An analysis of the invariance and conservation laws of some classes of nonlinear Parabolic and Ostrovsky equations and related systems

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Abstract

We study various classes of the nonlinear dynamics of some ‘high’ order parabolic equations (pdes) like the Benjamin-Bona-Mahony-Peregrine-Burger and the Oskolkov-Benjamin-Bona-Mahony-Burgers equations that arise in the study of some wave phenomena. Also, a large class of pdes arising in the modelling of ocean waves are due to Ostrovsky. We determine the invariance properties (through the Lie point symmetry generators) of the nonlinear systems and construct classes of conservation laws for some of the models above and show how the relationship leads to double reductions of the systems. This relationship is determined by a recent result involving ‘multipliers’ that lead to ‘total divergence’.
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1 Introduction and background

In our analysis, we consider two broad classes of models that have been and are of interest.

I. A large class of equations in mathematical physics are special cases of the Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) [18] equation

\[ u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta uu_x + \beta u_{xxx} = 0. \]  

Equation (1.1) features a balance between nonlinear and dispersive effects and takes account of dissipation and the problem arises in, both, bore propagation and the water waves. Furthermore, it describes long waves on the surface of water in a channel with small-amplitude. A special case is a version of the KdV equation [18, 3]

\[ u_t + u_{xxt} + u_x + \theta uu_x = 0. \]  

If \( \alpha = \beta = 0 \) and \( \beta = 0 \), we obtain the standard BBM and the Oskolkov-Benjamin-Bona-Mahony-Burgers (OBBMB) equations, respectively. With specific choices of the parameters, other cases that arise are the Oskolkov equation describing the dynamics of an incompressible viscoelastic KelvinVoigt fluid [15] and Camassa-Holm equation which is a bi-Hamiltonian model for waves in shallow water [8]. An extension of this is given by

\[ u_t - u_{xxt} + \alpha u_x + 3\theta u^2 u_x - \gamma u_x u_{xx} + 2\beta uu_x - uu_{xxx} = 0 \]  

leading to, with appropriate choices of parameters, the Degasperis-Procesi [19] and Fornberg-Whitham [20] equations, inter alia.

II. The work of Ostrovsky in the modelling of wave phenomena is well known with particular forms involving ‘nonlinear surface and internal waves’ ([16, 17, 13, 4]. For
the case of the ‘rotating ocean’, the equation is

\[(u_t + c_0 u_x + puu_x + qu_{xxx})_x = \gamma u\]  \hspace{1cm} (1.4)

where \(c_0\) is the velocity of dispersion linear waves, \(p\) is the coefficient of nonlinearity, \(q\) is the Boussinesq dispersion and \(\gamma\) is the Coriolis dispersion coefficient (measuring the effect of rotation). For suitable choices of the parameters, the equation is the Vakhnenko equation \((u_t + uu_x)_x + u = 0\) which transforms to

\[uu_{xxt} - u_xu_{xt} + u^2u_t = 0\]  \hspace{1cm} (1.5)

(the details of the transformation is given in [5]). As a limiting case of ‘very long waves’, \(q = 0\) so that (1.4) becomes

\[(u_t + c_0 u_x + puu_x)_x = \gamma u\]  \hspace{1cm} (1.6)

The general equation has been analyzed by a number of people with varying approaches (see [9, 4] and references therein). Furthermore, if the wave motion has weak transverse effects along the \(y\)-axis, the model can be expressed, generally, as

\[(u_t + cu_x + \alpha uu_x + \alpha_1 u^2u_x - \beta u_{xxx})_x = \gamma u - \frac{c}{2}u_{yy}.\]  \hspace{1cm} (1.7)

As pointed out above, a number of approaches have been adopted in the analyses of various versions of these equations - none, to our knowledge, deals with a Lie symmetry analysis and/or conservation law approach. In this paper, we will aim to achieve this with regard to a special case of (1.6), for illustrative reasons, the transformed Vakhnenko equation (1.5) and the following system that arises in the modelling of surface waves to the two-layer fluid due to Ostrovsky, see [17]. The complicated system

\[v_t + (vu)_x = 0, \quad u_t + uu_x + gv_x = D_s, \quad D_s = \left(\frac{h^2}{3}D_x + vv_x\right)(u_{xt} + uu_x - u_x^2)\]  \hspace{1cm} (1.8)
would require some cumbersome calculations for its analysis, be it the invariance
properties or the conservation laws. In the latter case, the ‘higher order’ ones, viz.,
those that arise from derivative dependent multipliers - the higher the order of the
multiplier, the more tedious the calculations.

The Lie symmetry approach is now an established route for the reduction of differen-
tial equations and its advantages in the analysis of nonlinear partial differential
equations (pdes) is vast. The method centres around the algebra of one parameter
Lie groups of transformations that are admitted by the pde; once known, the reduc-
tion of the pde is standard and may lead to exact (symmetry invariant) solutions
(see [7, 14]).

There are a number of reasons to find conserved densities of pdes. Some conservation
laws are physical (e.g., conservation of momentum and energy) and others facilitate
analysis of the pde and predicts integrability. Also, some reasons are related to
the numerical solution of pdes. For e.g., one should check whether the conserved
quantities are in fact constant (see [6]). That is, if for e.g., \( u = u(x,t) \) and \( u \to 0 \)
for \( |x| \to \pm \infty \), then the conserved form \( D_t \Phi^t + D_x \Phi^x = 0 \) is \( \partial_t \Phi^t + \partial_x \Phi^x = 0 \) so that

\[
\int_{-\infty}^{\infty} (\partial_t \Phi^t + \partial_x \Phi^x) \, dx = 0
\]

so that differentiating with respect to \( t \) leads to

\[
\int_{-\infty}^{\infty} \Phi^t \, dx = \text{constant},
\]

for all solutions of the pde.

Lastly, the use of symmetry properties of a given system of partial differential equa-
tions to construct or generate new conservation laws from known conservation laws
has been investigated [10, 11].

We now present some preliminaries. Consider an \( r \)th-order system of partial dif-
ferential equations (pdes) of \( n \) independent variables \( x = (x^1, x^2, \ldots, x^n) \) and \( m \)
dependent variables $u = (u^1, u^2, \ldots, u^m)$

$$G^\mu(x, u, u_{(1)}, \ldots, u_{(r)}) = 0, \quad \mu = 1, \ldots, \tilde{m},$$ (1.9)

where $u_{(1)}, u_{(2)}, \ldots, u_{(r)}$ denote the collections of all first, second, \ldots, $r$th-order partial derivatives, that is, $u_i^{\alpha} = D_i(u^\alpha), u_{ij}^{\alpha} = D_jD_i(u^\alpha), \ldots$ respectively, with the total differentiation operator with respect to $x^i$ given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^{\alpha} \frac{\partial}{\partial u^{\alpha}} + u_{ij}^{\alpha} \frac{\partial}{\partial u_j^{\alpha}} + \ldots, \quad i = 1, \ldots, n,$$ (1.10)

where the summation convention is used whenever appropriate. A current $\Phi = (\Phi^1, \ldots, \Phi^n)$ is conserved if it satisfies

$$D_i \Phi^i = 0$$ (1.11)

along the solutions of (1.9). It can be shown that every admitted conservation law arises from multipliers $Q_\mu(x, u, u_{(1)}, \ldots)$ such that

$$Q_\mu G^\mu = D_i \Phi^i$$ (1.12)

holds identically (i.e., off the solution space) for some current $\Phi^i$.

We determine the conserved flows by first constructing the multipliers $Q_\mu$ which are obtained by noting that the Euler(-Lagrange) operator, $\delta \delta u^{\alpha}$, annihilates total divergences, i.e., a defining equation, for $Q_\mu$, would be

$$\frac{\delta}{\delta u^{\alpha}} [Q_\mu G^\mu] = 0.$$ (1.13)

The conserved flow $\Phi$ is then obtained by a well known ‘homotopy’ formula.

Furthermore, if $X = \xi \partial_x + \tau \partial_t + \eta \partial_u$ is a Lie point symmetry that leaves a scalar pde in (1.9), say,

$$G(x, t, u, u_x, u_t, u_{xx} \ldots) = 0,$$ (1.14)
invariant with

\[ XG = RG \quad (1.15) \]

and

\[ \lambda = D_t \tau + D_x \xi \quad (1.16) \]

such that

\[ XQ + (R + \lambda)Q = 0 \quad (1.17) \]

then \( X \) is associated with the corresponding conserved flow \( \Phi = (T, \phi) \) and, via \( X \) and \( \Phi \), double reduction may be obtained [1, 2]. Here, \( D_t T + D_x \phi = 0 \mid_{(1.14)} \).

Moreover, in transformed coordinates \((\zeta, \rho, U)\), \( \bar{X} = \partial_\rho \) so that \( \bar{X}(\rho) = 1 \), \( \bar{\Phi} = (\bar{T}, \bar{\phi}) \) and \( D_\zeta \bar{T} + D_\rho \bar{\phi} = 0 \). This procedure leads to a double reduction of the original system since

\[ k = \bar{T} \mid_{\zeta, u, u', \ldots} \quad (1.18) \]

where \( k \) is a constant - for details, see [2].

\section{Symmetries and Conservation laws}

We now study the symmetries and conservation laws and their relationship for a special case of (1.6), for illustrative reasons, the transformed Vakhnenko equation (1.5) and the system (1.8) - due to Ostrovsky.

\textbf{Equation (1.6)} with \( p = 1, \gamma = -1, c_0 = 0 \) has the original Vakhnenko form

\[ u_{xt} + u_x^2 + uu_{xx} = -u. \quad (2.1) \]

It has no obvious conserved form nor is it variational - one can check this by noting that the Frechet derivative is not self adjoint. Its Lie point symmetry generators are

\[ X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = t \partial_t - x \partial_x - 2u \partial_u. \]
The multipliers $Q$ leading to, possible, conservation laws are identified by (1.13) which in this case has the form
\[ \frac{\delta}{\delta u} (Q[u_{xt} + u^2 + uu_{xx} + u]) = 0. \]

For illustrative purposes we assume a first-order derivative in $u$ dependence of $Q$, i.e., $Q = f(x, t, u, u_x, u_t)$. Expanding the Euler operator and separating by second and higher derivative combinations lead to an over determined system of pdes whose solution leads to a single, nontrivial multiplier
\[ Q = uu_x + u_t. \]

The corresponding conserved vector $(\Phi^x, \Phi^t)$ for which
\[ D_x \Phi^x + D_t \Phi^t = (uu_x + u_t)(u_{xt} + u^2_x + uu_{xx} + u) \]
is given via a homotopy formula involving a complicated integral (see [14]). We get
\[
\begin{align*}
\Phi^x &= \frac{1}{12}(4u^3 + 3u_t^2 + u(-3u_{tt} + 8u_tu_x) + u^2(6u_x^2 - 2u_{xt})) , \\
\Phi^t &= \frac{1}{12}(3u_tu_x + u(4u_x^2 + 3u_{xt}) + 2u^2(3 + u_{xx})).
\end{align*}
\]

A detailed procedure and analysis of the invariance properties of the conservation law through invariance properties of its corresponding multiplier is done is the main aim of the paper and is detailed below. Suffice to say here is that $Q$ is strictly invariant under time and space translations and $X_3Q = -3Q$ implying ray invariance of $Q$ under $X_3$.

Note. The conserved vector/s may be determined using the partial Lagrangian approach developed in [12] using the partial Lagrangian
\[ L_p = \frac{1}{2} u_x u_t + \frac{1}{2} uu_x^2 - \frac{1}{2} u^2 \]
to determine the possible Noether-type generators leading to conserved flow/s $(\Phi^x_p, \Phi^t_p)$. Then, the total divergence would be
\[ D_x \Phi^x_p + D_t \Phi^t_p = W(u_{xt} + u^2_x + uu_{xx} + u) \]
which, as always via the Noether approach, leads to, at best, first-order dependence of the multiplier $W$. This may be limited as will be seen in the following examples.

2.1 Parabolic equations

2.1.1 BBMBP equations

With $\theta = 1$, (1.2) becomes

$$u_t + u_{xxt} + u_x + uu_x = 0$$

(2.2)

which admits a three dimensional algebra of point symmetry generated by

$$\partial_t, \quad \partial_x, \quad t\partial_t - (1 + u)\partial_u$$

(2.3)

so that travelling solutions, which have been studied, are generated by $X_1 = \partial_t + c\partial_x$.

From (1.13), we obtain two multipliers $Q_1 = 1$ and $Q_2 = u$. For $X_1$, $\lambda_1 = R_1 = 0$ so that (1.17) is satisfied for $Q_1$ and $Q_2$. For the scaling symmetry $X_2 = t\partial_t - (1 + u)\partial_u$, $\lambda_2 = 1$ and $R_2 = 0$ so that (1.17) is not satisfied. In this case, towards obtaining a scaling invariant solution, we will reduce (2.3) in the usual way.

We, firstly, perform two double reductions of (2.3). Note that

$$\frac{\partial T}{\partial t} + \partial_x \phi = \frac{\partial T}{\partial \zeta} \frac{\partial \zeta}{\partial t} + \frac{\partial T}{\partial \rho} \frac{\partial \rho}{\partial t} + \frac{\partial \phi}{\partial \zeta} \frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial \rho} \frac{\partial \rho}{\partial x}.$$  

(2.4)

For $X_1$, $\zeta = x - ct$ ($c$ being the speed of the travelling wave) and $U(\zeta) = u$. Thus,

$$\bar{T} = -cT + \phi, \quad \bar{\phi} = \frac{1}{c}\phi.$$  

(2.5)
The multiplier $Q_1$ and $Q_2$ leads to the conserved flows

\[
T_1 = \frac{1}{3} u_{xx} + u, \\
\phi_1 = \frac{2}{3} u_{xt} + \frac{1}{2} u^2 + u, \\
T_2 = \frac{1}{3} uu_{xx} - \frac{1}{6} u_x^2 + \frac{1}{2} u^2, \\
\phi_2 = \frac{1}{3} u^3 + \frac{1}{2} u^2 - \frac{2}{3} uu_{xt} - \frac{1}{3} u_x u_t
\]  

Double reductions based on these can only be performed with $X_1$ which from (2.5)(a), respectively,

\[
k_1 = U'' + \frac{c-1}{c} U - \frac{1}{2c} U^2, \\
k_2 = -cUU'' + \frac{c}{2} U'^2 + \frac{1}{2} (1 - c) U^2 + \frac{1}{3} U^3.
\]

By letting $U = V$ and $U' = W$, these reduce to first-order odes. In (2.7)(a), if $k_1 = 0$, the solution of the ode is

\[
3cW^2 = V^3 + 3(1 - c)V^2 + k.
\]  

With $k_2 = 0$ in (2.7)(b), we obtain a solution

\[
W = -\frac{1 - c}{3c} V - \frac{2}{15c} V^2 + k.
\]

Even though there is no association of a conservation law with $X_2$, a direct reduction is possible; we obtain invariants $\zeta = x$ and $U = t(1 + u)$ so that another reduction leading to an exact solution of (2.3) is obtained by

\[
U'' + U U' - U = 0.
\]

By letting $U = V$ and $U' = W$, (2.10) reduces to a first-order ode with solution

\[
-2W - \ln(1 - W)^2 = V^2 + k.
\]
2.1.2 Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB)

We now seek (double) reductions of (1.1) with a similar procedure. The Lie point symmetry algebra is generated by

\[ \partial_t, \quad \partial_x, \quad \text{(for } \alpha = 0) \quad X_2 = -\beta t \partial_x + t \partial_x - \left( \frac{\beta + \gamma}{\theta} + u \right) \partial_u \]  

(2.12)

Travelling solutions are obtained by \( X_1 = \partial_t + c \partial_x \). From (1.13), we obtain two multipliers \( Q_1 = 1 \) and \( Q_2 = u \). For \( X_1 \), \( \lambda_1 = R_1 = 0 \) so that (1.17) is satisfied for \( Q_1 \) and \( Q_2 \). For the symmetry \( X_2 \), \( \lambda_2 = 1 \) and \( R_2 = 0 \) so that (1.17) is not satisfied. The multiplier \( Q_1 \) and \( Q_2 \) leads to the conserved flows

\[ T_1 = \frac{1}{3} u_{xx} - u, \]
\[ \phi_1 = \frac{2}{3} u_{xt} - \beta u_{xx} + \alpha u_x - \frac{\theta}{2} u^2 - \gamma u, \]

\[ T_2 = -\frac{1}{3} uu_{xx} + \frac{1}{6} u_x^2 + \frac{1}{2} u^2, \]
\[ \phi_2 = \frac{\theta}{3} u^3 + \frac{\gamma}{2} u^2 + \beta uu_{xx} - \frac{\beta}{2} u_x^2 - \frac{2}{3} uu_{xt} + \frac{1}{3} u_x u_t. \]

(2.13)

For \( X_1 \), \( \zeta = x - ct \) and \( U(\zeta) = u \), double reductions can be performed with \( X_1 \) which from (2.5)(a), respectively, is

\[ k_1 = -(\beta + c) U'' + (c - \gamma) U + \alpha U' - \frac{\theta}{2} U^2, \]
\[ k_2 = (\beta + c) UU'' - \frac{1}{2} (c + \beta) U'^2 + \frac{1}{2} (1 + \gamma) U'^2 + \frac{\theta}{2} U^3. \]

(2.14)

Using \( X_2 \) with invariants \( \zeta = x + \beta t \) and \( U = t (u + \frac{\beta + \gamma}{\theta}) \), a direct reduction of (1.1) for \( \alpha = 0 \) is

\[ U'' + \theta UUU' - U = 0. \]

(2.15)

The three second-order odes are all easily reducible to first-order odes.
2.2 II. Nonlinear Ostrovsky equations

(1). Equation (1.5)

The symmetry generators are given by

\[ Y_1 = \partial_x, \quad Y_2 = F(t)\partial_t, \quad Y_3 = x\partial_x - 2u\partial_u. \]

For reduction, we consider \( Y = \partial_t + c\partial_x \) and \( Z = x\partial_x + \mu t\partial_t - 2u\partial_u \) and the multiplier \( Q = 1 \) yielding the conserved flow

\[ \begin{align*}
T &= \frac{1}{3}u^3 + \frac{7}{12}uu_{xx} - \frac{5}{12}u_x^2 \\
\phi &= \frac{5}{12}uu_{xt} - \frac{7}{12}u_xu_t,
\end{align*} \tag{2.16} \]

The condition for association (1.17) is satisfied for \( Y \) but not for scaling \( Z \). In the first case, we use the conserved flow (2.16) to obtain the reduced equation

\[ k = UU'' + \frac{1}{3}U^3 - U'^2. \tag{2.17} \]

Via \( Z \), a direct reduction of (1.5) is obtainable with invariants \( \zeta = tx^\mu \) and \( t^2 U = u \).

Notes. For a further elaborate discussion, we suppose a second-order derivative dependence of \( Q \) in which case (1.13) becomes

\[ \frac{\delta}{\delta u} \left( Q(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})[uu_{xxt} - u_xu_{xt} + u^2u_t] \right) = 0. \]

The Euler operator is extended and the corresponding expansion and separation is much more cumbersome and its analysis requires the aid of software. The analysis shows that in addition to the obvious multiplier \( Q = 1 \), we get the following multipliers and corresponding components of the conserved flow (via the homotopy formula).
a. \( Q_1 = \frac{1}{6}u^{-4}(3u^2u_t a'(t) + a(t)(6u^2u_{tt} + 3u_{xt}^2)) \):

For \( a(t) = 1 \), we get the density

\[
T_1 = \int_0^1 \frac{1}{12\lambda u^4}(-15tu_x^2u_{xt}^2 + 3\lambda u^3(4tu_{xt}^2 + u_x(u_{xt} + 2tu_{xt}) + u_tu_{xx} + 2tu_{tt}u_{xx} + 6tu_{xx}) \\
+ tu_{xt}(5u_xu_{xx} + 14u_xu_{xxt}) + 4\lambda u^4(3\lambda u_t^2 - u_{xx}^2) - u^2(3\lambda u_t u_x + 6tu_{xt}) \\
+ 2t(3\lambda u_{tt}u_x^2 + u_{xxt}^2 + u_{xtu_{xxxt}}))d\lambda
\]

b. \( Q_2 = u^{-4}(6u^3u_{xx} - 3u^2u_x^2 + 4u^5) \):

\[
\phi_2 = \left[ -3u_tu_x^3 - 3uu_x^2u_{xt} + u^2(6u_xu_{xx} + 6u_xu_{xxt} - 4u_tu_{xxx}) - 4u^3(2u_tu_x + u_{xxt}) \right]/4u^2, \\
T_2 = u^4 + 3u^4 + 4u^2u_{xx} - \frac{9mu^2u_{xx}}{4u} + \frac{1}{2}(3u_x^2 - u_xu_{xxx}) + u(-u_x^2 + u_{xxxx})
\]

c. \( Q_3 = u^{-2} \):

\[
\phi_3 = \int_0^1 \frac{u_tu_x - uu_{xt}}{2\lambda u^2}d\lambda, \\
T_3 = \int_0^1 \frac{2\lambda u^3 - u_x^2 + uu_{xx}}{2\lambda u^2}d\lambda
\]

As an example of a combination, the multiplier \( Q_* = u^{-4}(6ku^4 + 3mu^2) \) leads to the conserved density

\[
T_* = \int_0^1 \left( 6k\lambda^2u^3 - 5k\lambda u_x^2 - \frac{3mu_x^2}{2\lambda u^2} + \frac{3mu_{xx}}{2\lambda u} + u(3m + 7k\lambda u_{xx}) \right)d\lambda
\]

A symmetry analysis of the conservation laws (conserved vectors) above is given by the action of the symmetry on the multipliers. Firstly, it is clear that

\[
Y_1Q_i = 0, \quad i = 1, 2, 3, * \\
Y_2Q_j = 0, \quad j = 2, 3, * \\
Y_2Q_1 = F(t)\partial_tQ_1 = \tilde{Q}_1.
\]

Thus, all the conserved vectors quoted above are strictly invariant under \( Y_1 \) whilst (b), (c) and combined one are strictly invariant under \( Y_2 \). From what follows, we
see that the conserved vectors in (a), (b) and (c) are ray invariant under $Y_3$; the calculations show that

$$Y_3Q_1 = 2Q_1, \quad Y_3Q_2 = -2Q_2, \quad Y_3Q_3 = 4Q_3$$

It can be shown that the ray invariance condition based on $Y_3$, in general, is

$$Y_3Q_i = (\lambda + 5)Q_i, \quad i = 1, 2, 3,$$

so that $\lambda$ is $-3, -7, -1$, respectively.

(2). Equation (1.8)
Here, since (1.8) is a system of pdes in $u$ and $v$, the characteristic equation (1.13) becomes

$$\frac{\delta}{\delta(u,v)}(Q_1G^1 + Q_2G^2) = 0, \quad (2.18)$$

where $G^1$ and $G^2$ are the respective equations in the system (1.8) and $Q = (Q_1, Q_2)$. Since $Q_1$ and $Q_2$ are chosen to be dependent on derivative upto second-order, the calculations are extremely cumbersome and may only be done by software. The nontrivial solutions obtained, with the corresponding conserved flows are given below.

a. $(Q_1^1, Q_2^1) = (tv, tu - x)$:

$$\Phi^1_1 = -\frac{1}{12}v(-72tu^2 - v(3t(12k + 2vu_x - uu_x))$$
$$+ 2v(u_x + 12tux^2 - 11tu_{xx})) + 3u(24x + 8tv^2ux + 2tvux + v(ux + tvx))$$
$$\Phi^1_1 = -\frac{1}{12}v(72x + 3tvuxv_x + 2tv^2uxx - 3tu(24 + 2v_x^2 + vx_{xx}))$$
b. \((Q_1^2, Q_2^3) = (tv, tu - x + t(vu_{xx} + uv_{xx} + 2u_xv_x + v_{xt})):\)

\[
\Phi_2^x = \frac{1}{72} (18tv_t^2 + 36tu^2(2v + v_x^2) + 6v(v_t(-3 + 8tu_x) - t(3v_t + 2u_tv_x))
+ 3v^2(2(-2 + tv_x)u_x + 12tu_x^2 + t(12k - u_tv_x - 4u_{xt})) + 2v^3(u_x
+ 12tu_x^2 - 11tu_{xt}) - 3u(8tv^3u_x - 20vtv_x + v^2(v_x + tv_{xt})
+ 2v((2 + tv_t - 12tu_x)v_x + 2(6x + tv_{xt})),
\]

\[
\Phi_2^t = \frac{1}{72} (6tv_x(3v_t + 2uv_x) - 2tv^3u_{xx} + 3tv^2(-u_xv_x + 4u_{xx} + uv_{xx})
+ 6v(3(-4x + 2tu_xv_x + tv_{xt}) + tu(12 + v_x^2 + 2v_{xx})))
\]

It is clear that a reduction using the conserved flows is not feasible as the multipliers
are not associated with the only symmetries \(\partial_t\) and \(\partial_x\); a direct reduction of the
system (1.8) is attainable using these symmetries or the travelling wave reduction,
viz.,

\[
-cV' + UV' + VU' = 0,
\]

\[
-cU' + UU' + gV' = \frac{1}{3} h^2 (-cU''' + U^2 + UU'' - 2U''')
+ VV'(-cU'' + UU' - U'^2)
\]

(2.19)

which is reducible to a system second-order odes.

3 Optimal System of Subalgebras Admitted by
Parabolic Equations and Reduction

In this section, we now construct optimal system of one-dimensional subalgebra of
the main algebra admitted by Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB)
equations. Constructing optimal system provides us the set of non-equivalent classes
of symmetry generators that leads us to the construction of general classes of in-
variant solutions. To construct optimal system, we will find all the non-equivalent
symmetry generators under action of the adjoint representation, explained by Olver
The adjoint representation is given by
\[
Ad(\exp(\epsilon X_i))(X_j) = X_j - \epsilon [X_i, X_j] + \frac{\epsilon^2}{2!}[X_i, [X_i, X_j]] - \cdots
\] (3.20)
where \(\epsilon\) is a real number and \([X_i, X_j]\) denotes the Lie product defined by
\[
[X_i, X_j] = X_i X_j - X_j X_i.
\] (3.21)

### 3.1 BBMBP equations

First we construct the optimal system for BBMBP equation discussed in section (2) given by
\[
u_t + u_{xxt} + u_x + uu_x = 0,
\]
which admits a three dimensional algebra of point symmetry generated by
\[
X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_t - (1 + u)\partial_u,
\] (3.22)
with nonzero commutator \([X_2, X_3] = X_2\). The adjoint representation corresponding to above algebra is presented in table (1) that help us in constructing the optimal system of subalgebra for equation (2.2).

<table>
<thead>
<tr>
<th>(Ad(\exp(\epsilon X_i))(X_j))</th>
<th>(X_1)</th>
<th>(X_2)</th>
<th>(X_3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(X_1)</td>
<td>(X_1)</td>
<td>(X_2)</td>
<td>(X_3)</td>
</tr>
<tr>
<td>(X_2)</td>
<td>(X_1)</td>
<td>(X_2)</td>
<td>(X_3 - \epsilon X_3)</td>
</tr>
<tr>
<td>(X_3)</td>
<td>(X_1)</td>
<td>(X_2 e^{\epsilon})</td>
<td>(X_3)</td>
</tr>
</tbody>
</table>

Table 1: Table of Adjoint Representation

Consider \(X \in \mathcal{L}_3\), we have
\[
X = a_1 X_1 + a_2 X_2 + a_3 X_3.
\] (3.23)
Now executing the adjoint operator (3.20) on $X$ and using the adjoint Table 1, we obtain an optimal system which is as follows. Non-similar symmetry generators are given by

$$X^1 = Ad(e^{aX_2})X = X_1 + X_3 \quad ; \quad a_3 \neq 0, a_1 \neq 0$$

$$X^2 = Ad(e^{aX_2})X = X_3 \quad ; \quad a_3 \neq 0, a_1 = 0$$

$$X^3 = Ad(e^{aX_3})X = c_1X_1 \pm X_2 \quad ; \quad a_3 = 0, a_2 \neq 0$$

$$X^4 = Ad(e^{aX_3})X = X_1 \quad ; \quad a_3 = 0, a_2 = 0$$

We have received four elements $X^1, X^2, X^3$ and $X^4$ in optimal set of non-equivalent symmetry generators admitted by equation (2.2). Reduction of $X^2$ and $X^4$ has already performed in section (2). At the conclusion of this section, reduction under optimal system is performed using the direct method which solve the invariance surface conditions explicitly by solving the corresponding characteristic equation given by

$$\frac{dt}{\xi_1(t, x, u)} = \frac{dx}{\xi_2(t, x, u)} = \frac{du}{\eta_1(t, x, u)}. \quad (3.24)$$

To follow the reduction of new symmetries obtained in optimal system, first consider the symmetry generator $X^1 = X_1 + X_3$ has invariant solution $u = F(x)$ which is the invariant solution of (2.2) and satisfies the ODE $F'(x) = 0$. Invariant solution corresponding to symmetry generator $X^3 = cX_1 + X_2$ holds the form

$$u(x, t) = F(-cx + t), \text{ where } \alpha = -cx + t$$

and satisfy the ODE

$$F'('\alpha)+F''('\alpha)c^2 - F'(\alpha)c - F(\alpha)F'(\alpha)c = 0 \quad (3.25)$$
3.2 Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB)

The Lie point symmetry algebra is generated by

\[ X_1 = \partial_t, \quad X_2 = \partial_x, \quad (\text{for } \alpha = 0) \quad X_3 = -\beta t \partial_x + t \partial_t - \left( \frac{\beta + \gamma}{\theta} + u \right) \partial_u \]

with the only nonzero commutator \([X_2, X_3] = X_2 - \beta X_1\). The adjoint representation corresponding to above algebra is presented in table (2) that help us in constructing the optimal system of one-dimensional subalgebra for equation (2.12).

<table>
<thead>
<tr>
<th>(Ad(\exp(\epsilon X_i)) X_j)</th>
<th>(X_1)</th>
<th>(X_2)</th>
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</tr>
</thead>
<tbody>
<tr>
<td>(X_1)</td>
<td>(X_1)</td>
<td>(X_2)</td>
<td>(X_3)</td>
</tr>
<tr>
<td>(X_2)</td>
<td>(X_1)</td>
<td>(X_2, X_3 - \epsilon(X_2 - \beta X_1))</td>
<td></td>
</tr>
<tr>
<td>(X_3)</td>
<td>(X_1)</td>
<td>(e^\epsilon X_2 + (1 - e^\epsilon)\beta X_1)</td>
<td>(X_3)</td>
</tr>
</tbody>
</table>

Table 2: Table of Adjoint Representation

Consider \(X \in \mathcal{L}_3\), we have

\[ X = a_1 X_1 + a_2 X_2 + a_3 X_3. \] \hspace{1cm} (3.26)

Now executing the adjoint operator (3.20) on \(X\) and using the adjoint Table 1, we obtain an optimal system which is as follows. Non-similar symmetry generators are given by

\[ X^1 = Ad(e^{aX_1}) X = (1 + \beta) X_1 + X_3 \quad ; \quad a_3 \neq 0, a_1 \neq 0 \]
\[ X^2 = Ad(e^{aX_2}) X = X_3 \quad ; \quad a_3 \neq 0, a_1 = 0 \]
\[ X^3 = Ad(e^{aX_3}) X = X_2 \quad ; \quad a_3 = 0, a_2 \neq 0 \]
\[ X^4 = Ad(e^{aX_3}) X = X_1 \quad ; \quad a_3 = 0, a_2 = 0 \]
For Benjamin-Bona-Mahony-Peregrine-Burgers, we again have received four sym-
metries $X^1, X^2, X^3$ and $X^4$ in optimal system which are non-equivalent. Reduction
under optimal system for the symmetry generator

$$X^1 = (1 + \beta)X_1 + X_3,$$

has invariant solution of the form

$$u = F \left( \ln(\beta + 1 + t)\beta^2 - \beta t + \ln(\beta + 1 + t)\beta - x \right),$$

and satisfies the ODEs

$$- (F''(\alpha)\beta + \alpha F''(\alpha) + F'(\alpha)(\theta F(\alpha) + \gamma)) (1 + \beta) = 0, \quad (3.27)$$

$$-\alpha F''(\alpha) - F'(\alpha)(\theta F(\alpha) + \beta + \gamma) = 0. \quad (3.28)$$

The corresponding invariant variables are given by

$$\alpha = \ln(\beta + 1 + t)\beta^2 - \beta t + \ln(\beta + 1 + t)\beta - x \text{ and } u = F(\alpha).$$

4 Conclusion

Complex nonlinear dynamical systems are usually dealt with very well by invari-
ance methods in conjunction with conservation laws. This approach has not been
previously use, to the best of our knowledge, on the classes of equations studied
here.

We have shown how a study of the relationship between symmetries and conservation
laws are attained by the notion of multipliers; once established, we can obtain double
reduction of the original equations which is a double step in determing solutions. a
large of class of parabolic equations were studied in this way.
Furthermore, the models for certain wave phenomena, developed by Ostrovsky, are interesting and range from scalar to the more complex systems of pdes. Their symmetry properties are tied in with the underlying conservation laws which are varied and usually involves multipliers which are derivative dependent - particularly, the higher-order derivative dependent multipliers. These are somewhat cumbersome to determine but, it is clear, that obtaining these lead to new, nontrivial conserved flows.

References


