Physical Bounds on the Time-Domain Response of a Linear Time-Invariant System

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Abstract

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Index Terms—time-domain (TD) analysis, physical bounds, electromagnetic theory.

I. INTRODUCTION

For a given reflection coefficient, the Bode-Fano criterion provides an upper bound on the bandwidth over which the desired match can be achieved (see [1, Ch. XIII], [2], [3, Sec. 5.9]). This result is, in fact, a sum rule that can be derived using integral representations and asymptotic expansions of Herglotz functions [4]. A similar approach is followed in this letter where early-time and late-time physical bounds on the response of a linear time-invariant systems are derived using its analytic properties under the one-sided Laplace transform.

It is anticipated that such time-domain (TD) physical bounds will be of key importance for assessing the performance of time-varying systems and devices (e.g., [5]).

In contrast to the present work that deals with the TD response of a general causal system, our previous study on the subject is dominantly focused on (the early-time bound of) a special class of system functions [6].

II. PROBLEM DEFINITION

In the analysis that follows, \( t \) is the time coordinate. The time-convolution operator is denoted by \( \ast \). The Heaviside unit-step function is denoted \( H(t) \) and the Dirac delta distribution is \( \delta(t) \). Consequently, the time-integration operator can be defined as \( \partial_t^{-1}f(t) = f(t) \ast H(t) = \int_{t-\tau}^{t} f(\tau) d\tau \).

We shall analyze the response function of a linear time-invariant system, the output signal, say \( y(t) \), can be expressed via a time-convolution integral as

\[
y(t) = \Gamma(t) \ast x(t) = \int_{\tau=0}^{t} \Gamma(t-\tau)x(\tau)d\tau,
\]

where \( \Gamma(t) \) is the (non-anticipatory) impulse response and \( x(t) \) denotes the (causal) input signal. In our applications, \( \Gamma(t) \) typically represents the TD reflection coefficient, or the dielectric relaxation function of a dispersive medium, for example. The analysis that follows makes use of the one-sided Laplace transform that is defined as

\[
\hat{\Gamma}(s) = \int_{t=0}^{\infty} \exp(-st)\Gamma(t)dt,
\]

where \( s = \sigma + i\omega \) is the Laplace transform parameter (= complex frequency) with \( \sigma > 0 \), thereby accounting for the universal property of causality. Consequently, the response function \( \hat{\Gamma}(s) \) is an analytic function in \( \text{Re}(s) > 0 \) (including infinity) with possible simple poles at \( s = 0 \), \( s = \pm i\omega_n \), and simple poles at in \( \text{Re}(s) < 0 \) (i.e., the poles of a response function have negative real parts; those on the imaginary axis must be simple). The response function \( \hat{\Gamma}(s) \) is a real function of \( s \), i.e., the response function takes on real values when \( s \) is real-valued. Consequently, Schwartz’s reflection principle applies, \( \hat{\Gamma}(s^*) = [\hat{\Gamma}(s)]^* \). Hence, its TD original, \( \Gamma(t) \), is real-valued and its poles are either real or occur in complex-conjugate pairs [7, Ch. 6]. The \( s \)-domain response function can be then expressed as

\[
\hat{\Gamma}(s) = \gamma_{\infty} + \sum_{n=1}^{N} \frac{\gamma_n}{s^2 + \omega_n^2} + \hat{\Omega}(s)
\]

where \( \gamma_{\infty}, \gamma_0 \) and \( \gamma_n \) are real-valued coefficients,

\[
\hat{\Omega}(s)
\]

thus represents a function that is analytic in \( \text{Re}(s) > 0 \) and its entire boundary \( V \cup C^\infty \) (see Fig. 1). Assuming, in addition,
\( \hat{\Omega}(s) = o(1) \) as \( |s| \to \infty \), the integration around the semicircle \( C^\infty \) \( \{ z = \Delta \exp(i\theta); \Delta > 0, \pi/2 \geq \theta \geq -\pi/2 \} \) vanishes as \( \Delta \to \infty \), so that Cauchy’s integral formula leads to

\[
\hat{\Omega}(s) = \frac{2}{\pi} \int_{v=0}^{\infty} \operatorname{Re} \left[ \hat{\Omega}(iv) \right] \frac{s}{s^2 + v^2} dv
\]

for \( \operatorname{Re}(s) > 0 \). \hspace{1cm} (4)

Here, we have used the property \( \hat{\Omega}(-iv) = \hat{\Omega}^*(iv) \), since \( \hat{\Omega}(s) \) is a real function of \( s \). Note that the terms pertaining to the (simple) pole singularities along the \( \omega \)-axis can be with (4) incorporated in a single Stieltjes integral (see [4, pp. 494–495]). Similarly, we can obtain

\[
\hat{\Omega}(s) = -\frac{2}{\pi} \int_{v=0}^{\infty} \operatorname{Im} \left[ \hat{\Omega}(iv) \right] \frac{v}{s^2 + v^2} dv
\]

for \( \operatorname{Re}(s) > 0 \). \hspace{1cm} (5)

Integral representations (4) and (5) can be substituted in (3) to get general representations for the response function \( \hat{\Gamma}(s) \).

### III. Time-Domain Physical Bounds

Integral representations (4) and (5) will next be employed to establish physical bounds on the TD response function

\[
\Gamma(t) = \gamma_\infty \delta(t) + \gamma_0 H(t) + 2 \sum_{n=1}^{N} \gamma_n \cos(\omega_n t) H(t) + \Omega(t).
\]

These bounds are to be determined using the asymptotic expansions

\[
\hat{\Omega}(s) = a_1 s^{-1} + a_2 s^{-2} + o(s^{-2}) \text{ as } |s| \to \infty,
\]

\[
\hat{\Omega}(s) = b_0 + b_2 s^2 + o(s^2) \text{ as } s \to 0.
\]

#### A. Early-Time Bound

To establish an early-time bound on the impulse function, we employ the property that \( \hat{\Omega}(s) \) is an analytic function in the right half of the complex \( s \)-plane and its boundary, which on account of Cauchy’s theorem [8, Eq. (B.1-21)] leads to

\[
\frac{1}{2\pi i} \oint_{s \in B \cup C^\infty} \hat{\Omega}(s) ds = 0,
\]

where \( B \) and \( C^\infty \) are shown in Fig. 2. Using (7) to evaluate the contribution of integration along \( C^\infty \) and combining the integrations along \( \{ -\infty < \omega < 0 \} \) and \( \{ 0 < \omega < \infty \} \), we end up with

\[
\frac{2}{\pi} \int_{\omega=0}^{\infty} \operatorname{Re} \left[ \hat{\Omega}(i\omega) \right] d\omega = a_1.
\]

Owing to (8), the integral around a small indentation around the origin is vanishingly small. As \( s/(s^2 + v^2) \) corresponds to \( \cos(\omega t) \hat{H}(t) \) for which \( \cos(\omega t) \leq 1 \), the TD counterpart of (4) with (10) yields

\[
\Omega(t) \leq a_1 \hat{H}(t) \text{ for } a_1 > 0,
\]

Assuming \( \operatorname{Re}[\hat{\Omega}(i\omega)] > 0 \) for all \( \omega > 0 \). A similar inequality applies to \( \operatorname{Re}[\hat{\Omega}(i\omega)] < 0 \), i.e.,

\[
-\Omega(t) \leq -a_1 \hat{H}(t) \text{ for } a_1 < 0.
\]

The use of (11) or (12) in (6) yields the corresponding (early-time) bound on the response function.

#### B. Late-Time Bound

To establish a late-time bound on the impulse function, we now employ (cf. (9))

\[
\frac{1}{2\pi i} \oint_{s \in B \cup C^\infty} \hat{\Omega}(s) \frac{ds}{s} = 0.
\]

On account (7) the integral around \( C^\infty \) is vanishingly small as \( \Delta \to \infty \). Consequently, accounting for the (nonzero) contribution from the indent using (8), we arrive at

\[
-\frac{2}{\pi} \int_{\omega=0}^{\infty} \operatorname{Im} \left[ \hat{\Omega}(i\omega) \right] d\omega = b_0.
\]

Since \( v/[s(s^2 + v^2)] \) corresponds to \( [1 - \cos(\omega t)]v^{-1} \hat{H}(t) \) for which \( [1 - \cos(\omega t)]v^{-1} \leq 2v^{-1} \), the integral representation (5) with (14) yields

\[
\frac{\partial}{\partial t} \hat{\Omega}(t) \leq 2b_0 \hat{H}(t) \text{ for } b_0 > 0,
\]

provided that \( \operatorname{Im}[\hat{\Omega}(i\omega)] < 0 \) for all \( \omega > 0 \). Finally, for \( \operatorname{Im}[\hat{\Omega}(i\omega)] > 0 \) we get

\[
-\frac{\partial}{\partial t} \hat{\Omega}(t) \leq -2b_0 \hat{H}(t) \text{ for } b_0 < 0.
\]

The use of (15) or (16) (in the time-integrated version of) (6) yields the corresponding (late-time) bound on the response function.

For example, the response function of a first-order system that is governed by \( \partial_t \hat{y}(t) + \tau^{-1} \hat{y}(t) = \pm x(t) \) with \( \tau > 0 \) can be expressed as \( \hat{\Gamma}(s) = \hat{y}(s)/\hat{x}(s) = \pm 1/(s + 1/\tau) = \hat{\Omega}(s) \), respectively. Consequently, \( \operatorname{Im}[\hat{\Omega}(i\omega)] = \mp \omega/(\omega^2 + 1/\tau^2) \) is either negative or positive for all \( \omega > 0 \), depending on the polarity of the input signal (see also (22) pertaining to the reflection against a serial \( RL \) load).

If \( b_0 = 0 \) the late-time bounds (15) and (16) are not applicable. In such a case, one can express the TD counterpart of \( \hat{\Omega}(s)/s^2 \) via (4) and compare the result with

\[
\frac{2}{\pi} \int_{\omega=0}^{\infty} \operatorname{Re} \left[ \hat{\Omega}(i\omega) \right] d\omega = b_1,
\]

that is found upon integrating \( \hat{\Omega}(s)/s^2 \) around \( B \cup C^\infty \) (cf. (13)). This way leads for \( \operatorname{Re}[\hat{\Omega}(i\omega)] > 0 \) to

\[
\frac{\partial}{\partial t} \hat{\Omega}(t) \leq 2b_1 \hat{H}(t) \text{ for } b_0 = 0 \text{ and } b_1 > 0,
\]

Fig. 2. Complex \( s \)-plane with the pertaining integration contours.
and
\[- \partial_t \hat{\Omega}(t) \leq -2b_1 H(t) \text{ for } b_0 = 0 \text{ and } b_1 < 0. \tag{19}\]
provided that \(\text{Re}[\hat{\Omega}(\omega)] < 0\) for all \(\omega > 0\). This procedure can be further continued along similar lines of reasoning.

Finally note that the time-integrated version of the early-time bound (11), i.e.,
\[- \hat{\Omega}(t) \leq a_1 t^2 H(t), \]
represents a straight line that intersects with the corresponding late-time constant bound (15) at the “corner time” \(t_c = 2(b_0/a_1)\). If \(b_0 = 0\), the intersection of (18) with \(\hat{\Omega}(t) \leq a_1 t^2 H(t)/2\) occurs at \(t_c = 2(b_1/a_1)^{1/2}\).

IV. ILLUSTRATIVE EXAMPLES

The results presented in Sec. III will be demonstrated on illustrative problems.

A voltage pulse traveling along a transmission line of characteristic impedance, \(Z_0 > 0\), is at the terminal load of impedance, say \(\hat{Z}(s)\), reflected back to its source. The reflection can be quantified by the reflection coefficient that is to be associated with the response function, i.e.,
\[\hat{\Gamma}(s) = \frac{\hat{Z}(s) - Z_0}{\hat{Z}(s) + Z_0},\tag{20}\]
First, let us assume that the TL is loaded by a serial \(RL\) load, that is, \(\hat{Z}(s) = R + sL\). The corresponding reflection coefficient reads
\[\hat{\Gamma}(s) = 1 - \frac{(2Z_0/L)/(s + \tau^{-1})}{\hat{Z}(s) + Z_0},\tag{21}\]
where we introduced \(\tau = L/(R + Z_0)\), for brevity. Upon inspection with (3) we find \(\gamma_\infty = 1\), \(\gamma_0 = \gamma_n = 0\) and
\[\hat{\Omega}(s) = -(2Z_0/L)/(s + \tau^{-1}),\tag{22}\]
which can be transformed to TD at once as
\[\Omega(t) = -(2Z_0/L) \exp(-t/\tau) H(t).\tag{23}\]
Consequently, we have
\[- \partial_t^{-1} \Omega(t) = -(2Z_0/(R + Z_0)) \left[ 1 - \exp(-t/\tau) \right] H(t).\tag{24}\]
From the asymptotic behavior of \(\hat{\Omega}(s)\), i.e.,
\[\hat{\Omega}(s) = -(2Z_0/L)s^{-1} + o(s^{-1}) \text{ as } |s| \rightarrow \infty,\tag{25}\]
\[\hat{\Omega}(s) = -2Z_0/(R + Z_0) + o(1) \text{ as } s \rightarrow 0,\tag{26}\]
we get \(a_1 = -(2Z_0/L)\) and \(b_0 = -2Z_0/(R + Z_0)\) (see (7) and (8)). Hence, the use of (12) leads to the early-time bound (cf. (23))
\[- \Omega(t) \leq (2Z_0/L)H(t),\tag{27}\]
the time-integrated version of which reads
\[- \partial_t^{-1} \Omega(t) \leq (2Z_0/L) t H(t).\tag{28}\]
Furthermore, since \(b_0 < 0\), the corresponding late-time bound follows from (16) as (cf. (24))
\[- \partial_t^{-1} \Omega(t) \leq 2Z_0/(R + Z_0) H(t).\tag{29}\]
A graphical representation of the bounds is shown in Fig. 3a.

Figure 3b shows the TD bounds.

REFERENCES


