sy}$ and centred in the corner of the slot.

Both the cross-section and the magnetic flux density $B_{l1}(y)$ are constant along the length of an infinitesimal flux tube. Hence, Ampere’s law can be expressed as in (1), where $N$ is the number of turns per phase, $i_{ph}$ is the phase current, and $d_{l1}(y)$ and $A_{enc-l1}(y)$ are the flux tube’s length and the linked area of conductors, respectively.

$$B_{l1}(y)\frac{d_{l1}(y)}{\mu_0} = 2NI_{ph}\frac{A_{enc-l1}(y)}{A_{ss}}$$

(1)

Expressions for $d_{l1}(y)$ and $A_{enc-l1}(y)$ are given in the following:

$$d_{l1}(y) = \left(\frac{\pi}{2} - \delta_{sy}\right)y$$

(2)

$$A_{enc-l1}(y) = \left(\frac{\pi}{2} - \delta_{sy}\right)\frac{y^2}{2}$$

(3)

The magnetic flux $d\phi_{l1}$ of an infinitesimal flux tube of cross section $L_{stk}dy$ can be determined by substituting (2) and (3) into (1):

$$d\phi_{l1}(y) = B_{l1}L_{stk}ydy = \frac{\mu_0NI_{ph}L_{stk}y}{A_{ss}}dy$$

(4)

Then, the infinitesimal flux linkage $d\Psi_{l1}$ is obtained by multiplying $d\phi_{l1}$ by number of linked conductors:

$$d\Psi_{l1}(y) = \frac{2NA_{enc-l1}(y)}{A_{ss}}d\phi_{l1}(y)$$

(5)

Consequently, flux linkage $\Psi_{l1}$ is obtained by integration as follows:

$$\Psi_{l1} = \int_{0}^{y_1} \frac{2N}{A_{ss}}A_{enc-l1}(y)\frac{\mu_0NI_{ph}L_{stk}y}{A_{ss}}dy$$

(6)

Finally, permeance $P_{l1}$ is equal to:

$$P_{l1} = \frac{\Psi_{l1}}{N^2i_{ph}} = \left(\frac{\pi}{2} - \delta_{sy}\right)\frac{\mu_0L_{stk}y^4}{A_{ss}^2}$$

(7)

B. Leakage Flux Tubes $l_2$

Fig. 4(b) shows an $l_2$ leakage tube. These are modelled as concentric circular arcs centred in the rotor’s axis of rotation. The flux tubes’ length $d_{l2}(y)$ can be expressed as in (8), whereas the enclosed conductors’ area $A_{enc-l2}(y)$ may be approximated as in (9).

$$d_{l2}(y) \approx \frac{D_s}{2}(\zeta_s - \beta_{st}) + (h_{st} - y)\zeta_s$$

(8)

$$A_{enc-l2}(y) \approx \frac{y}{4}(D_s(\zeta_s - \beta_{st}) + \zeta_s(2h_{st} - y))$$

(9)

The same procedure followed for $l_1$ may be followed to determine permeance $P_{l2}$, expressed as:

$$P_{l2} \approx \int_{y_1}^{y_2} \frac{\mu_0L_{stk}(D_s(\zeta_s - \beta_{st}) + \zeta_s(2h_{st} - y))^2}{2A_{ss}^2(D_s(\zeta_s - \beta_{st}) + 2\beta_{st}(h_{st} - y))} dy$$

IV. MAGNETISING FLUX TUBES

Magnetising flux tubes originate in the stator, cross the airgap and terminate in the rotor. In this paper, they are classified as lateral or frontal, depending on which part of the stator tooth they originate from. In order to keep the modelling straightforward, all magnetising tubes are split into two parts, one pertaining to the stator and the other to the rotor, allowing one to model the flux tubes as combinations of elliptic arcs and straight lines.

A. Lateral Magnetising Flux Tubes

Lateral flux tubes are shown in Fig. 5. All stator parts are represented with elliptic arcs that subend a 90° angle, with their centre in $x_0$, on the mid-airgap line and underneath the stator tooth corner. Conversely, rotor parts are as follows:

- $m_1$: are elliptic arcs centred in $x_3$, on the mid-airgap line and above the rotor tooth corner,
- $m_2$: are elliptic arcs also centred in $x_3$,
- $m_3$: are straight lines.

Then, $x_3$ is the breakpoint between $m_1$ and $m_2$, which corresponds to $y_3$, and $x_4$ is the breakpoint between $m_2$ and $m_3$ corresponding to $y_4$. Lateral magnetising tubes on the other tooth side are modelled in the same way.

B. Frontal Magnetising Flux Tubes

Frontal flux tubes are shown in Fig. 6. All stator parts are straight lines extending from the tooth frontal edge to the mid-airgap lines (distinct for the left and right sides of the tooth), whereas rotor parts are modelled as follows:

- $m_4$: are elliptic arcs centred in $x_3$,
- $m_5$: are straight lines,
- $m_6$: are elliptic arcs centred in $x_7$.

Then, $x_5$ is the breakpoint between $m_4$ and $m_5$, and $x_6$ is the breakpoint between $m_5$ and $m_6$. 
Hence, between the stator tooth and the corner of the rotor tooth.

In the following subsections.

Associated to straight line and elliptic arc flux tubes are derived formulas.

In order to obtain a general formulation suitable for any SRM geometry, \( \epsilon \) is evaluated empirically by tracing the FEA field map of a linearised SRM with infinitely high teeth. In accordance with [26], the value is 1.18.

\[ \Gamma_y = \frac{\Gamma_x^2}{\epsilon} + \frac{l_y}{2} \]  

(13)

In order to obtain the permeance contribution from stator elliptic-arc flux tubes, \( m2 - sta \) is considered and shown in Fig. 7. The \( x \) horizontal coordinate lies on the mid-airgap line, with the origin set in \( x_0 \). Based on the model proposed in [26], lengths of the semi-axes of an infinitesimal tube, \( \Gamma_x \) and \( \Gamma_y \), are related to each other as in (13), where \( \Gamma_x \) coincides with \( x \).

\[ \Gamma_y = \frac{\Gamma_x^2}{\epsilon} + \frac{l_y}{2} \]  

(13)

As an example of stator elliptic-arc flux tube, \( m2 - sta \) is considered and shown in Fig. 7. The \( x \) horizontal coordinate lies on the mid-airgap line, with the origin set in \( x_0 \). Based on the model proposed in [26], lengths of the semi-axes of an infinitesimal tube, \( \Gamma_x \) and \( \Gamma_y \), are related to each other as in (13), where \( \Gamma_x \) coincides with \( x \).

\[ \Gamma_y = \frac{\Gamma_x^2}{\epsilon} + \frac{l_y}{2} \]  

(13)

Fig. 6. Frontal magnetising tubes.

Fig. 7. Example of elliptic stator tube and infinitesimal tube.

Fig. 8. Ellipses x- vs. y-semi-axis length (bisector plotted in dash).

E. Permeance Contributions from Stator Elliptic-Arc Flux Tubes

As an example of a stator elliptic-arc flux tube, \( m2 - sta \) is considered and shown in Fig. 7. The \( x \) horizontal coordinate lies on the mid-airgap line, with the origin set in \( x_0 \). Based on the model proposed in [26], lengths of the semi-axes of an infinitesimal tube, \( \Gamma_x \) and \( \Gamma_y \), are related to each other as in (13), where \( \Gamma_x \) coincides with \( x \).

\[ \Gamma_y = \frac{\Gamma_x^2}{\epsilon} + \frac{l_y}{2} \]  

(13)

In order to obtain a general formulation suitable for any SRM geometry, \( \epsilon \) is evaluated empirically by tracing the FEA field map of a linearised SRM with infinitely high teeth. In accordance with [26], the value is 1.18.

\[ \Gamma_y \] length is plotted as a function of \( \Gamma_x \) in Fig. 8. It can be noted that for \( \Gamma_x \) equal to 0, \( \Gamma_y \) is equal to \( l_y/2 \), corresponding to a straight flux tube passing through \( x_0 \). Additionally, considerably eccentric shapes are encountered only for small values of \( \Gamma_x \), i.e., in the vicinity of \( x_0 \). On the other hand, flux tubes turn practically into circular arcs for \( \Gamma_x \) greater than \( 3l_y \).

For the evaluation of the permeance contribution from \( m2 - sta \), \( P_{m2 - sta} \), it has to be observed that the flux tube cross-section \( dA \) is not constant, thereby requiring a 2-variable formulation [27]. However, this work proposes a reasonable assumption whereby \( dA \) varies linearly along the flux tube length, so that \( dA \) can be replaced by its average \( dA_{AVG} \):
\[ dA_{avg}(x) \approx \frac{L_{stk}}{2} (d\Gamma_x + d\Gamma_y) \]
\[ \approx \frac{L_{stk}}{2} \left( 1 + \frac{\partial\Gamma_y}{\partial x} \right) dx \quad (14) \]

It is well known that the length of an elliptic arc cannot be expressed exactly in closed-form. In the case of stator elliptic arcs which span 90°, the following simplified expression (15) is used:

\[ d_{m2-sta}(x) \approx \frac{\pi}{4} (x + \Gamma_y(x)) \quad (15) \]

Both \( dA_{AVG} \) and \( d_{m2-sta}(x) \) are now expressed as functions of \( x \), so that the need for a 2-variable formulation has been removed. Eventually, permeance contribution can be calculated via (16), which is solvable in a closed-form expression (not included in this work due to its length).

\[
P_{m2-sta} = \int_{x_4}^{x_3} \frac{\mu_0}{d_{m2-sta}(x)} \frac{\partial A_{avg}(x)}{\partial x} dA_{avg}(x) \\
= \int_{x_4}^{x_3} \frac{2\mu_0 L_{stk}}{\pi} \frac{x(x + l_g)}{(x + l_g)(2x + l_g)} \frac{1}{x + \frac{l_g^2}{2} + \frac{x}{x + \frac{l_g^2}{2}}} dx \quad (16)\]

\section*{F. Permeance Contributions from Rotor Elliptic-Arc Flux Tubes}

As an example of a rotor elliptic-arc flux tube, \( m2-rot \) is considered and shown in Fig. 9. The formulation is the same as in (16), with the only difference being the expression of the flux tube’s length with respect of \( x \).

Firstly, \( \Gamma_x \) is now equal to:

\[ \Gamma_x(x) = x_3 - x \quad (17) \]

Secondly, rotor flux tubes do not span 90°. As a compromise between accuracy and simplicity, the length \( d_{m2-rot}(x) \) is approximated in this work as the difference between an elliptic arc spanning 90° and a circular arc having a radius \( R(x) \) equal to \( \Gamma_y - l_g/2 \), centred in the rotor tooth corner and subtending an angle equal to \( \theta \):

\[
\begin{align*}
\frac{d_{m2-rot}(x)}{l_g} & \approx \frac{\pi}{4} (\Gamma_x + \Gamma_y) - R(x)\theta \\
R(x) & = \Gamma_y - \frac{l_g}{2} \quad (18)
\end{align*}
\]

\section*{V. Analytical Calculation of Unsaturated Inductance}

Leakage and magnetising permeances are given by summing the individual contributions shown in Fig. 10:

\[ P_l = \sum_{i=1}^{4} P_{l[i]} \quad (19) \]

\[ P_m = \sum_{i=1}^{9} P_{m[i]} \quad (20) \]

The inductance \( L \) is obtained by summing the leakage \( L_l \) and magnetising inductances \( L_m \), which derive directly from \( P_l \) and \( P_m \):

\[ L = L_l + L_m = 2MN^2 (P_l + P_m) \quad (21) \]

where \( N \) is the number of turns and \( M \) the number of stator teeth pairs per phase.

It is important to highlight that all permeance contributions (and therefore inductance) are functions of \( \theta \). For example, in the case of (16) and (18), permeance dependency from \( \theta \) is found in:

- the integral’s extremes, i.e., the breakpoints \( x_4 \) and \( x_3 \) change with \( \theta \); \( x_3 \) changes due to the movement of the rotor itself, whereas \( x_4 \) changes as a result of the self-tailoring process (see section VI),
- the airgap \( l_g \) on either side of the stator changes with \( \theta \).

In a bid to maintain the manuscript readability at its maximum, notation indicating dependency on \( \theta \), e.g. \( l_g(\theta) \), has been omitted.

Furthermore, it is well-known that flux lines/tubes should cross an air/ideal iron interface at 90°. In the model being proposed, this constraint is always met with straight and stator elliptic-arc tubes. On the other hand, the fact that rotor elliptic-arc tubes span angles different from 90° violates this principle. This approximation has been deemed to be a reasonable compromise to keep the model simplicity at its lowest whilst maintaining sufficient accuracy.
VI. THE SELF-TAILORING PROCESS

Leakage and magnetising flux tubes need to be adjusted, or ‘tailored’, to the specific SRM candidate at a given $\theta$. In this section, it is shown how to make this process automatic, with no need for FEA simulations or generic geometric rules.

Initially, it is assumed that all flux tube types and breakpoints shown in Fig. 10 exist. Subsequently, all those flux tubes that do not approximate any ‘real’ flux tube of the specific SRM candidate at the given $\theta$ need to be discarded and the breakpoints of all remaining tubes calculated.

A. Frontal Magnetising Flux Tubes

In order to keep the explanation at its easiest, frontal magnetising flux tubes in non-overlap conditions are initially considered, Fig.11(a) and 11(c). In this case, the objectives of the self-tailoring are:

1) define the existence or absence of $m4$, $m5$ and $m6$,
2) evaluate $x_5$ and $x_6$.

To this end, it is reasonable to assume that the type of flux tube with the shortest path from stator to rotor should be selected at any $x$.

A first example is shown in Fig. 11(a), where $m4$, $m5$ and $m6$ exist. As a reference, their length $d(x)$ is plotted in Fig. 11(b). As it can be seen, at $x_0$, the shortest flux tube is selected, i.e. $m4$. Then, breakpoint $x_5$ is determined where an $m5$ type flux tube becomes shorter than an $m4$ type. Analogously, breakpoint $x_6$ is determined where an $m6$ type flux tube becomes shorter than an $m5$ type. Having determined $x_5$ and $x_6$, permeance contributions from the three tubes can be calculated through the guidelines of section IV.

A second example is provided in Fig. 11(c), where flux tube $m5$ doesn’t exist, i.e. there are no values of $x$ where $m5$ is the shortest flux tube type. Consequently, $x_5$ and $x_6$ are coincident.

In terms of an automated implementation of the above, the process consists in identifying all possible flux tube combinations and enclosing them in an if-else tree. For the frontal flux tubes, it may be observed that $m4$ exists for any SRM of engineering interest. Then, apart from the cases discusses in Fig.11, two other, albeit unrealistic cases should be added: one where only $m4$ and $m5$ exist, and another where only $m4$ exists. To construct the if-else tree, values of $x$ for which two or more tube types are equal in length must be found:

\[
\begin{align*}
x_{45} & \quad \text{solves} \quad d_{m4}(x) = d_{m5}(x), \quad x \in [x_0, x_1] \\
x_{46} & \quad \text{solves} \quad d_{m4}(x) = d_{m6}(x), \quad x \in [x_0, x_1] \\
x_{56} & \quad \text{solves} \quad d_{m5}(x) = d_{m6}(x), \quad x \in [x_0, x_1]
\end{align*}
\]

Solutions outside the interval $[x_0, x_1]$ need to be discarded (set to $NaN$). Based on that, all possible cases, conditions and corresponding breakpoints are summarised in Table II.

B. Leakage and Lateral Magnetising Flux Tubes

The determination of leakage and lateral magnetising tube types and their breakpoints must be performed together. In line with the principle of maximising the compromise between model accuracy and simplicity, it is reasonable to maintain the criterion of selecting the shortest flux tube type at any $y$. Therefore, the process is analogous to that described for the frontal flux tubes. The only difference pertains to breakpoints $y_5(x_3)$ and $y_7(x_7)$, which are defined directly by the $x$ coordinates of the rotor tooth corners, as illustrated in subsection IV.A, and hence there is no need to include them in the if-else tree.

VII. VALIDATION AND DISCUSSION OF THE PROPOSED METHOD

A. SRM Designs and Prototype Description

In order to validate the applicability and high accuracy of the proposed model, four SRM designs have been taken into account to cover different applications of modern SRMs, such as automotive (SRM-A and SRM-D) and flywheel energy storage (SRM-B and SRM-C). Their geometric parameters are reported in Table III. Their cross sections are shown Fig. 12.

Considering that the proposed model is conceived for an early design stage, validation is mostly conducted against FEA. Nonetheless, experimental results from SRM-A are included, in order to validate the FEA results and assess the discrepancy between the analytical and experimental values. The SRM-A prototype is shown in Fig. 13. Inductances have been measured by an N4L-PSM1735-1AI impedance analyser set to 50Hz, with the rotor being locked to the graduated flange shown in Fig. 13(b) and position measured through the Smartsyn TS2620N1051E11 resolver.

B. Unsaturated Inductance Validation

In this subsection, the accuracy in calculating unsaturated inductances in the non-overlap region is assessed. Validation begins with SRM-A. Table IV compares the analytical results against FEA and experimental results, where $\Delta_{\text{FEA}}$ and $\Delta_{\text{exp}}$
TABLE II
SELF-TAILORING: FRONTAL FLUX TUBE CONDITIONS AND BREAKPOINTS

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Breakpoints</th>
</tr>
</thead>
<tbody>
<tr>
<td>m4, m5 and m6</td>
<td>( x_{45} &lt; x_{46} &lt; x_{56} )</td>
<td>( x_5 = x_{45} ) and ( x_6 = x_{56} )</td>
</tr>
<tr>
<td>m4 and m6</td>
<td>( x_{56} &lt; x_{46} &lt; x_{45} ) or ( x_{56} &lt; x_{46} ) and ( x_{45} = NaN )</td>
<td>( x_5 = x_6 = x_{56} )</td>
</tr>
<tr>
<td>m4 and m5</td>
<td>( x_{45} &lt; x_{46} ) and ( x_{56} = NaN )</td>
<td>( x_5 = x_6 = x_{56} )</td>
</tr>
<tr>
<td>m4</td>
<td>Any other</td>
<td>( x_5 = x_6 = x_1 )</td>
</tr>
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</table>

TABLE III
GEOMETRIC PARAMETERS OF THE SRM DESIGNS

<table>
<thead>
<tr>
<th>Bore Diameter [mm]</th>
<th>SRM-A</th>
<th>SRM-B</th>
<th>SRM-C</th>
<th>SRM-D</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>82.5</td>
<td>120</td>
<td>200</td>
<td>219</td>
</tr>
<tr>
<td>Stack Length [mm]</td>
<td>80</td>
<td>158</td>
<td>250</td>
<td>121</td>
</tr>
<tr>
<td>Rotor Tooth Angle</td>
<td>15.2</td>
<td>20.0</td>
<td>11.6</td>
<td>16.5</td>
</tr>
<tr>
<td>Rotor Tooth Angle</td>
<td>17.1</td>
<td>42.7</td>
<td>12.7</td>
<td>21.8</td>
</tr>
<tr>
<td>Number of Phases</td>
<td>3</td>
<td>3</td>
<td>8</td>
<td>3</td>
</tr>
<tr>
<td>Number of Turns</td>
<td>4x13</td>
<td>2x4</td>
<td>2x22</td>
<td>4x6</td>
</tr>
<tr>
<td>Rated Power [kW]</td>
<td>1.12</td>
<td>40</td>
<td>50</td>
<td>20</td>
</tr>
</tbody>
</table>

TABLE IV
SRM-A NON-OVERLAP UNSATURATED INDUCTANCE VALIDATION

<table>
<thead>
<tr>
<th>Case</th>
<th>( \theta ) [º]</th>
<th>Analytic [mH]</th>
<th>FEA [mH]</th>
<th>Exper. [mH]</th>
<th>( \Delta_{FEA} ) [%]</th>
<th>( \Delta_{exp} ) [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>m4</td>
<td>16.25</td>
<td>0.376</td>
<td>0.394</td>
<td>0.340</td>
<td>-4.5</td>
<td>-12.5</td>
</tr>
<tr>
<td>m5</td>
<td>17.5</td>
<td>0.302</td>
<td>0.329</td>
<td>0.341</td>
<td>-8.2</td>
<td>-14.4</td>
</tr>
<tr>
<td>m6</td>
<td>18.75</td>
<td>0.276</td>
<td>0.299</td>
<td>0.308</td>
<td>-7.6</td>
<td>-10.3</td>
</tr>
<tr>
<td>m7</td>
<td>20.0</td>
<td>0.263</td>
<td>0.281</td>
<td>0.291</td>
<td>-5.8</td>
<td>-8.0</td>
</tr>
<tr>
<td>m8</td>
<td>21.25</td>
<td>0.257</td>
<td>0.272</td>
<td>0.280</td>
<td>-5.5</td>
<td>-6.5</td>
</tr>
<tr>
<td>m9</td>
<td>22.5</td>
<td>0.256</td>
<td>0.269</td>
<td>0.275</td>
<td>-4.8</td>
<td>-6.9</td>
</tr>
</tbody>
</table>

As it can be noted, errors lower than 9% with respect to FEA models are attained, along with a peak discrepancy of 12.9% against the experimental measurement at \( \theta_2 \). On the other hand, discrepancy between FEA and experiments at this point is also quite considerable.

At this point, given that the above analysis also proved the validity of the FEA model, the SRM-B to SRM-D are taken into account. Fig. 14(a)-(d) compare the non-overlap unsaturated inductances obtained analytically and via FEA of the four SRM designs, where a very close similarity between analytical (solid lines) and FEA (dashed lines) results can be observed. For the sake of completeness, experimental values of SRM-A are also included. Errors are limited to approximately -8% and +10%, demonstrating that the model has high accuracy for widely different SRMs. By comparing the error vs. \( \theta \) trends, different behaviours are visible: a concave shape for SRM-A, a decreasing trend for SRM-B and SRM-C and an increasing trend for SRM-D. Hence, a common pattern or a systematic error mechanism cannot be identified.

C. Flux Tube Shape and Breakpoint Validation

The field map obtained with the proposed model for SRM-A at \( \theta_2 \) and \( \theta_u \) (rotor positions where non-overlap respectively commences and terminates) is compared against FEA in Fig. 15. Here, for visualisation purposes only, flux tubes defined by the proposed model have been generated in Matlab. As it can be seen, an acceptable similarity is attained. In particular, accuracy in the prediction of breakpoints at \( \theta_2 \) and \( \theta_u \) is assessed in Table V and Table VI respectively (breakpoints in Table VI obtainable by symmetry have been omitted).

Overall, an extremely accurate prediction is obtained for \( y_1 \), \( y_2 \) and \( x_5 = x_6 \) at \( \theta_2 \), as well as for \( y_1 \) at \( \theta_u \), whereas a larger discrepancy occurs for \( y_2 \) and \( y_6 \) at \( \theta_2 \) and \( y_2 \) at \( \theta_u \). This fact highlights the difficulty in modelling flux tubes crossing long air paths without solving the Maxwell’s equations. Nonetheless, as demonstrated by the results shown in subsection VII-B, these errors have a relatively low impact on the calculation of unsaturated inductance. Finally, Table VI also reports \( y_3 \), which is determined directly from the
Fig. 14. Analytical vs. FEA unsaturated inductances: (a) SRM-A (b) SRM-B (c) SRM-C (d) SRM-D.

geometry. Its accurate estimation shows the relatively high precision of the proposed model in representing flux tubes in the vicinity of the stator tooth corner.

D. Proposed vs. Standard FTB Method

In this subsection, the improvement brought by the use of the proposed method is assessed by comparing it against the traditional approach, where the flux tubes are represented by straight lines and circular arcs, as illustrated in Fig. 16. In particular, in order to make a fair comparison and not rely on FEA, breakpoints for the standard approach are assumed to lie at fixed positions [21]:

- $y_1$ lies at the middle of the stator side,
- $x_1$ lies at one quarter of the stator tooth front.

The comparison is conducted based on the inductance $L_u$ of the four SRMs in the fully unaligned position. Table VII compares the errors against FEA, $\Delta_{\text{FEA}}[\%]$, incurred with both methods, showing the remarkable improvement (7% to 17%) attained through the combined utilisation of elliptic arc/straight lines and self tailoring.

E. FTB Method in an Automated Analytical Design Process

This subsection reports a benchmark design exercise, with the objective of showing the reduction in computation time attained through the FTB model in lieu of FEA. A genetic algorithm is used, comprising of 50 generations of 15 individuals.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$\theta_2$ Breakpoints Validation & \hline
\hline
$y_1$ & -0.31 \hline
$y_2$ & -22.67 \hline
$y_3$ & -6.89 \hline
$y_4$ & -15.14 \hline
$x_5$ & -5.49 \hline
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$\theta_u$ Breakpoints Validation & \hline
\hline
$y_1$ & 0.42 \hline
$y_2$ & -26.89 \hline
$y_3$ & -2.38 \hline
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Proposed FTB, $\Delta_{\text{FEA}}[\%]$ & Traditional FTB, $\Delta_{\text{FEA}}[\%]$ & \hline
SRM-A & -4.8 & -22.1 \hline
SRM-B & -3.7 & -10.9 \hline
SRM-C & -5.8 & -14.5 \hline
SRM-D & -2.0 & -15.0 \hline
\hline
\end{tabular}
\end{table}
 mains and running in Matlab on a workstation with an i7-3630 processor @2.40 GHz, 24 GB RAM.

For each candidate, the $L(\theta)$ curve is attained from a three piece function [8], which requires the evaluation of $L(\theta_2)$, $L(\theta_3)$, along with the inductance in the fully aligned position. The computation times needed for the $L(\theta)$ curves of the 750 candidates are extracted and reported in Table VIII.

The analytical model completes the process in one minute and a half, i.e. it is between 448 and 1682 times faster than FEA.

### Table VIII: Computation Times Comparison

<table>
<thead>
<tr>
<th>SRM</th>
<th>Analytical [mins]</th>
<th>FEA [mins]</th>
<th>Mesh Elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>SRM-A</td>
<td>1.56</td>
<td>2624</td>
<td>59855</td>
</tr>
<tr>
<td>SRM-B</td>
<td>1.56</td>
<td>1312</td>
<td>32258</td>
</tr>
<tr>
<td>SRM-C</td>
<td>1.56</td>
<td>699</td>
<td>9132</td>
</tr>
<tr>
<td>SRM-D</td>
<td>1.56</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### VIII. Conclusion

This work proposed a novel FTB model using permeances based on the twofold novelty: improved use of elliptic arcs for flux tube modelling and self-tailoring. The model has been implemented to estimate inductance in the non-overlap regions of four SRMs and has been validated against their FEA models, along with the experimental results of a prototype. The following key conclusions can be drawn:

- The method is extremely robust and accurate, with errors below 9% against FEA results.
- Compared to the traditional straight lines and circular arcs approach, elliptic flux tubes slightly increase the complexity of the mathematical formulation, although non-iterative solutions are maintained.
- Compared to the traditional approach, the proposed method has shown a remarkable improvement in accuracy (7% to 17%).
- A reduction in computation time of three orders of magnitude compared to the FEA process can be attained.

### References


