Existence results for nonlinear fractional differential equations with integral initial value conditions involving Hilfer fractional derivatives

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Abstract

In this paper, we consider the existence results for nonlinear Hilfer fractional differential equations with integral initial value conditions on finite interval $[0, T]$. Sufficient conditions for the existence of solution for the initial value problem are obtained by Schaefer’s fixed point theorem in constructed Banach Space.
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Abstract: In this paper, we consider the existence results for nonlinear Hilfer fractional differential equations with integral initial value conditions on finite interval [0, T]. Sufficient conditions for the existence of solution for the initial value problem are obtained by Schaefer’s fixed point theorem in constructed Banach Space.

Keywords: Nonlinear fractional differential equations; integral initial value conditions; Schaefer’s fixed point theorem

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1 Introduction

We investigate the following nonlinear Hilfer fractional differential equations with integral initial value conditions:

\[
\begin{align*}
D_{0+}^{\alpha,\beta} x(t) &= f \left( t, x(t), D_{0+}^{\mu,v} x(t) \right), \quad t \in J = (0, T], \\
\gamma(t) &= \int_0^T x(s) ds,
\end{align*}
\]

where \( D_{0+}^{\alpha,\beta} (\cdot) \) and \( D_{0+}^{\mu,v} (\cdot) \) are the Hilfer fractional derivatives defined by Hilfer in [1, 2] of order 1 < \( u < \alpha < 2 \), and type 0 ≤ \( \beta, v \leq 1 \). \( \gamma = \alpha + 2\beta - \alpha\beta, \gamma' = u + 2v - uv \) and \( \gamma' < \alpha \).

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In recent years, fractional calculus plays a crucial role in the fields of biology, medical science and mathematical model for COVID-19 in particular, physics, engineering, etc. See for example [3–7] and references therein. The most common of multifarious fractional derivatives are Riemann-Liouville and Caputo sense, please refer to [3] for further understanding of their properties and applications. In this paper, we consider the Hilfer fractional derivative that interpolates the Riemann-Liouville and the Caputo fractional derivative and have been widely used in analysis of steady state in science [8–12].

Very recently, numerous monographs have appeared concerning on the results of existence for nonlinear differential equations with integral initial value condition involving Hilfer fractional derivatives. Furati et al [13] proved the existence and uniqueness for the nonlinear integral initial value problem (1.2) in a weighted space of continuous functions.

\[
\begin{align*}
D_{a}^{\alpha,\beta} y(x) &= f(x, y), \quad x > a, 0 < \alpha < 1, 0 \leq \beta \leq 1, \\
I_{a}^{1-\gamma} y(a) &= y(a), \quad \gamma = \alpha + \beta - \alpha \beta,
\end{align*}
\]

where \(D_{a}^{\alpha,\beta}(\cdot)\) is the Hilfer fractional derivative, \(I_{a}^{1-\gamma}(\cdot)\) is the Riemann-Liouville fractional integral.

Subashini et al [14] utilized Mönch fixed point theorem concerned the existence of Hilfer fractional integro-differential equations with integral initial value condition.

\[
\begin{align*}
D_{0}^{\lambda,\mu} y(u) &= Ay(u) + K(u, y(u), \int_{0}^{u} h_{1}(u, s, y(s))ds, \int_{0}^{u} h_{2}(u, s, y(s))ds), \quad u \in [0, b_{1}], \\
I_{0}^{(1-\lambda)(1-\mu)} y(0) &= \sum_{n=1}^{\mu} c_{n} y(u_{n}), \quad u \in [-r, 0],
\end{align*}
\]

where \(D_{0}^{\lambda,\mu}(\cdot)\) is Hilfer fractional derivative with order \(0 < \lambda < 1\), and type \(0 \leq \mu \leq 1\); \(I_{0}^{(1-\lambda)(1-\mu)}(\cdot)\) is Riemann-Liouville fractional integral; \(A\) is the infinitesimal generator on the Banach space.

Meanwhile, inspired by questions mentioned in [13–20] and references therein, we will discuss existence results by means of Schaefer’s fixed point theorem [21, 22] to explore the existence results for the nonlinear differential equation (1.1) in relation to Hilfer fractional derivative.

This paper is organized as follows, In Section 2, as preliminaries, the fundamental definitions and properties are given. In Section 3, the existence results for the nonlinear Hilfer fractional differential equation (1.1) on finite interval in Banach spaces are proved.

## 2 Preliminaries

In this section, we will introduce Banach space and some related results about Hilfer fractional derivative and Riemann-Liouville fractional integral and derivative to be used in Section
3.

We define a special working space

$$\mathbb{X} = \left\{ y \middle| y(t) \in L^1([a, b], \mathbb{R}), D_0^{\alpha \omega} y(t) \in L^1([a, b], \mathbb{R}) \right\},$$

$$\rho \in (n - 1, n], n = \lfloor \alpha \rfloor + 1, \lfloor \alpha \rfloor \text{ means the integral part of } \alpha \text{ in equation (1.1). } \omega \in [0, 1].$$

The space $\mathbb{X}$ with the associated norm

$$\| y \|_\mathbb{X} = \left\{ \sup_{t \in [a, b]} \| y(t) \|, \sup_{t \in [a, b]} \| D_0^{\alpha \omega} y(t) \| \right\},$$

where $\| \cdot \|$ represent norm in view of continuous function space $C[a, b]$, defined as $\| y \| = \max_{t \in [a, b]} |x(t)|$, according to [23, 24], we know that $(\mathbb{X}, \| \cdot \|_\mathbb{X})$ is a Banach space.

**Definition 2.1.** [Riemann-Liouville fractional integral] The left-sided Riemann-Liouville fractional integral of order $\alpha > 0$ of the function $f \in L^1(a, b)$ is defined by

$$I_+^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - s)^{\alpha - 1} f(s)ds, x > a,$$

provided that the right-hand side is pointwise defined.

**Lemma 2.2.** [Riemann-Liouville fractional derivative] Let $\alpha \geq 0, \beta \geq 0$, and $f \in L^1(a, b)$. Then

$$I_+^\alpha I_+^\beta f(x) = I_+^{\alpha + \beta} f(x).$$

**Definition 2.3.** [Riemann-Liouville fractional derivative] The left-sided Riemann-Liouville fractional derivative of order $\alpha$ of the function $f$ is defined by

$$D_+^\alpha f(x) = D^n I_+^{n - \alpha} f(x) = \frac{1}{\Gamma(n - \alpha)} D^n \int_a^x (x - s)^{n - \alpha - 1} f(s)ds, x > a,$$

$n - 1 < \alpha < n, n \in \mathbb{Z}^+$. $D^n = \left( \frac{d}{dx} \right)^n$. The right-hand side derivative $D_0^\alpha (\cdot)$ is defined in a similar form.

**Definition 2.4.** [Hilfer fractional derivative] The left-sided Hilfer fractional derivative of order $\alpha \in (n - 1, n)$, and type $\beta \in [0, 1]$ of $f$ is defined by

$$D_+^{\alpha, \beta} f(x) = I_+^{\beta(n - \alpha)} D^n I_+^{(1 - \beta)(n - \alpha)} f(x) = I_+^{\beta(n - \alpha)} D^n I_+^{\gamma} f(x) = I_+^{\beta(n - \alpha)} D_+^\gamma f(x), x > a,$$

$$\gamma = \alpha + n\beta - \alpha \beta, n = \lfloor \alpha \rfloor + 1.$$  

**Lemma 2.5.** Let $n - 1 < \alpha < n, 0 \leq \beta \leq 1$ and $\gamma = \alpha + n\beta - \alpha \beta$. If $f \in \mathbb{X}$, then

$$I_+^{\gamma} D_+^{\gamma} f(x) = I_+^{\alpha, D_+^{\alpha, \beta}} f(x) = f(x) - \sum_{j=1}^{n} \frac{D_+^{\gamma - j} f(a)}{\Gamma(\gamma - j + 1)} (x - a)^{\gamma - j}, n = \lfloor \alpha \rfloor + 1.$$

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Lemma 2.6.[2] If $f \in X$ and $D^{\beta(n-\alpha)}_{a^+} f$ exists, then

$$D^{\alpha,\beta}_{a^+} I^{\alpha}_{a^+} f(x) = I^{\beta(n-\alpha)}_{a^+} D^{\beta(n-\alpha)}_{a^+} f(x), n = [\alpha] + 1.$$ 

Theorem 2.7. (Schaefer’s fixed point theorem)[21] Let $F : X \to X$ completely continuous operator. If the set

$$\Omega = \{ x \in X : x = \lambda F x \ for \ some \ \lambda \in (0,1) \},$$

is bounded, then $F$ has fixed points.

3 Main results

In this section, we will prove the existence for the nonlinear fractional differential equation involving Hilfer fractional derivative by means of Schaefer’s fixed point theorem. To prove our results, we will make the following assumptions:

(H1) Let $L_1(\cdot), L_2(\cdot)$ are nonnegative continuous functions. Suppose that function $f \in C(J \times X \times X \to X)$ satisfy the following condition

$$\| f(t, x, y) - f(t, x', y') \| \leq L_1(t) \| x(t) - x'(t) \| + L_2(t) \| y(t) - y'(t) \|, \forall t \in J.$$

(H2) There exist $Q, N \in \mathbb{R}^+$ such that functions $L_1(t), L_2(t)$, and $f(t, 0, 0)$ satisfying the following conditions, for any $t \in J$

$$\max_{t \in J}(L_1 + L_2)(t) \leq Q, \| f(t, 0, 0) \| \leq N.$$

Lemma 3.1 Suppose that function $x(t) \in X$ be the solution of system (1.1), is equivalent to the following integral equation.

$$x(t) = \frac{\int_0^T f(s, x(s), D^{\alpha,\beta}_{0^+} x(s))ds}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), D^{\alpha,\beta}_{0^+} x(s)) ds$$

$$+ \frac{t^{\gamma-2}}{p(\gamma)\Gamma(\gamma-1)} \left[ T^{\gamma} \int_0^T f(s, x(s), D^{\alpha,\beta}_{0^+} x(s)) ds \right] \frac{1}{\Gamma(\gamma+1)} + \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1} f(s, x(s), D^{\alpha,\beta}_{0^+} x(s)) ds dt].$$

Proof. According to Definition 2.3 and $x \in X$ be a solution of system (1.1), we know that $x(t), D^{\alpha}_{0^+} x(t), D^{\alpha,\beta}_{0^+} x(t) \in L^1(J, \mathbb{R})$. Applying Riemann-Liouville fractional integral operator $I^{\alpha}_{0^+}(\cdot)$ on both sides of (1.1) simultaneously, we have

$$x(t) = \frac{D^{\gamma-1}_{0^+} x(0)}{\Gamma(\gamma)} t^{\gamma-1} + \frac{t^{\gamma-2}}{\Gamma(\gamma+1)} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), D^{\alpha,\beta}_{0^+} x(s)) ds.$$
On the one hand, we carrying out the integrals on \([0, T]\) for above equation and combining with integral initial value conditions, we can get the following equation.

\[
\int_0^T x(t)dt = \frac{T^\gamma \int_0^T f(s, x(s), D_0^u x(s))ds}{\Gamma(\gamma + 1)} + \frac{T^{\gamma-1} \int_0^T x(s)ds}{\Gamma(\gamma)} + \frac{1}{\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1} f(s, x(s), D_0^\alpha x(s)) ds dt,
\]

so, we can derive that

\[
\int_0^T x(t)dt = \frac{T^\gamma \int_0^T f(s, x(s), D_0^u x(s))ds}{p(\gamma)\Gamma(\gamma + 1)} + \frac{1}{p(\gamma)\Gamma(\gamma)} \int_0^T \int_0^t (t-s)^{\alpha-1} f(s, x(s), D_0^\alpha x(s)) ds dt,
\]

where \(p(\gamma) = 1 - \frac{T^{\gamma-1}}{\Gamma(\gamma)}, \gamma = 1 + 2\beta - \alpha\beta \in [1, 2].\)

Then for all \(t \in J,\) we have,

\[
x(t) = \frac{\int_0^T f(s, x(s), D_0^u x(s))ds}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), D_0^\alpha x(s)) ds + \frac{t^{\gamma-2}}{p(\gamma)\Gamma(\gamma-1)} \left[\frac{T^\gamma \int_0^T f(s, x(s), D_0^u x(s))ds}{\Gamma(\gamma + 1)} + \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^t (t-s)^{\alpha-1} f(s, x(s), D_0^\alpha x(s)) ds dt\right].
\]

Based on the argument above, it is obvious that if a continuous function \(x \in C(J, B_M)\) satisfies initial value system (1.1), if and only if the function satisfies the integral equation.

Here, we will deal with existence of solution to initial value problem (1.1) by virtue of Schaefer’s fixed point theorem.

**Theorem 3.2** Assume that conditions (H1) and (H2) hold. Then the integral initial value problems (1.1) have at least one solution.

**Proof.** We will utilize the Schaefer’s fixed point theorem to prove system (1.1) have at least one solution. we consider the operator \(T : \mathcal{X} \rightarrow \mathcal{X}\) that implies \(T(x) \in \mathcal{X}\) for each \(x \in \mathcal{X}\).

\[
F(x) = \frac{\int_0^T f(s, x(s), D_0^u x(s))ds}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s), D_0^\alpha x(s)) ds + \frac{t^{\gamma-2}}{p(\gamma)\Gamma(\gamma-1)} \left[\frac{T^\gamma \int_0^T f(s, x(s), D_0^u x(s))ds}{\Gamma(\gamma + 1)} + \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^t (t-s)^{\alpha-1} f(s, x(s), D_0^\alpha x(s)) ds dt\right].
\]

Further, let \(\gamma’ = u + 2v - uv\) and \(\gamma’ < \alpha\). According to Definition 2.4, we can get

\[
D_0^u F(x) = \frac{\int_0^T f(s, x(s), D_0^u x(s))ds}{\Gamma(\gamma)} \frac{\gamma’ - u}{\Gamma(\gamma)} t^{\gamma-1} + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\alpha-1} f(s, x(t), D_0^\alpha x(t)) ds + \frac{t^{\gamma-2}}{p(\gamma)\Gamma(\gamma-1)} \left[\frac{T^\gamma \int_0^T f(s, x(s), D_0^u x(s))ds}{\Gamma(\gamma + 1)} + \frac{1}{\Gamma(\gamma)} \int_0^T \int_0^t (t-s)^{\alpha-1} f(s, x(s), D_0^\alpha x(s)) ds dt\right].
\]
From Definition 2.1, 2.3 and Lemma 2.2, we deduce

\[
D_{0+}^{\alpha, \nu} Fx(t) = \frac{f(s, x(s), D_{0+}^{\alpha, \nu} x(s))}{\Gamma(\gamma - u)} \int_0^T f(t, x(t), D_{0+}^{\alpha, \nu} x(t)) \, dt
\]

\[
= Fx(t) + \frac{t^{\gamma - u - 2}}{\Gamma(\gamma - u - 1)} \int_0^T f(s, x(s), D_{0+}^{\alpha, \nu} x(s)) \, ds
\]

Now we will divide the proof into several steps.

**Step 1:** \( F \) is continuous. Let \( x_n \) be a sequence such that \( x_n \to x \) in \( X \), that means \( x_n, x, D_{0+}^{\alpha, \nu} x_n, D_{0+}^{\alpha, \nu} x \in L^1(J, \mathbb{R}) \). For each \( t \in J \), we have

\[
\|Fx_n(t) - Fx(t)\| \leq \frac{t^{\gamma - 1}}{\Gamma(\gamma)} \int_0^T \|f(s, x_n(s), D_{0+}^{\alpha, \nu} x_n(s)) - f(s, x(s), D_{0+}^{\alpha, \nu} x(s))\| \, ds
\]

\[
+ \frac{t^{\gamma - 2}}{\Gamma(\gamma - 1)\Gamma(\gamma + 1)} \int_0^T \|f(s, x_n(s), D_{0+}^{\alpha, \nu} x_n(s)) - f(s, x(s), D_{0+}^{\alpha, \nu} x(s))\| \, ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^T (t - s)^{\alpha - 1} \|f(s, x_n(s), D_{0+}^{\alpha, \nu} x_n(s)) - f(s, x(s), D_{0+}^{\alpha, \nu} x(s))\| \, ds
\]

\[
\leq \frac{t^{\gamma - 1}}{\Gamma(\gamma)} \int_0^T L_1(s) \|x_n(s) - x(s)\| + L_2(s) \|D_{0+}^{\alpha, \nu} x_n(s) - D_{0+}^{\alpha, \nu} x(s)\| \, ds
\]

\[
+ \frac{t^{\gamma - 2}}{\Gamma(\gamma - 1)\Gamma(\gamma + 1)} \int_0^T L_1(s) \|x_n(s) - x(s)\| + L_2(s) \|D_{0+}^{\alpha, \nu} x_n(s) - D_{0+}^{\alpha, \nu} x(s)\| \, ds
\]

\[
+ \frac{1}{\Gamma(\alpha)} \int_0^T (t - s)^{\alpha - 1} \|L_1(s) \|x_n(s) - x(s)\| + L_2(s) \|D_{0+}^{\alpha, \nu} x_n(s) - D_{0+}^{\alpha, \nu} x(s)\| \| \, ds
\]

\[
\leq \frac{T^{\gamma - 1}}{\Gamma(\gamma)} \|x_n - x\|_X \int_0^T (L_1 + L_2)(s) \, ds + \frac{T^{2\gamma - 2}}{\Gamma(\gamma - 1)\Gamma(\gamma + 1)} \int_0^T (L_1 + L_2)(s) \, ds
\]

\[
+ \frac{T^{\gamma - 2}}{\Gamma(\gamma - 1)\Gamma(\gamma + 1)} \int_0^T (t - s)^{\alpha - 1} \|L_1(s) \|x_n - x\|_X \| \, ds
\]

\[
+ \frac{\|x_n - x\|_X}{\Gamma(\alpha)} \int_0^T (t - s)^{\alpha - 1} \|L_1(s) \|x_n - x\|_X \| \, ds
\]

\[
\leq Q \left[ \frac{T^{\gamma}}{\Gamma(\gamma)} + \frac{T^{2\gamma - 2}}{\Gamma(\gamma - 1)\Gamma(\gamma + 1)} + \frac{T^{\alpha + \gamma - 1}}{\Gamma(\gamma - 1)\Gamma(\alpha + 2)} \right] \|x_n - x\|_X \to 0(n \to \infty).
\]
On the other hand,

\[ \|D_{0+}^{u,v}Fx_n(t) - D_{0+}^{u,v}Fx(t)\| \]

\[ \leq \frac{T^{\gamma-u-1}}{\Gamma(\gamma-u)} \int_0^T \|f(s,x_n(s),D_{0+}^{u,v}x_n(s)) - f(s,x(s),D_{0+}^{u,v}x(s))\| \, ds \]

\[ + \frac{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\gamma+1)}{T^{\gamma-u-2}} \int_0^T\int_0^t (t-s)^{\alpha-1}\|f(s,x_n(s),D_{0+}^{u,v}x_n(s)) - f(s,x(s),D_{0+}^{u,v}x(s))\| \, ds \, dt \]

\[ + \frac{1}{\Gamma(\alpha-u)} \int_0^t (t-s)^{\alpha-u-1}\|f(s,x_n(s),D_{0+}^{u,v}x_n(s)) - f(s,x(s),D_{0+}^{u,v}x(s))\| \, ds \]

\[ \leq \frac{T^{\gamma-u-1}}{\Gamma(\gamma-u)} \|x_n - x\|_\mathcal{X} \int_0^T (L_1 + L_2)(s)ds + \frac{T^{2\gamma-u-2}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\gamma+1)} \int_0^T (L_1 + L_2)(s)ds \]

\[ + \frac{T^{\gamma-u-2}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\gamma+1)} \|x_n - x\|_\mathcal{X} \int_0^t \int_0^t (t-s)^{\alpha-1}(L_1 + L_2)(s)ds \, dt \]

\[ + \frac{1}{\Gamma(\alpha-u)} \|x_n - x\|_\mathcal{X} \int_0^t (t-s)^{\alpha-u-1}(L_1 + L_2)(s)ds \]

\[ \leq Q \left[ \frac{T^{\gamma-u}}{\Gamma(\gamma-u)} + \frac{T^{2\gamma-u-1}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\gamma+1)} + \frac{T^{\alpha+\gamma-u-1}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\alpha+2)} + \frac{T^{\alpha-u}}{\Gamma(\gamma-u+1)} \right] \|x_n - x\|_\mathcal{X} \]

\[ \to 0(n \to \infty). \]

Hence, for all \( x, y \in X \), we conclude that

\[ \|Fx_n - Fx\|_\mathcal{X} \to 0(n \to \infty). \]

In order to draw the conclusion, we define nonempty closed convex subset \( B_M \subset \mathcal{X} \)

\[ B_M = \left\{ u | u(t) \in \mathcal{X}, \|u\|_\mathcal{X} \leq M \right\}, \]

\[ M = \frac{N \left[ \frac{\theta^\gamma}{\Gamma(\gamma)} + \frac{\theta^{\gamma+\theta-1}}{\Gamma(\gamma+1)\Gamma(\gamma+\theta-1)} + \frac{\theta^{\alpha+\theta-1}}{\Gamma(\gamma+1)\Gamma(\alpha+\theta-1)} + \frac{\theta^{\beta+\theta-1}}{\Gamma(\gamma+1)\Gamma(\beta+\theta-1)} \right]}{1-Q} \left[ \frac{T^{\gamma-u}}{\Gamma(\gamma-u)} + \frac{T^{\alpha+\gamma-u-1}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\alpha+2)} + \frac{T^{\alpha-u}}{\Gamma(\gamma-u+1)} \right] \] is a positive constant, where \( \theta = \gamma \) or \( \gamma - u \).

Step 2: \( F \) is uniformly bounded in \( B_M \). For any \( t \in J \), we derive

\[ \|Fx(t)\| \leq \frac{T^{\gamma-1}}{\Gamma(\gamma)} \int_0^T \|f(s,x(s),D_{0+}^{u,v}x(s) - f(s,0,0))\| + \|f(s,0,0)\| \, ds \]

\[ + \frac{T^{2\gamma-2}}{p(\gamma)\Gamma(\gamma-1)\Gamma(\gamma+1)} \int_0^T \|f(s,x(s),D_{0+}^{u,v}x(s) - f(s,0,0))\| + \|f(s,0,0)\| \, ds \]

\[ + \frac{T^{\gamma-2}}{p(\gamma)\Gamma(\gamma-1)\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1}\|f(s,x(s),D_{0+}^{u,v}x(s) - f(s,0,0))\| + \|f(s,0,0)\| \, ds \, dt \]

\[ + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha u-1}\|f(s,x(s),D_{0+}^{u,v}x(s) - f(s,0,0))\| + \|f(s,0,0)\| \, ds \]

\[ \to 0(n \to \infty). \]
\[
\leq T^{\gamma-1} \frac{\|x\|_X}{\Gamma(\gamma)} \int_0^T (L_1 + L_2)(s)ds + T^{\gamma-1} \frac{\|f(s,0,0)\|_X}{\Gamma(\gamma)} \int_0^T ds \\
+ \frac{T^{2\gamma-2}}{p(\gamma)\Gamma(\gamma-1)\Gamma(\gamma+1)} \|x\|_X \int_0^T (L_1 + L_2)(s)ds + \frac{T^{2\gamma-2}}{p(\gamma)\Gamma(\gamma-1)\Gamma(\gamma+1)} \int_0^T f(s,0,0)\|_X ds \\
+ \frac{T^{\gamma-2}}{p(\gamma)\Gamma(\gamma-1)\Gamma(\alpha)} \|x\|_X \int_0^T \int_0^t (t-s)^{\alpha-1}(L_1 + L_2)(s)dsdt \\
+ \frac{T^{\gamma-2}}{p(\gamma)\Gamma(\gamma-1)\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1} f(s,0,0)\|_X dsdt \\
+ \frac{1}{\Gamma(\alpha)} \|x\|_X \int_0^t (t-s)^{\alpha-1}(L_1 + L_2)(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s,0,0)\|_X ds \\
\leq (QM + N) \left[ \frac{T^{\gamma}}{\Gamma(\gamma)} + \frac{T^{2\gamma-1}}{p(\gamma)\Gamma(\gamma-1)\Gamma(\gamma+1)} + \frac{T^{\alpha+\gamma-1}}{p(\gamma)\Gamma(\gamma-1)\Gamma(\alpha+2)} + \frac{T^{\alpha}}{\Gamma(\gamma+1)} \right] = M_1
\]

Similarly, we obtain

\[
\|D_0^{\alpha,v}Fx(t)\| \\
\leq \frac{T^{\gamma-u-1}}{\Gamma(\gamma-u)} \int_0^T \|f(s,x(s),D_0^{\alpha,v}x(s) - f(s,0,0))\| + \|f(s,0,0)\|_X ds \\
+ \frac{T^{2\gamma-u-2}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\gamma+1)} \int_0^T \|f(s,x(s),D_0^{\alpha,v}x(s) - f(s,0,0))\| + \|f(s,0,0)\|_X ds \\
+ \frac{T^{2\gamma-u-2}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1} \left[ \|f(s,x(s),D_0^{\alpha,v}x(s) - f(s,0,0))\| + \|f(s,0,0)\|_X \right] dsdt \\
+ \frac{1}{\Gamma(\alpha-u)} \int_0^t (t-s)^{\alpha-u-1} \left[ \|f(s,x(s),D_0^{\alpha,v}x(s) - f(s,0,0))\| + \|f(s,0,0)\|_X \right] ds \\
\leq \frac{T^{\gamma-u}Q}{\Gamma(\gamma-u)} \|x\|_X + \frac{T^{\gamma-u}N}{\Gamma(\gamma-u)} + \frac{T^{2\gamma-u-1}Q}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\gamma+1)} \|x\|_X + \frac{T^{2\gamma-u-1}N}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\gamma+1)} \\
+ \frac{Q\|x\|_X}{\Gamma(\gamma-u-1)\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1} dsdt + \frac{NT^{\gamma-u-2}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\alpha)} \int_0^T \int_0^t (t-s)^{\alpha-1} dsdt \\
+ \frac{Q}{\Gamma(\alpha-u)} \|x\|_X \int_0^t (t-s)^{\alpha-u-1} ds + \frac{N}{\Gamma(\alpha-u)} \int_0^t (t-s)^{\alpha-u-1} ds \\
\leq (QM + N) \left[ \frac{T^{\gamma-u}}{\Gamma(\gamma-u)} + \frac{T^{2\gamma-u-1}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\gamma+1)} + \frac{T^{\alpha+\gamma-u-1}}{p(\gamma)\Gamma(\gamma-u-1)\Gamma(\alpha+2)} + \frac{T^{\alpha-u}}{\Gamma(\gamma-u+1)} \right] = M_2.
\]

That means \( \|Fx\|_X \leq M \). Thus, \( T \) is uniformly bounded in \( B_M \).

Step 3: \( T \) is equicontinuous. Let \( 0 \leq t_1 < t_2 \leq T, x \in B_M \), we have
\[ \|Fx(t_2) - Fx(t_1)\| \]
\[ \leq \frac{1}{\Gamma(\gamma)} \int_0^T f(s, x(s), D^u_{0+} x(s))ds (t_2^{-\gamma} - t_1^{-\gamma}) + \frac{1}{\Gamma(\alpha - u)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} f (s, y(s), D^u_{0+} x(s)) ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} f (s, y(s), D^u_{0+} x(s)) ds \]
\[ + \frac{1}{p(\gamma)\Gamma(\gamma - 1)} \left[ \int_0^T f(s, x(s), D^u_{0+} x(s))ds \right] \]
\[ \leq C_1(t_2^{-\gamma} - t_1^{-\gamma}) + C_2(t_2^{-\gamma} - t_1^{-\gamma}) \]
\[ + \frac{1}{\Gamma(\alpha - u)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - 1} f (s, y(s), D^u_{0+} x(s)) ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_2 - s)^{\alpha - 1} - (t_1 - s)^{\alpha - 1} f (s, y(s), D^u_{0+} x(s)) ds \]
\[ \leq C_1(t_2^{-\gamma} - t_1^{-\gamma}) + C_2(t_2^{-\gamma} - t_1^{-\gamma}) + QM + N \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)} (t_2^{-\alpha} - t_1^{-\alpha}) \rightarrow 0(t_2 \rightarrow t_1), \]

where \( \tau = \gamma \) or \( \gamma - u \).

On the other hand,

\[ \|D^u_{0+} Fx(t_2) - D^u_{0+} Fx(t_1)\| \]
\[ \leq \frac{1}{\Gamma(\gamma - u)} \int_0^T f(s, x(s), D^u_{0+} x(s))ds (t_2^{-\gamma-u} - t_1^{-\gamma-u}) + \frac{1}{\Gamma(\alpha - u)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - u - 1} f (s, x(s), D^u_{0+} x(s)) ds \]
\[ + \frac{1}{\Gamma(\alpha - u)} \int_0^{t_1} (t_2 - s)^{\alpha - u - 1} - (t_1 - s)^{\alpha - u - 1} f (s, y(s), D^u_{0+} x(s)) ds \]
\[ + \frac{1}{p(\gamma)\Gamma(\gamma - u - 1)} \left[ \int_0^T f(s, x(s), D^u_{0+} x(s))ds \right] \]
\[ \leq C_1(t_2^{-\gamma-u} - t_1^{-\gamma-u}) + C_2(t_2^{-\gamma-u} - t_1^{-\gamma-u}) \]
\[ + \frac{1}{\Gamma(\alpha - u)} \int_{t_1}^{t_2} (t_2 - s)^{\alpha - u - 1} f (s, y(s), D^u_{0+} x(s)) ds \]
\[ + \frac{1}{\Gamma(\alpha - u)} \int_0^{t_1} (t_2 - s)^{\alpha - u - 1} - (t_1 - s)^{\alpha - u - 1} f (s, y(s), D^u_{0+} x(s)) ds \]
\[ \leq C_1(t_2^{-\gamma-u} - t_1^{-\gamma-u}) + C_2(t_2^{-\gamma-u} - t_1^{-\gamma-u}) + QM + N \frac{\Gamma(\alpha - u + 1)}{\Gamma(\alpha - u + 1)} (t_2^{-\alpha-u} - t_1^{-\alpha-u}) \rightarrow 0(t_2 \rightarrow t_1). \]

Hence, \( T \) is equicontinuous.
Step 4: Set $\Omega = \{x \in X : x = \lambda Fx \text{ for some } \lambda \in (0, 1)\}$, is bounded. Let $x \in \Omega$, then $x = \lambda Fx$ for some $\lambda \in (0, 1)$, therefore, we can get the following conclusions, for any $t \in J$.

\[
\frac{1}{\lambda} \|x(t)\| \leq (QM + N) \left[ \frac{T^\gamma}{\Gamma(\gamma)} + \frac{T^{2\gamma-1}}{p(\gamma)\Gamma(\gamma - 1)\Gamma(\gamma + 1)} + \frac{T^{\alpha+\gamma-1}}{p(\gamma)\Gamma(\gamma - 1)\Gamma(\alpha + 2)} + \frac{T^\alpha}{\Gamma(\gamma + 1)} \right],
\]

at the same time

\[
\frac{1}{\lambda} \|D_{0+}^\alpha x(t)\| \leq (QM + N) \left[ \frac{T^{\gamma-u}}{\Gamma(\gamma - u)} + \frac{T^{2\gamma-u-1}}{p(\gamma)\Gamma(\gamma - u - 1)\Gamma(\gamma + 1)} + \frac{T^{\alpha+\gamma-u-1}}{p(\gamma)\Gamma(\gamma - u - 1)\Gamma(\alpha + 2)} + \frac{T^{\alpha-u}}{\Gamma(\gamma - u + 1)} \right].
\]

Therefore, for some $\lambda \in (0, 1)$, we can acquire

\[
\|x(t)\|_X \leq \lambda M \leq M.
\]

That means $\Omega$ is bounded. Thus, by Theorem 2.7, there exists at least a fixed point $x$ of $F$ in Banach space $X$ on $[0, T]$.

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