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Solving the unsolvable non-stationary $M/E_k/1$ queue’s state variable open problem

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Abstract

This paper is a continuation on my revolutionary theory of solving the pointwise fluid flow approximation model for time-varying queues. Thus, the long-standing simulative approach has now been replaced by an exact solution by using a constant ratio $\beta$ (Ismail’s ratio), offering an exact analytical solution. The stability dynamics of the time-varying $M/E_k/1$ queueing system are then examined numerically in relation to time, $\beta$, and the queueing parameters.

Keywords: Time Varying $M/E_k/1$ queueing system.

1 Introduction

The field of transient/non-stationary analysis has limited literature, which can be categorized into simulation, transient analysis, analysis, and applications techniques. These categories encompass various approaches to studying systems that change over time, including simulations, analysing transient behaviour, and exploring non-stationary phenomena. In certain cases, mathematical transformations can be used to obtain a closed form expression for analysing non-stationary queueing systems. However, evaluating these expressions can be computationally complex. As a result, there has been a focus on numerically determining the transient behaviour of such systems instead of deriving closed form expressions.

The current exposition contributes to solving for first time ever, the longstanding unsolved problem of obtaining the state variable of the time varying $M/E_k/1$ queueing system.

The following flowchart shows how this paper is organized.

2 PSFFA

Let $f_{in}(t)$ and $f_{out}(t)$ serve as the temporal flow in, and flow out, respectively. Therefore,
\[
\frac{dx(t)}{dt} = x(t) = -f_{out}(t) + f_{in}(t), \text{as the state variable}
\]

(2.1)

\( f_{out}(t) \) links server utilization, \( \rho(t) \) and the time-dependent mean service rate, \( \mu(t) \) by:

\[ f_{out}(t) = \mu(t)\rho(t) \]

(2.2)

For an infinite queue waiting space:

\[ f_{in}(t) = \text{Mean arrival rate} = \lambda(t) \]

(2.3)

Thus (2.1) rewrites to:

\[ x(t) = -\mu(t)\rho(t) + \lambda(t), \quad 1 > \rho(t) = \frac{\lambda(t)}{\mu(t)} > 0 \]

(2.4)

The stable subcase of (2.4)

i.e.,

\[ x(t) = 0 \]

(2.5)

\[ x = G_1(\rho) \]

(2.6)

The numerical invertibility of \( G_1(\rho) \), yields

\[ \rho = G_1^{-1}(x) \]

(2.7)

Hence,

\[ x(t) = -\mu(t)(G_1^{-1}(x(t))) + \lambda(t) \]

(2.7)

Therefore, the time varying \( M/E_k/1 \) queueing system’s -PSFFA model reads:

\[ x' = -\mu \left(\frac{k(x+1)}{k-1} - \frac{\sqrt{(k^2x^2+2kx+k^2)}}{k-1}\right) + \lambda \]

(2.8)

Time varying queue’s life example [6] is depicted by figure 1.

![Figure 1](image-url)
3 Solving the non-stationary $M/E_k/1$ queueing system’s PSFFA (c.f., (2.8))

Theorem 3.1 Ismail’s ratio, $\beta$, solves (2.9), with a closed form expression to read as:

$$
\left(1 - \frac{a(x+\frac{1}{k})}{\sqrt{(1-\frac{1}{k^2})+(1-\frac{1}{k^2}+x^2)}}\right)^B \left(1 - \frac{b(x+\frac{1}{k})}{\sqrt{(1-\frac{1}{k^2})+(1-\frac{1}{k^2}+x^2)}}\right)^B = \eta e^{-\frac{k}{(k-1)t} \mu dt}
$$

(3.1)

Proof

We have

$$
\lambda = -\mu \left(\frac{k(x+1)}{k-1} - \sqrt{k^2x^2 + 2kx + k^2}\right) + \lambda, k, \beta = \frac{\lambda(t)}{\mu(t)} > 0 \quad \text{(c.f., (2.8))}
$$

Let $x = \sqrt{\left(1 - \frac{1}{k^2}\right) \text{csch}y - \frac{1}{k}}$, then $x = -\sqrt{\left(1 - \frac{1}{k^2}\right) y \text{coth} \text{csch}y}$. Thus, we have

$$
-\sqrt{\left(1 - \frac{1}{k^2}\right) \text{coth} \text{csch}y dy = \frac{\beta(k-1)}{k} \frac{\beta(k-1)}{k - 1} dt}
$$

(3.2)

This means

$$
\frac{\text{coth} \text{csch}y dy}{-(1 + \sinh y - \cosh y) + \left(\frac{(\beta - 1)(k-1)}{\sqrt{(k^2 - 1)}}\right) \sinh y \sinh y} = -\frac{\mu k}{k - 1} dt
$$

Equivalently

$$
\frac{2}{[(\beta - 1)(k-1)]^{(e^{2y} + e^y) dy}} = -\frac{\mu k}{k - 1} dt
$$

(3.3)

Define $\frac{[(\beta - 1)(k-1)]}{\sqrt{(k^2 - 1)}} = \zeta$

$$
\left[e^{2y} + \frac{2e^y}{\zeta} - 1 \right] = 0 \Rightarrow e^y = \frac{\left(\frac{1}{\zeta} \pm \frac{1}{\zeta} \sqrt{-4\zeta^2 - 4}\right)}{2} = \frac{\left(\frac{1}{\zeta} \pm \frac{1}{\zeta} \sqrt{-8\zeta^2 - 4}\right)}{2} = a, b
$$
where

\[
a = \frac{\left(\frac{1}{\zeta} + \sqrt{\frac{1}{\zeta^2} - \frac{8}{\zeta} + 4}\right)}{2}
\]

\[
b = \frac{\left(\frac{1}{\zeta} - \sqrt{\frac{1}{\zeta^2} - \frac{8}{\zeta} + 4}\right)}{2}
\]

where

\[
\frac{\frac{2}{\zeta}(e^{3y} + e^y)dy}{[e^{2y} - \frac{e^y}{\zeta} + [\zeta - 1][e^{2y} - 1]} = -\frac{\mu dt}{(1 - \xi)}
\]

\[
= \left[ \frac{A}{(e^y - a)} + \frac{B}{(e^y - b)} + \frac{C}{(e^y - 1)} + \frac{D}{(e^y + 1)} \right]
\]

Hence, it is implied that:

\[
\therefore A + B + C + D = \frac{2}{\zeta} \implies A = \frac{2}{\zeta} - (B + C + D) \quad (3.4)
\]

implies

\[
A = \frac{2}{\zeta} = \frac{(1 + \frac{2}{\zeta} + 2a)(\frac{1}{\zeta} + a)}{(\frac{2}{\zeta} + a)}
\]

which yields:

\[
C = \frac{\frac{4}{\zeta^2}(\frac{1}{\zeta} + a) - [(1 + \frac{2}{\zeta} + 2a)(\frac{1}{\zeta} + a) + (ab - (a + b))(\frac{1}{\zeta} + a)]D}{(\frac{2}{\zeta} + a + [ab - (a + b)](\frac{1}{\zeta} + a)}
\]

\[
bA + abC - abD = 0
\]

Therefore, we have
\[ D = \frac{2b}{\zeta} + a \left( ab - \frac{b(1 + \frac{2}{\zeta})}{\left(\frac{2}{\zeta} + a\right)} + a \left( \frac{(1 - a)}{\left(\frac{2}{\zeta} + a\right)} \right) \right) + \frac{4}{\zeta} \left( \frac{2}{\zeta} + a \right) \left( a + [ab - (a + b)] \left(\frac{2}{\zeta} + a\right) \right) \]

\[ B = \frac{(1 - a)(C - D)}{\left(\frac{2}{\zeta} + a\right)} \]

Integrating both sides of the resulting (3.3), will transform final solution, for some non-negative real constant parameter \( \eta \), to:

\[
\left( \frac{1}{1 - ae^{-csch^{-1}(\frac{x+1}{\sqrt{1-k^2}})}} \right) |^A \left( \frac{1}{1 - be^{-csch^{-1}(\frac{x+1}{\sqrt{1-k^2}})}} \right) |^B \left( \frac{1}{1 - e^{-csch^{-1}(\frac{x+1}{\sqrt{1-k^2}})}} \right) |^C
\]

\[ = \eta e^{-\frac{k}{(x-1)f} \mu dt} \]

By mathematical analysis, it is well known that:

\[
\text{csch}^{-1} \left( \frac{x + \frac{1}{k}}{\sqrt{1 - \frac{1}{k^2}}} \right) = \ln \left( 1 + \sqrt{1 + \left( \frac{x + \frac{1}{k}}{\sqrt{1 - \frac{1}{k^2}}} \right)^2} \right) = \ln \left( \sqrt{1 - \frac{1}{k^2}} + \sqrt{1 - \frac{1}{k^2}} + \left( \frac{x + \frac{1}{k}}{\sqrt{1 - \frac{1}{k^2}}} \right)^2 \right) \left( \frac{x + \frac{1}{k}}{\sqrt{1 - \frac{1}{k^2}}} \right)\]

Provided that, \( x \neq -\frac{1}{k}, \frac{1}{k} \in (0,1) \)

Thus, one gets:
\[ \left( \frac{1}{\sqrt{1 - \frac{1}{k^2}}} + \sqrt{1 - \frac{1}{k^2} + \frac{x}{k^2}} \right) \right)^{B} \]

\[ \left( \frac{1}{\sqrt{1 - \frac{1}{k^2}}} + \sqrt{1 - \frac{1}{k^2} + \frac{x}{k^2}} \right) \right)^{D} \]

\[ = \eta e^{-\frac{k}{k-1} \int \mu(t)} \]

This completes the proof.

**Numerical experiment**

Let \( k = 2 \) (number of states), \( \eta = 1, \beta = 1.1 \). Hence, \( \zeta = 0.05773502692 \). Let \( \mu(t) = \frac{1}{t} \).

\[ a = \frac{\left( \frac{1}{\zeta} + \sqrt{\frac{1}{\zeta^2} - \frac{\theta}{\zeta^4}} \right)}{2} = 15.09134911, b = \frac{\left( \frac{1}{\zeta} + \sqrt{\frac{1}{\zeta^2} - \frac{\theta}{\zeta^4}} \right)}{2} = 2.229158964 \]

\[ A = 41.83226741, B = -1.330027162, C = -0.4795616949, D = -5.17360455 \]

Therefore, one gets:

\[ \left( \frac{\left( 1 - \frac{15.09134911(x + 0.5)}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right)}{\left( 1 + \frac{x + 0.5}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right)} \right)^{41.83226741} \]

\[ \left( \frac{\left( 1 - \frac{2.229158964(x + 0.5)}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right)}{\left( 1 + \frac{x + 0.5}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right)} \right)^{-1.330027162} \]

Consequently,

\[ t = \frac{\left( \frac{1}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right)^{2.586802275} \left( 1 - \frac{15.09134911(x + 0.5)}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right) \left( 1 - \frac{2.229158964(x + 0.5)}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right) \left( 1 - \frac{x + 0.5}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right) \left( 1 - \frac{0.4795616949(x + 0.5)}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right) \left( 1 - \frac{0.665013581(x + 0.5)}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right) \left( 1 - 0.2397808475 \right) \right) \]

\[ t = \frac{20.91613371}{0.665013581} \]

\[ \frac{0.665013581}{0.2397808475} \]
The continuous increase of the underlying queue’s state variable is impacted by the progressive decrease of time, as visualized by the above three figures.

To show this analytically,

\[
\lim_{x(t) \to \infty} t = \lim_{x(t) \to \infty} \left( \frac{1 - \frac{15.09134911(x + 0.5)}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}}}{\left(1 + \frac{(x + 0.5)}{\sqrt{0.75 + \sqrt{0.75 + [x + 0.5]^2}}} \right)^{2.586802275}} \right)^{0.2397808475} = 0
\]

4 Closing remarks with next phase of research

In this work, a challenging topic in queueing theory is examined; more precisely, the underlying queue’s state variable is determined. Future work will concentrate on resolving open research issues and investigating applications of non-stationary queues in other scientific areas. The study also examines the effects of time, and other specific parameters of the underlying queue on its stability.

References


