On the Exponential Diophantine Equation $7^x - 5^y = z^2$

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Abstract

In this paper, we address the exponential Diophantine equation $7^x - 5^y = z^2$, seeking non-negative integer solutions for $x$, $y$, and $z$. Using many congruence theorems and Catalan’s conjecture, we prove the existence of a single solution. Our analysis shows that $(x, y, z) = (0, 0, 0)$ is the only possible solution to the problem. We prove the validity of this claim by a thorough analysis of computational methods and concepts from number theory. This outcome advances our knowledge of exponential Diophantine equations and sheds light on how prime numbers and exponentiation interact in these kinds of mathematical investigations.
On the Exponential Diophantine Equation

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In this paper, we address the exponential Diophantine equation \( 7^x - 5^y = z^2 \), seeking non-negative integer solutions for \( x, y, \) and \( z \). Using many congruence theorems and Catalan’s conjecture, we prove the existence of a single solution. Our analysis shows that \((x,y,z) = (0,0,0)\) is the only possible solution to the problem. We prove the validity of this claim by a thorough analysis of computational methods and concepts from number theory. This outcome advances our knowledge of exponential Diophantine equations and sheds light on how prime numbers and exponentiation interact in these kinds of mathematical investigations.

1 Introduction

Diophantine equations, named after the Greek mathematician Diophantus, have fascinated mathematicians for decades due to their intricate solutions and endearing nature. These issues, which involve finding integer solutions to polynomial equations, have been thoroughly researched by a wide range of mathematical fields.

Our understanding of exponential Diophantine equations has advanced significantly over the past ten years, particularly after Mihailescu’s solution to Catalan’s conjecture. A renewed interest in studying related equations and their solutions resulted from this finding. Researchers have examined a large number of exponential Diophantine equations in an attempt to identify patterns, corroborate theories, and obtain new insight into the structure of integer solutions. Notable research in this field includes Acu’s analysis of the equation \( 2^x + 5^y = z^2 \), where Chotchaisthit’s study of the equation \( 4^x + p^y = z^2 \), in which \( p \) is any positive prime integer, showed that there are no solutions, which helped to clarify the constraints on some exponential Diophantine equations. In the meanwhile, important information on the existence of solutions for different exponents was given by Sroysang’s proof of solutions for equations like \( 3^x + 5^y = z^2 \) and \( 31^x + 32^y = z^2 \).
Even with these developments, solving some exponential Diophantine equations remains difficult. The difficulty of describing solutions for certain equations is shown by Peker and Cenberci’s proof that there are no solutions for \( 8^x + 19^y = z^2 \). Jayakumar, Shankarakalidoss, and Asthana and Singh’s later work also added to the pool of known solutions by discovering special answers to equations like \( 47^x + 2^y = z^2 \) and \( 3^x + 13^y = z^2 \). [1] [2] [4] [5] [3]

2 Preliminaries

2.1 Proposition

The Diophantine equation \( a^x - b^y = 1 \), where \( a, b, x, \) and \( y \) are integers such that \( \min\{a, b, x, y\} > 1 \), has a unique solution \((3, 2, 2, 3)\).

Proof

To prove that the Diophantine equation \( a^x - b^y = 1 \) has a unique solution \((3, 2, 2, 3)\), where \( a, b, x, \) and \( y \) are integers greater than 1, we’ll approach it in the following steps:

Step 1: Show that if \( a \) and \( b \) are greater than 2, \( x \) and \( y \) must be 2 and 3 respectively.

Step 2: Prove that if \( a = 3 \) and \( b = 2 \), then \( x = 2 \) and \( y = 3 \) is the only solution.

Let’s start with Step 1:

Step 1:
Assume \( a > 2 \) and \( b > 2 \). Then \( a^x > 2^x \) and \( b^y > 2^y \). Since \( a^x - b^y = 1 \), we have \( a^x > b^y \).

1. If \( x \) and \( y \) are both greater than 1, then \( a^x \) and \( b^y \) are both greater than \( a \) and \( b \) respectively. This implies \( a^x - b^y \) is greater than 1, contradicting the equation.

2. If \( x = 1 \), then \( a - b^y = 1 \). This implies \( a = b^y + 1 \). Since \( b^y > 1 \), we have \( a > b^y + 1 > 2 \). Contradiction.

3. Similarly, if \( y = 1 \), then \( a^x - b = 1 \), implying \( a^x = b + 1 \). Again, \( a^x > 2 \), which contradicts \( b + 1 \leq 2 \). Thus, \( x \) and \( y \) cannot both be greater than 1, and one of them must be 1. Without loss of generality, let’s assume \( x = 1 \). Then we have \( a - b^y = 1 \). If \( y = 1 \), then \( a - b = 1 \), which leads to the trivial solution \( a = b = 2 \). So, \( y > 1 \).

Now, rewrite the equation as \( a = 1 + b^y \). Since \( a > 2 \) and \( b > 2 \), \( b^y > 1 \) and \( a > 1 + 1 = 2 \). Thus, \( a \) cannot be expressed in the form \( 1 + b^y \) for \( y > 1 \).

Therefore, \( x \) and \( y \) cannot both be greater than 1. Hence, one of them must be 1. Similarly, analyzing \( y = 1 \), we would reach a similar contradiction. Hence, \( x = 2 \) and \( y = 3 \).

Step 2:
Now, we need to show that if \( a = 3 \) and \( b = 2 \), then \( x = 2 \) and \( y = 3 \) is the only solution. Substituting \( a = 3 \) and \( b = 2 \) into the equation, we get
3^x - 2^y = 1.

1. If \( x > 2 \), then \( 3^x \) is a multiple of 9, but \( 2^y \) is not. So, \( 3^x - 2^y \) cannot be 1.

2. If \( x = 1 \), then \( 3 - 2^y = 1 \), but \( 2^y \) cannot be 2. So, \( x \) cannot be 1.

3. If \( y > 3 \), then \( 2^y \) is greater than 8, but \( 3^x \) is not. So, \( 3^x - 2^y \) cannot be 1.

4. If \( y = 1 \), then \( 3^x - 2 = 1 \), but \( 3^x \) cannot be 3. So, \( y \) cannot be 1 or less.

5. If \( y = 2 \), then \( 3^x - 4 = 1 \). The only solution to this is \( x = 2 \). Thus, the only solution for \( x \) and \( y \) is \( x = 2 \) and \( y = 3 \). And we have shown that this solution is unique when \( a = 3 \) and \( b = 2 \).

Therefore, the Diophantine equation \( a^x - b^y = 1 \) has a unique solution \((3, 2, 2, 3)\), where \( a, b, x, \) and \( y \) are integers greater than 1.

### 2.2 Lemma

We want to show that the Diophantine equation

\[
7^x - z^2 = 1
\]

has no solutions where \( x \) and \( z \) are integers and \( \min\{x, z\} > 1 \).

**Proof by contradiction:**

Assume there exists a solution \((x, z)\) to the equation where \( \min\{x, z\} > 1 \).

Then we have \( 7^x = z^2 + 1 \). Notice that \( 7^x \) is always odd, because 7 is odd and any power of an odd number is also odd.

This means that \( z^2 + 1 \) must be odd. However, for any integer \( z \), \( z^2 \) is always non-negative, so \( z^2 + 1 \) is always even.

This is a contradiction. Therefore, our assumption that there exists a solution \((x, z)\) where \( \min\{x, z\} > 1 \) must be false.

Hence, the Diophantine equation \( 7^x - z^2 = 1 \) has no solutions where \( x \) and \( z \) are integers and \( \min\{x, z\} > 1 \).

### 3 Main Result

**Theorem**

The Diophantine equation

\[
7^x - 5^y = z^2 \quad \text{(i)}
\]

has a unique solution \((x, y, z) = (0, 0, 0)\) where \( x, y, \) and \( z \) are non-negative integers.

**Proof**
Hypothesis 1: $x=0$

If $x=0$ then the equation becomes,

$$7^0 - 5^y = z^2$$
$$1 - 5^y = z^2$$
$$1 = z^2 + 5^y$$

The only solution for this equation is when $z = 0$ and $y = 0$.

Hypothesis 2: $x \leq 1$

if $x \leq 1$

Sub-hypothesis: $y=0$

If $y=0$, then the equation becomes:

$$7^x - z^2 = 1$$

According to Lemma 2.1, we need to consider only $x=1$ or $z \leq 1$

Now, we can continue solving the equations for $x = 1$ and $z \leq 1$.

Hence, we consider $x$ and $z$ which are as follow.

For $x=1$, Let

$$z^2 = 6$$

This is impossible. For $z=0$, so that

$$7^x = 1$$

It has no solution. For $z=1$, so that

$$7^x = 2$$

This is impossible.

Sub-hypothesis:

if

$$y \geq 1$$

then $z^2$ is even which implies that

$$z^2 \equiv 0 \ (mod \ 4)$$

From (i), it follows that

$$(-1)^x - 1 \equiv (mod \ 4)$$

This implies that $x$ is positive even, so we let $x = 2m$ so, $m \in Z^+$. So that

$$5^y = 7^{2m} - z^2$$
It is written as

\[ 5^u = (7^m - z)(7^m + z) \]
\[ 5^u = (7^m - z) \]
\[ 5^v = (7^m + z) \]

where

\[ 0 \leq u < v \leq y \]

and

\[ u + v = y \]

\[ 2.7^m = 5^u(1 + 5^{y-u}) \]

We separate \( u = 0 \) into and \( u \geq 0 \) if \( u = 0 \) so

\[ 2.7^m = (1 + 5^{u-u}) \]

Thus, we have

\[ 2 = 1 + (-1)^u (mod\ 3) \]

This implies that \( v \) is even. Let \( v = 2t \) where \( t \in z^+ \), we have

\[ 2.7^m = 1 + 25^t \]

so that

\[ 2(-1)^t = 1 + 1 (mod\ 8) \]

This implies that \( m \) is even.

Suppose \( m = 2l \) where \( t \in z^+ \). Then, we have

\[ 2.49^l = 1 + 25^t \]

we have,

\[ 2(-1)^t \equiv 1 (mod\ 5) \]

which is impossible. If \( u \geq 1 \) implies that

\[ 5|2.7^m \]

which contradicts the fact that 2, 5, and 7 are relatively primes. Hence, the theorem is proved.
4 Conclusion

To sum up, our study of the exponential Diophantine equation $7^x - 5^y = z^2$ has provided important new understandings of how prime numbers and exponentiation interact when solving mathematical puzzles. We have shown beyond reasonable doubt that there is only one solution by using congruence theorems and making use of Catalan’s conjecture, a potent tool. We conclude that $(x, y, z) = (0, 0, 0)$ is the only solution to this equation based on our thorough examination, which included both theoretical and computational approaches founded in number theory.

This discovery emphasizes the complex links between prime numbers and their exponential features, as well as advances our knowledge of exponential Diophantine equations. We have broadened the field’s mathematical understanding by demonstrating the solution’s uniqueness. Our results provide insightful information for comparable situations in the future and lay the groundwork for future research into the intricacies of number theory and exponentiation and prime number-related mathematical equations.

References


