Analog Measurement of the Median, and other Percentiles, of Periodic Signals

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March 14, 2024

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Index Terms—median measurement, percentiles, analog signal

I. INTRODUCTION

The value for which a continuous-time signal, over a time interval, spends equal amounts of time above and below is the median of the signal. More generally, the P-th percentile of the signal is the level the signal spends P% of the time above, and (100-P)% of the time below. The median is the 50th percentile of the signal.

The median of discrete signals is commonly used, e.g., in image processing, while the median of continuous-time signals is seldom considered. The median of a discrete-time signal of finite odd-length $2N + 1$, $[s_1, \ldots, s_{N+1}, \ldots, s_{2N+1}]$, is usually obtained digitally by first ordering (either in a nondecreasing or a nonincreasing fashion) the vector of the values taken by the signal, which results in the ordered vector $[s(1), \ldots, s(N+1), \ldots, s(2N+1)]$, and then choosing the component $s_{(N+1)}$.

For a continuous-time signal $s : [0, T] \to \mathbb{R}$, of finite length $T$, the set $V := \{s(t) : t \in [0, T]\}$ of the values taken by the signal can be “ordered” in a nondecreasing fashion using measure theoretical tools [1]. In this way, a nondecreasing signal $s^\ast(\tau), \tau \in [0, T]$ results, with properties

- $s^\ast(\tau_1) \leq s^\ast(\tau_2)$, for each $0 \leq \tau_1 < \tau_2 \leq T$, and,
- $\mu(s^{-1}(I)) = \mu(s^\ast(I))$, for any interval $I \subset V$, where $\mu$ denotes Lebesgue measure. This ordered version $s^\ast$ of $s$ gives the median of $s$ as $s^\ast(\frac{T}{2})$; see [2] and [3].

An analog computation of $s^\ast$ is not straightforward, however. A more practical approach for the computation of the median of a continuous-time signal, that does not require the ordering of the set of the values the signal takes, is used in [4] and [5]. In this manner, we use the function

$$\eta_{\leq}(x) := \text{sgn}(x - F),$$

where $\text{sgn}$ is the signum function.

Now, the equation

$$\frac{1}{T} \int_0^T \eta_{\leq}(s(t))dt = 0$$

implies, as $F$, the median of the finite-length signal $s : [0, T] \to \mathbb{R}$ since, when the average of $\eta_{\leq}(s(\tau))$ is zero, the signal $s$ is spending equal amounts of time, above and below the level $F$. We solve Equation 2 electronically, via a feedback analog circuit. In the circuit, $\eta_{\leq}(s(t))$ is computed with a comparator and, for the average, we assume that $s(t)$ is periodic, and use an RC circuit. In Section II-A, other values besides zero for the average of $\eta_{\leq}(s)$ are considered, which allows for the measurement of other percentile levels besides the median. The period of the signal should be approximately known in order to adjust the time constants of the circuit.

If the signal is only approximately periodic, e.g., a biological signal, the circuit correspondingly gives a fluctuating output. If the median of a finite-length, analogously recorded segment of a signal is desired, the segment needs to be periodized beforehand.

II. AVERAGE, $\eta_L$, AND PERCENTILES

A geometric intuition for the average $\overline{s}$ of a signal $s(t)$ over a time interval of length $T$ comes from the fact that the average times the duration of the interval gives the integral (signed area) of the signal over the interval:

$$\overline{s}T = \int_0^T s(t) dt.$$ 

The overall average $\overline{s}$ of a signal $s(t)$ is defined as

$$\overline{s} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T s(t) dt;$$

if the signal $s$ is $T$-periodic\(^4\), its overall average is also the average over a single period (or over an integral number of periods):

$$\overline{s} = \lim_{n \to \infty} \frac{1}{nT} \int_0^{nT} s(t) dt = \frac{1}{T} \int_0^T s(t) dt.$$ 

The average of a periodic signal can be approximately obtained with a low-pass filter having a cutoff frequency below the fundamental frequency of the signal.

\(^3\text{sgn}(x) = 1, \text{ for } x > 0, \text{sgn}(0) = 0, \text{ and } \text{sgn}(x) = -1, \text{ for } x < 0.\)

\(^4\text{I.e. } \forall t, s(t + T) = s(t).\)
A. Binarized version

For a periodic signal $s(t)$, and with $\eta_F$ defined as in Equation 1, we call $\eta_F[s(t)]$ the \textit{+1-binarized version of} $s$ \textit{with threshold} $F$. The percentiles of the signal $s$ can be measured by averaging $\eta_F[s(t)]$, which is also $T$-periodic. Denote the average of $\eta_F[s(t)]$ as $E$:

$$E := \frac{\eta_F[s(t)]}{T_s}.$$  \hspace{1cm} (3)

Note that $E \in [-1, 1]$, $E = -1$ for $F \geq \max(s)$, and $E = 1$ for $F \leq \min(s)$. Electrically, $E$ can be obtained with little ripple using a first-order, low-pass RC circuit. Since the cutoff frequency $\frac{1}{2\pi FR}$ of the filter must be below the fundamental frequency $\frac{1}{T_s}$ of $s$, you need $RC > \frac{1}{2\pi F}$. On the other hand, the larger the value of the time constant $RC$, the slower the circuit.

Given a level $E$, with $s$ $T$-periodic, let $\tau_a$ and $\tau_b = 1 - \tau_a$, be the fractions of $T$ that $s(t)$ spends above and below $F$, respectively; i.e. the fractions of time $\eta_F[s(t)]$ takes the values $+1$ and $-1$. Thus, the average of $\eta_F[s(t)]$ is

$$E = \tau_a - \tau_b = 2\tau_a - 1.$$  \hspace{1cm} (4)

In particular, for $E = 0$, $\tau_a = \tau_b$, and $F$ is the median of $s$. In general, given $E \in [-1, 1]$, Equation (3) says that $F$ is the $100\frac{1+E}{2}$, $50(1+E)$th percentile $P$ of $s$, and

$$P = 50(1 + E).$$  \hspace{1cm} (5)

For example, with $E = \frac{1}{2}$, the signal spends $\tau_a = \frac{3}{4}$ of the time above $F$, and one fourth of the time below, and thus $F$ is then the 75th percentile level of $s$.

Given $E$, Equation 3 implicitly gives the value of $F$. Equation 3 can be solved for $F$ electronically, with the help of a feedback circuit.

III. CIRCUITIAL IMPLEMENTATION

Feedback circuits that measure any desired percentile $P$ of a periodic analog signal, that solve Equations 2 and 3, can be designed on the basis of these considerations. The principle is illustrated first with the two ideal prototype circuits of Figs. 1 and 2; then, an implementable circuit is given in Fig. 3.

A. Prototype Circuit I

In the idealized circuit of Figure 1, called Prototype Circuit I, the comparator at left compares the periodic signal $s(t)$ with the feedback level $F$, which is the integral of the average level of the comparator; since the comparator gives output levels of $\pm 1$, the output of the comparator is $\eta_F[s(t)]$. The comparator output is averaged via a low-pass RC circuit, producing the signal $\overline{\eta}[s(t)]$, assumed to give negligible ripple. The output of the integrator is inverted for clarity, although a simpler circuit without the inverter could be used; also for clarity, a buffer makes independent the stages of averaging and integration. The integral of the difference between the average $\overline{\eta}[s(t)]$ and the preset value of $E$ (negative of the integrator output,) is fed back to the comparator; in the stationary case, this difference is zero, and the output of the integrator remains at a constant value $-F$. The level $F$ makes the average of $sgn(s(t)-F)$ be equal to $E$, forcing $F$ to be the $50(1-E)$th percentile of $s$. The feedback signal $F$ sent to the inverting input of the comparator is then the desired percentile level $P = 50(1+E)$. For e.g. $E = 0$, the output of the circuit (the output of the inverter at right) is the median of $s$. With $E$ the preset level at the noninverting input of the comparator, the average of the binarized signal is

$$E = \tau_a - \tau_b = \tau_a - (1 - \tau_a) = 2\tau_a - 1.$$  \hspace{1cm} (6)

Or, derivating

$$\int_{t_0}^{t} \eta_F[s(\tau)] - E \, d\tau = F,$$

you get

$$\overline{\eta}[s(t)] = E.$$  \hspace{1cm} (7)

B. Prototype Circuit II

The integrator and averaging stages can be swapped, as in the circuit of Fig. 2; we call this circuit Prototype Circuit II. In this circuit there is no need of a buffer between stages but we still invert the output of the integrator. The integral of the difference between the comparator output and $E$ is an irregular sawtooth type signal that is averaged by the RC circuit.

Let $E$ be the preset level at the noninverting input of the integrator, and let the slope of the sawtooth signal when the output $\eta[s(t)]$ of the comparator is at level $+1$ be $m_a = 1 - E$ and, corresponding to level $-1$, be $m_b = -1 - E$. The sawtooth is averaged and fed back to the comparator, as level $F$. In stationary state, $m_a \tau_a = m_b \tau_b$, and $\frac{\tau_a}{\tau_b} = \frac{\tau_a}{1-\tau_a} = \frac{1+E}{1-E}$, or

$$\tau_a = \frac{1+E}{2}$$  \hspace{1cm} (8)

and

$$E = 2\tau_a - 1,$$  \hspace{1cm} (9)

is the 50th percentile level of $s$. The feedback signal $F$ sent to the inverting input of the comparator is then the desired percentile level $P = 50(1+E)$. For e.g. $E = 0$, the output of the circuit (the output of the inverter at right) is the median of $s$. With $E$ the preset level at the noninverting input of the comparator, the average of the binarized signal is

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Or, derivating

$$\int_{t_0}^{t} \eta_F[s(\tau)] - E \, d\tau = F,$$

you get

$$\overline{\eta}[s(t)] = E.$$  \hspace{1cm} (7)
as above.

Both prototype feedback circuits compute the desired feedback level $F$. Yet, in the corresponding open loop systems, the responses are different since, with $h$ denoting the impulse response of the averaging RC circuit,

$$\int_{-\infty}^{t} \eta_F [s(\tau) - E] \, d\tau \ast h(t) \neq \int_{-\infty}^{t} \eta_F [s(\tau)] \ast h(\tau) - E \, d\tau,$$

Incidentally, in a related, linear case, you do have

$$\int_{-\infty}^{t} x(\tau) \, d\tau \ast h(t) = \int_{-\infty}^{t} x(\tau) \ast h(\tau) \, d\tau.$$

C. Implementable Circuit

A practical, implementable circuit that measures the median is shown in Fig. 3; it can be easily modified to compute other percentiles. Even though the median operator does not obey superposition, it is a homogeneous operator, and median($\alpha x$) = $\alpha$ median($x$); thus median($x$) = $-\text{median}(-x)$, a fact taken advantage of in this circuit. Other percentiles share a similar yet different property.

The circuit in Figure 3 was designed by one of the authors for its application in power electronics, without initially realizing its alternate application as a median measurement circuit. The circuit uses low-noise JFET Operational Amplifiers, and switching silicon diodes; it has been proven to be accurate. The integrator stage is placed before the average stage, as in Prototype Circuit No. 2; an additional stage, that low-pass filters ($f_{\text{cut}} \approx 20$ Hz) the feedback signal $F$ has been added, giving a smoother output for the circuit. Level $E$ is set to 0 and the output gives the median.

The circuit includes a precision comparator of output $\pm 5$V, alleviating the fact that the output of the comparator at the top left part of the circuit, though close to its polarizing voltages of $\pm 15$, is hard to predict with precision. The output of this comparator is fed to the precision comparator, that gives an accurate output of $\pm 5$V. The references of $\pm 5$ are obtained at the inferior, left part of the circuit. The precision comparator can be seen as a nonlinear voltage divider, between a resistor of 8.2 kΩ, and the parallel of two nonlinear resistances. These nonlinear resistances have L-shaped $i$-$v$ characteristics, with knees at $i = 0$, and $v = \pm 5$ (see e.g. [6], Sec. 3.2.D).

The output $v_i$ of the first comparator, normally at $\pm 15$, momentarily transitions between levels. The output $v_i$ of the precision comparator feeds the integrator, and is related to $v_c$ via the nonlinear divisor, as

$$v_i = 5, \text{ if } v_c > 10,$$

$$v_i = -5, \text{ if } v_c < -10, \text{ and }$$

$$v_i = \frac{1}{2} v_c, \text{ for } -10 < v_c < 10 \text{ (momentarily.)}$$

The slopes of the sawtooth signal at the output of the integrator are $\pm \frac{5}{822}$ V/ms = $\pm 0.609$ V/ms. The time constant of the averaging stage is 8.2 ms ($f_{\text{cut}} = 19.4$ Hz.) The time constant of the filter at the output, that smooths again the feedback signal, producing the output of the circuit, is 0.82 ms ($f_{\text{cut}} = 194$ Hz.)

**Fig. 3.** Implemented circuit. The circuit measures the median of a signal applied at the left. For the measurement of other percentiles, the noninverter input of the integrator should be set accordingly.

**Fig. 4.** The signal $s(t') = 3 \cos \omega_0 t' + 2 \cos 2\omega_0 t'; \omega_0 = 8000$. It has a minimum of $-2.5625$ at e.g. $\omega_0 t'' = -0.384396$. The median level is indicated with red.

IV. Examples

We ran several simulations of the implemented circuit. Consider the signal $s(t) = 3 \sin \omega_0 t - 2 \cos 2\omega_0 t$, which has an average of 0, and a minimum of $-2.5625$ at e.g. $\omega_0 t = -0.384396$. For $f_0 = \frac{\omega_0}{2\pi} = 1273$ Hz, the application of the signal to the circuit gives an output that attains the median value after approximately 50 ms, or 60 periods of the signal. See Figs. IV and IV.

Likewise, the application of the signal $s(t) = [3 \sin \omega_0 t]$, which has an average of 1.909, with $f_0 = \frac{\omega_0}{2\pi} = 200$ to the circuit (see Fig. IV ) gives the median, of $3 \sin \frac{\pi}{4} = 2.121$, after a transient of approximately 60 ms, or 24 periods of the signal.

V. Concluding Remarks

We have presented a simple and practical approach for the electronic measuring of the median, and other percentile levels, of analog periodic signals. With little additional circuitry, you may compute e.g. the range, midrange, midranges and quasiranges as well. As mentioned in [4], “... analog devices consume much less energy and are therefore more suitable to operate in autonomous conditions, such as mobile communications, space missions, prosthetic devices, etc.”

In the discrete case, the classical example that compares the sample median and the sample mean (average) of a vector of data, is the salaries of the employees of a company. If the median salary is $\nu$, half the people earn at most $\nu$ and half the people earn at least $\nu$, no matter if three employees earn
The most common measures of centrality are surely the average, the median and the midrange. Of these three functionals, only the average is linear, which is another reason for its frequent use; yet, the median often replaces advantageously the average as a measure of centrality. On the one hand, it may be less noisy. It is considered a more robust (against spurious spikes) estimator of centrality than the average.

The average is sensitive to all variations of the signal while the median is more sensitive to variations near the central values of the signal, and insensitive to variations at the extremes.

Percentiles are useful as guidelines in the control of processes, including biological ones. If the 95th percentile of a pressure signal reaches a certain value, an alarm could be set off; likewise, if the 5th percentile of an oxygen saturation signal falls below a certain level.

We have considered the computation of the median and percentiles of finite-length or periodic signals. Given a window length, you may be also interested in the analog median filtering of an indefinite-length signal. One approach is to consider piecewise linear signals, as has been done in [7].

VI. ACKNOWLEDGEMENTS

We would like to thank Prof. Germán Yamhure Kattah for his help with the figures and the simulations run.

The idea for this circuit resulted while writing [7], to be published shortly.

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APPENDIX A
ON THE ORDERING OF CONVEX EVEN FUNCTIONS WITH ZERO MINIMUM

Suppose \( \min f = 0, \ y = f(x) \)
\[ \mu f^{-1}[0, y] = \mu f^*^{-1}[0, y] \]
\[ \mu f^*^{-1}[0, y] = f^*^{-1}(y) \]
then
\[ f^*(\mu f^{-1}[0, y]) = y = f(x) \]
or
\[ f^*(\mu f^{-1}[0, f(x)]) = f(x) \]
for \( f \) even, \( f^*(2x) = f(x) \)
or
\[ f^*(\xi) = f(\frac{\xi}{2}) \]

Examples,
for \( f(x) = x^2 \) with \( x \in [-1, 1] \), \( f^*(\xi) = \xi^2, \ \xi \in [0, 2] \)
for \( f(x) = |\sin x| \) with \( x \in [-\pi, \pi] \), \( f^*(\xi) = \sin \frac{\xi}{2}, \ \xi \in [0, 2\pi] \)

APPENDIX B
THE MEAN AND THE MEDIAN AS STATISTICAL PARAMETERS

The average is an estimator of the mean parameter of a distribution, and the (sample) median is an estimator of the median parameter of a distribution. For a random variable with cumulative probability function \( F_X(a) \), the mean parameter \( \mu \) is given by
\[ \mu = \int_{-\infty}^{\infty} a \ dF_X(a) \quad (6) \]
while the median parameter is any value \( \nu \) with
\[ \nu \in F^{-1}(0.5); \]
in fact, for discrete or mixed random variables, \( \nu \) may not exist, as \( F^{-1}(0.5) \) may be empty. Strictly speaking, the integral in (6) may not exist, and the distribution have no mean parameter, as in the case of a Cauchy distribution. The sample median will always exist of course, and will be one of the elements in the sample.