Directivity of Radiating Quantum Source Systems

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Abstract

We explore essential factors pertaining to the spatial directivity of quantum radiating source systems (QSSs), encompassing quantum antennas and quantum sensors. Our primary focus is on their capacity to control the emission of photons in specific spatial directions. We present a comprehensive definition of quantum directivity, drawing inspiration from Glauber's photon detection theory. This definition closely parallels the framework of analogous concepts in classical antenna theory. By conducting thorough conceptual and mathematical analysis, we address the challenge of characterizing the directive properties of a general QSS. Essentially, our approach presents a computational model that relies solely on the radiation field operator's density as input.
Abstract—We explore essential factors pertaining to the spatial directivity of quantum radiating source systems (QSSs), encompassing quantum antennas and quantum sensors. Our primary focus is on their capacity to control the emission of photons in specific spatial directions. We present a comprehensive definition of quantum directivity, drawing inspiration from Glauber’s photon detection theory. This definition closely parallels the framework of analogous concepts in classical antenna theory. By conducting thorough conceptual and mathematical analysis, we address the challenge of characterizing the directive properties of a general QSS. Essentially, our approach presents a computational model that relies solely on the radiation field operator’s density as input. Additionally, we establish theorems that illustrate the non-local nature of quantum radiating system directivity, manifesting a decay pattern proportional to $|\mathbf{x} - \mathbf{x}'|^{-4}$.

Index Terms—Quantum antennas, quantum technologies, open quantum systems, master equations, directivity.

I. INTRODUCTION

Directive quantum source systems (QSSs), for example quantum antennas and quantum sensors [1]–[7], are important elements of quantum information processing systems and are currently investigated for numerous applications ranging from sensing and tomography to nanoelectronics, quantum plasmonics, and quantum information transmission and reception [8]–[19]. For information transmission, a quantum source system is expected to be able to direct energy, i.e., quantized radiation particles in this case, toward specific directions, while suppressing transmission (and sometimes reception) along other directions. Since energy in the quantum domain is proportional to the number of particles [20], [21], a natural strategy for estimating the directive properties of source systems is relying on photon statistics [13], [18]. Another strategy is to treat the QSS as an open dynamical system [22]–[24], then compute the directivity using accurate stochastic dynamic method such as the GKSL master equation [25] popular in quantum dynamics and information theory [26]–[29]. This strategy is best suited for situations where noise and interference are essential as in quantum communications and quantum information processing in general [30], [31].

Currently there are exists two possible theoretical frameworks for thinking about the directivity of a generic quantum source system (QSS), where the latter is defined as an abstract source function controlling the spatio-temporal radiation properties of the system under consideration [11]. First, few words about the model itself. We are intentionally avoiding using the term ‘quantum current’ instead of ‘quantum source’ in order to keep the nature of how the system is excited or prepared as flexible as possible. In particular, it is not necessary that the quantum source is a (quantized) current comprised of moving electrons (or other charged particles or molecules.) A quantum source might be such a current, but also it can be some externally-controlled potential, an external laser or microwave electromagnetic field, a heat pump, or something completely different based on other physical phenomena. The only important formal requirements that a source function in a QSS must satisfy are that it is an externally-controlled structure (so information can be injected for transmission through radiation) and capable of modifying or effecting the directive properties of the emitted quantum radiation [21]. For instance, in classical antenna theory, the directivity of the antenna is a functional of the radiating current source (surface or volume current distributions) [32]–[34].

A key issue in defining the directivity of QSS is whether this should be done in space-time or momentum space. Since calculations in quantum theory are considerably easier in momentum space, the latter approach was advocated in some recent approaches [9], [23], [35]–[39], while space-time measures were propounded in few other studies [2], [11], [13], [18]. Both spatio-temporal approaches (using the Feynman propagator method) and momentum space were proposed in [11] within the context of relativistic quantum field theory. However, the main examples considered there (neutral Klein-Gordon field theory) involved massive particles with spin 0. It is of interest to reexamine theoretically this topic from the viewpoint of nonrelativistic quantum electrodynamics, where the main radiation consists of massless bosons (with spin 1), i.e., photons [40]. Certainly, it is the latter scenario that will engage our attention in the subsequent discussion. To streamline matters, we adopt a non-relativistic framework, eschewing the use of a manifestly covariant formalism. Instead, we employ the conventional Coulomb gauge second quantization, a customary approach within contemporary quantum optics research [21], [41]–[44].
One of the fundamental conceptual challenges concerning the directivity of electromagnetic source systems, whether they operate in the microwave or optical domain, revolves around the concept of photon \textit{localizability} in space [41]. In essence, this refers to the well-documented issue of the inadequate definitions of photon position or spatial density operators [45–47]. This topic is intertwined with the preceding question of whether to define directivity in momentum space or position space. This duality arises due to the Heisenberg uncertainty principle, resulting in the two depictions of the radiating quantum system’s directional characteristics being inherently complementary [48], [49]. In the forthcoming discussion, we will explore the proposed position-space directivity formula of a QSS, presented in the context of photon detection theory [43], and demonstrate its nonlocal nature by undertaking a rigorous mathematical framework for nonlocality in general QSSs. To analyze nonlocality, we derive the commutator of measurements executed at spatially distant points. This commutator serves as a pivotal tool to quantitatively assess the extent of nonlocality inherent in QSSs. We contend that this nonlocality stands as an essential—perhaps inescapable—attribute across all photonic QSSs. Despite this nonlocality, a certain continuity persists in the vector potential is given by

\begin{equation}
E^{(+)}(x) = \sum_{k_s} \hat{e}_{ks} i \sqrt{\frac{\hbar \omega_k}{2 \varepsilon_0 V}} a_{k_s} e^{i k_s x},
\end{equation}

where \( \hbar \) is the reduced Planck constant; \( k \) is the box quantization wavevector; \( V \) is the box volume; \( \varepsilon_0 \) the electric permittivity of free space; \( \omega_k = ck \) is the \( k \)th mode frequency with \( k := |k| \), while \( c \) is the vacuum speed of light; \( s \in \{0, 1\} \) labels the polarization (orthonormal) set \( \hat{e}_{ks} \); and \( a_{k_s} \) is the \( k \)th mode annihilation operator. The relation between the positive frequency components of the electric field and the vector potential is given by

\begin{equation}
E^{(+)}(x) = ic \left( -\nabla^2 \right)^{1/2} A^{(+)}(x).
\end{equation}

The following commutation relations are valid in vacuum:

\begin{equation}
\left[ A_i^{(+)}(x), A_j^{(-)}(x') \right] = \frac{\hbar}{2 \varepsilon_0 c} \left( -\nabla^2 \right)^{-1/2} \delta_{ij} (x - x'),
\end{equation}

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\left[ E_i^{(+)}(x), E_j^{(-)}(x') \right] = \frac{\hbar c}{2 \varepsilon_0} \left( -\nabla^2 \right)^{1/2} \delta_{ij} (x - x'),
\end{equation}

\begin{equation}
\left[ A_i^{(+)}(x), A_j^{(+)}(x') \right] = \left[ A_i^{(-)}(x), A_j^{(-)}(x') \right] = 0,
\end{equation}

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\left[ E_i^{(+)}(x), E_j^{(+)}(x') \right] = \left[ E_i^{(-)}(x), E_j^{(-)}(x') \right] = 0,
\end{equation}

where we note that these relations hold for all \( i, j = 1, 2, 3 \). The transverse Dirac delta function used above is defined by

\begin{equation}
\delta_{ij} (x - x') := \int \frac{d^3 k}{(2\pi)^3} \left[ \delta_{ij} - \frac{k_i k_j}{k^2} \right] e^{ik \cdot (x - x')}.
\end{equation}

Furthermore, it can be shown that [50]

\begin{equation}
\delta_{ij} (x - x') = \delta_{ij} \delta (x - x') + \frac{1}{4\pi} \nabla_i \nabla_j \frac{1}{|x - x'|}.
\end{equation}

Differential operators in the form \( (-\nabla^2)^{\pm 1/2} \) can best be defined by the formula

\begin{equation}
g \left( -\nabla^2 \right)^{\pm 1/2} F(x) := \int d^3 x' G_g (x - x') F(x'),
\end{equation}

where

\begin{equation}
G_g (x - x') := \int \frac{d^3 x}{(2\pi)^3} g \left( k^2 \right)^{\mp 1/2} e^{ik \cdot (x - x')}.
\end{equation}

This definition is valid for any measurable function \( g(\cdot) \). For example, the special case \( g \left( -\nabla^2 \right) \exp(ik \cdot x) = g \left( k^2 \right) \exp(ik \cdot x) \) is important. Note that the operator \( g \left( -\nabla^2 \right) \) can be moved to the right through the following useful integration by parts identify:

\begin{equation}
\int d^3 x F_1^*(x) g \left( -\nabla^2 \right) F_2(x) = \int d^3 x \left[ g \left( -\nabla^2 \right) F_1^*(x) \right] F_2(x),
\end{equation}

where \( F_1(x) \) and \( F_2(x) \) are two generic functions.

II. PRELIMINARY CONSIDERATIONS AND THE BASIC THEORETICAL MODEL

A. Field Quantization

The total electric vector potential and electric field operators, labeled by the position \( x \in \mathbb{R}^3 \), are expanded as \( A(x) = A^{(+)}(x) + A^{(-)}(x) \) and \( E(x) = E^{(+)}(x) + E^{(-)}(x) \), respectively. The superscripts (\( \pm \)) denotes the positive/negative frequency components, respectively, while \( A^{(-)}(x) = A^{(+)}(x) \), \( E^{(-)}(x) = E^{(+)}(x) \). In box mode quantization, the field operators expansions are

\begin{equation}
A^{(+)}(x) = \sum_{k_s} \hat{e}_{ks} \sqrt{\frac{\hbar \omega_k}{2 \varepsilon_0 V}} a_{k_s} e^{i k_s x},
\end{equation}

\begin{equation}
E^{(+)}(x) = \sum_{k_s} \hat{e}_{ks} i \sqrt{\frac{\hbar \omega_k}{2 \varepsilon_0 V}} a_{k_s} e^{i k_s x},
\end{equation}

...
B. Photon Detection Theory

In quantum optics, photodiodes and other devices are systematically deployed in order to perform local measurements of the quantized radiation fields [41], [51]. If the detector can be assumed to be localized at a single point \( \mathbf{x} \) in the QSS’s exterior domain \( D_{\text{ex}} \), then the signal output by the detector may be interpreted as a preliminary representation of the configuration of the quantum field at that spatial location [43]. For example, here we will focus on single counting statistics, which can be captured by the position-dependent signal [41]

\[
w(\mathbf{x}, t) := \text{Tr}\left\{ \rho \mathbf{E}(-)(\mathbf{x}, t) \cdot \mathbf{E}(+)(\mathbf{x}, t) \right\},
\]

where the time-dependent operators (interaction picture) are given by [42]

\[
A^{(+)}(\mathbf{x}, t) = \sum_{\mathbf{k}s} \hat{e}_{\mathbf{k}s} \sqrt{\frac{\hbar}{2\varepsilon_0 c \omega_k}} a_{\mathbf{k}s} e^{i\mathbf{k} \cdot \mathbf{x} - i\omega_k t},
\]

\[
E^{(+)}(\mathbf{x}, t) = \sum_{\mathbf{k}s} \hat{e}_{\mathbf{k}s} i \sqrt{\frac{\omega_k}{2\varepsilon_0 c}} a_{\mathbf{k}s} e^{i\mathbf{k} \cdot \mathbf{x} - i\omega_k t}.
\]

The density operator of the system, \( \rho \), is the reduced radiation density operator obtained in the steady state after tracing out all the matter degrees of freedom in the QSS operator \( \mathbf{J}(\mathbf{x}) \) (see Fig. 1) [24]. It represents the statistical mixture of all initial radiation field states entering into interaction with the photon detector [21].

In order to understand the physical meaning of the signal (13), we quickly recap quantum detection theory in quantum optics [41], [43]. Recall that using perturbation theory, one can compute the total probability of all atomic transitions from lower states to excited states after interaction with the illumination (measured) electromagnetic field using the following formula [41]:

\[
p(\mathbf{x}, t) = \sum_{ij} \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' D_{ij}(t_1 - t_2) G_{ij}^{(1)}(\mathbf{x}, t_1; \mathbf{x}, t_2),
\]

where the spatiotemporal field-field correlation tensor \( G_{ij}^{(1)} \) is defined by

\[
G_{ij}^{(1)}(\mathbf{x}, t_1; \mathbf{x}, t_2) := \text{Tr}\left\{ \rho \mathbf{E}(-)(\mathbf{x}, t_1) \mathbf{E}^{(+)}(\mathbf{x}, t_2) \right\}.
\]

The tensor \( D_{ij} : \mathbb{R} \rightarrow \mathbb{C}^3 \times \mathbb{C}^3 \) appearing in (16) represents the photodetector impulse response and is a unique characterization of each device, sometimes referred to as sensitivity function.\(^2\) The single photon counting rate \( w(\mathbf{x}, t) \) is then defined as the counting rate \( w^{(1)}(\mathbf{x}, t) := dp(\mathbf{x}, t)/dt \). Using (16), this becomes

\[
w^{(1)}(\mathbf{x}, t) = 2 \text{Re} \sum_{ij} \int_{t_0}^{t} dt' D_{ij}(t'-t) G_{ij}^{(1)}(\mathbf{x}, t'; \mathbf{x}, t). \tag{18}
\]

Consider now the special Fourier transform defined by

\[
X_{ij}(\omega, t) := \int_{t_0}^{t} dt' \mathcal{G}_{ij}^{(1)}(\mathbf{x}, t'; \mathbf{x}, t) \exp[i\omega(t-t')].
\]

Let its bandwidth be \( B_X \). We may then rewrite (18) as

\[
w^{(1)}(\mathbf{x}, t) = 2 \text{Re} \sum_{ij} \int \frac{d\omega}{2\pi} D_{ij}(\omega) X_{ij}(\omega, t), \tag{19}
\]

where \( D_{ij}(\omega) := \mathcal{F}_\omega \{ D_{ij}(\tau) \} \) is the Fourier transform of the detector impulse response (spectral device sensitivity function). In general, if the coherence time of the field is \( T_c \), then \( B_X \sim 1/T_c \) since the observation time \( \Delta t : = t - t_0 \) satisfies \( \Delta t \gg T_c \). Now let the device sensitivity bandwidth be \( B_D \), which is defined as the effective bandwidth of the function \( \| D_{ij}(\omega) \| \), where \( \| \| \) is a suitable matrix or tensor norm. Then we say our measurement of the radiation field is a broadband measurement if the condition \( B_D \gg B_X \) holds. In that case, it can be shown that (19) may be reduced to the considerably simpler relation [43]

\[
w^{(1)}(\mathbf{x}, t) = \sum_{ij} D_{ij} G_{ij}^{(1)}(\mathbf{x}, t; \mathbf{x}, t), \tag{20}
\]

where \( D_{ij} \) is now a constant device (sensitivity) tensor.\(^3\) We may also expand the hermitian tensor \( D_{ij} \) as a Hermitian matrix in the form \( \sum_{n=1}^{3} D_{n} \mathbf{u}_n \mathbf{u}_n^* \), where \( \mathbf{u}_n, n = 1, 2, 3 \), are orthonormal vectors. Using this form in (20), we obtain

\[
w^{(1)}(\mathbf{x}, t) = \sum_{n=1}^{3} D_{n} G_{n}^{(1)}(\mathbf{x}, t; \mathbf{x}, t), \tag{21}
\]

where

\[
G_{n}^{(1)}(\mathbf{x}, t; \mathbf{x}, t) := \text{Tr}\left\{ \rho \mathbf{u}_n \mathbf{E}(-)(\mathbf{x}, t) \mathbf{E}^{(+)}(\mathbf{x}, t) \cdot \mathbf{u}_n^* \right\}. \tag{22}
\]

The expression (21) proves to be highly advantageous for practical applications due to its ability to break down the counting rate into three distinct preferred spatial directions. This becomes particularly useful when considering scenarios involving the incorporation of a polarizer within the photodetection process. Such a setup allows for the precise measurement of quantum observables along specified spatial directions. In the special case when \( D_1 = D_2 = D_3 = D \), the expression (21) maybe reduced then to

\[
w^{(1)}(\mathbf{x}, t) = D \text{Tr}\left\{ \rho \mathbf{E}(-)(\mathbf{x}, t) \mathbf{E}^{(+)}(\mathbf{x}, t) \right\}. \tag{23}
\]

If we define \( w(\mathbf{x}, t) := w^{(1)}(\mathbf{x}, t)/D \), then the signal introduced in (13) is reproduced. In summary, \( w(\mathbf{x}, t) \) represents the rate of single-photon detection at time \( t \) when a localized point-like photodetector is positioned at \( \mathbf{x} \). The photodetector’s response exhibits isotropy in space, with sensitivity \( D \) along all directions. This rate is now normalized with respect to \( D \).

III. The Directivity Formula in Space and Time

The fundamental distinctions between considering quantum directivity in either frequency-momentum space or space-time are depicted in Figure 1. The momentum space approach

\(^2\)This tenor enjoys the important Hermitian symmetry condition \( D_{ij}(t) = D_{ij}^*(t) \).

\(^3\)Note that in deriving (20), the symmetry properties of \( D_{ij} \) and \( G_{ij}^{(1)} \) were used in addition to the one-sided sifting property of the Dirac delta function.
facilitates more direct and streamlined calculations, benefiting from the advantageous formulation of quantum field theory in Fourier or $k$-space representation [9], [23], [35]–[39]. However, within this framework, precise information about the exact photon positions cannot be obtained due to the inherent uncertainty relation of the Fourier transform [49]. Consequently, both classical [52] and quantum scenarios [11] suffer from the limitation of the lack of exact spatial localization of electromagnetic radiation. In practical terms, when seeking to assess the energy (number of emitted photons) guided by a given QSS along a specific spatial direction defined by spherical angles $\theta$ and $\varphi$, as shown in Figure 1(a), i.e., in momentum space, the resulting estimation becomes undetermined with respect to the radial distance or range from the QSS itself. Consequently, information regarding whether the directed energy predominantly falls within the deep near-field (NF) zone, intermediate NF zone, or far-field zone remains elusive. To obtain the missing position-space information, an alternative — in fact complementary — approach can be naturally proposed, aiming to assess the intensity of emitted radiation at specific spatial locations. Figure 1(b) illustrates the fundamental structure of this description. Unlike Figure 1(a), where angles $\theta$ and $\varphi$ are associated with the direction of momentum or wave vector $k$, in this case, they are instead correlated with the position vector $r$. Consequently, insights into the intensity of quantum radiation can be derived at any spatial position, encompassing both near-field and far-field zones.

The spacetime formulation introduced in [11] employed the concept of probability amplitude and Feynman propagators to directly assess the intensity of quantum radiation at specific points in spacetime, regardless of whether they are near or far from the source domain. Notably, this formulation demonstrated the convenient property of Lorentz covariance. In the subsequent discussion, we present an alternative approach that characterizes the directivity of the QSS by considering the averaged values of distinct quantum operators, which are readily measurable. Moreover, we present this tailored formulation specifically for photons, as opposed to the massive Klein-Gordon particles that were the central focus of the aforementioned study. Additionally, to enhance accessibility, we refrain from demonstrating manifest Lorentz covariance, as we utilize the widely-used Coulomb-gauge formulation.

However, a significant technical challenge arises when transitioning from the momentum space to the position space approach for analyzing directivity, particularly in the case of photons. This challenge stems from the inherent non-localizability of photons in position space. To grasp the essence of this issue, consider the scenario in momentum space where we aim to establish a spatial direction representation based on the $k$ vector direction. Since the momentum operator $p = \hbar k$ is well-defined for photons, this leads to relatively straightforward calculations, as demonstrated in works like [23], [24]. On the contrary, photons lack a well-defined position operator due to the absence of a general wavefunction for them [53]. Notably, a compelling theorem formulated by Newton and Wigner establishes that particles with zero mass cannot possess a well-behaved position operator defined in accordance with natural and reasonable expectations [45]. However, despite the implications of the Newton-Wigner theorem, extensive research spanning the past seven decades has been dedicated to addressing the challenge of photon localization from a foundational perspective. Despite these efforts, a universally accepted position-space-based photon wave function remains elusive, with proposals for such a formalism periodically emerging. For the context of our current paper, which primarily revolves around devising a practical and straightforward approach for characterizing the directivity of quantum antenna systems, we treat the non-localizability of photons as a foundational attribute that distinguishes quantum antennas from their classical counterparts.

Nevertheless, there exists an alternative avenue for addressing photon localization that avoids the requirement of photon position operators. This approach revolves around seeking suitable operators that exhibit two appealing attributes: 1) the ability to approximate photon localization behavior in space, and 2) measurability within the laboratory setting. In the realm of quantum optics, a prevalent option is to explore operators such as the number operator, with a particular focus on number densities [41]. In Fig. 2, we show a scenario where instead of asking about the directivity at a point in space, we attempt instead to estimate how many photons are concentrated in a specific volume in space. Let $V_{x_1}$ and $V_{x_2}$ be two arbitrary regions labeled by the positions $x_1 \in V_{x_1}$ and $x_2 \in V_{x_2}$, respectively. If the volumes of the regions $V_{x_1}$ and $V_{x_2}$ shrinks to zero while keeping $x_1$ and $x_2$ included, then we may consider the number operators associated with these limits as representation of the quantum radiation localization around $x_1$ and $x_2$, and hence make a comparison between the directivities of the QSS at different locations in space.

A natural alternative operator to consider is the number density operator associated with the number operator $N$ [41].

\begin{figure}
\centering
\includegraphics[width=\textwidth]{Figure_2.png}
\caption{The quantum source system (QSS). (a) Momentum space. (b) Spacetime.}
\end{figure}

\footnote{Note that this complication did not arise in the relativistic formulation presented in [11], where the calculations pertained to massive Klein-Gordon particles. Particles with non-zero mass can indeed be localized, as a well-defined position operator can be established in such cases.}

\footnote{Several notable works in this area include [41], [45]–[47], [54]–[60]. A particularly lucid conceptual exploration of the localization problem in quantum field theory can be found in [61]. A recent review (up to 2015) is given in [62].}
Recall that the number operator is given by [21]

\[
N = \frac{2\epsilon_0}{\hbar c} \int_{\mathbb{R}^3} d^3x \ E^(-)(x) \cdot (-\nabla^2)^{-1/2} E^+(x). \tag{24}
\]

For this approach to be viable, we need to determine suitable number operators associated with a given region \(V\), denoted as \(N(V)\). The most straightforward choice is to redefine the integration domain in (24) from \(\mathbb{R}^3\) to \(V\) [41], [63]. However, due to the non-positive definite integrand in (24), the average of this operator might be negative. Various proposals have been made to circumvent this issue by modifying the definition of the local number operator, such as the Mandel operator [41], [64]. In the subsequent discussion, we introduce a quantum antenna directivity definition based on an alternative approach to local number operators suggested in [65]. However, prior to that, we need to establish a general definition for what we consider an acceptable local operator.

**Definition 1.** (Local number operators) Consider a subset \(V \subset \mathbb{R}^3\). We define a local number operator \(N(V)\) as a quantum Hermitian operator with the property that its average value is always positive for any \(V \subset \mathbb{R}^3\). Moreover, this operator serves as an estimate for the number of photons localized within the region \(V\) modulo a linear operator.

**Remark 1.** As we have previously observed, the attempt to constrain the total number operator within finite subregions proves ineffective due to its inability to maintain positivity. For completeness, we summarize that notable fact by the following proposition: The operator \(N(V)\) defined by

\[
N(V) := \frac{2\epsilon_0}{\hbar c} \int_{x \in V} d^3x \ E^(-)(x) \cdot (-\nabla^2)^{-1/2} E^+(x) \tag{25}
\]

is not a local number operator.

Motivated by the approach proposed by Deutsch and Garrison to deal with quantum beam optics [65], we now propose the following form for the radiating QSS local number operator:

\[
N(V) = \frac{2\epsilon_0}{\hbar c} \int_{x \in V} d^3x \ E^-(x,t) \cdot E^+(x,t). \tag{26}
\]

This definition draws direct inspiration from Glauber’s theory of photon detection [43]. To see the rationale behind this choice, refer back to (23) and the corresponding discussion, where it was shown that the average of the operator (26) offers an estimation of the photon detection rate in the volume \(V\). The presence of a linear factor in this relation establishes a connection between the average value and the photon number, characterized by \(D^{-1}d/dt\) (cf. Sec. II-B). Additionally, it is evident that the integrand within (26) serves as a positive definite operator, ensuring that the average of \(N(V)\) remains positive for any \(V \subset \mathbb{R}^3\). Consequently, all conditions stipulated in Definition 1 are satisfied, culminating in the ensuing theorem:

**Theorem 1.** The operator \(N(V)\) defined through (26) is a local number operator.

\(\text{It can be shown that when } V = \mathbb{R}^3\), the average of \(N\) is always positive. Armed by this result, let us now proceed forward to employ the local photon detection operator (26) to establish a definition for the directivity of a radiating QSS. To begin, we will revisit the classical antenna directivity definition. Assume that the radiation intensity of a classical antenna at frequency \(\omega\) in the far-zone is described as \(U_{\text{rad}}(x;\omega) = |x|^2 U(\Omega;\omega)\). The overall time-harmonic radiated power across all solid angles \(\Omega\) is given by \(P_{\text{rad}}(\Omega;\omega) = |x|^2 \int_{\Omega} d\Omega U(\Omega;\omega)\). Within this context, the directivity is defined as follows [32], [66]:

\[
D(\Omega;\omega) := \frac{U(\Omega;\omega)}{P_{\text{rad}}(\Omega;\omega)/4\pi} = \frac{U(\Omega;\omega)}{\int_{4\pi} d\Omega U(\Omega;\omega)/4\pi}. \tag{27}
\]

It’s important to note that in the far-field zone, the directivity becomes an angular function and remains independent of the radial distance \(|x|\) from the origin [33]. Physically, the directivity provides a measure of the amount of energy the antenna can effectively radiate along the direction \(\Omega\), when compared to the radiation distribution of an ideal isotropic radiator characterized by a constant value of \(P_{\text{rad}}/4\pi\) [66].

Moving now to the quantum case, the local number density operator corresponding to (26) can be obtained with the help of the formula

\[
U_{\text{rad}}^q(x,t) := \lim_{\|V\| \to 0} \frac{N(V)}{\|V\|} = \frac{2\epsilon_0}{\hbar c} E^-(x,t) \cdot E^+(x,t), \tag{28}
\]

where \(\|V\|\) is the volume of the region \(V\). Inspired by the classical antenna directivity formula (27), we now define the quantum antenna directivity as follows.

**Definition 2.** (Quantum antenna directivity in space-time) Consider an QSS with Field operator \(E(x,t) = E^+(x,t) + E^-(x,t)\) and a radiation state density operator \(\rho\). Then the angular directivity at distance \(|x|\) from origin is defined by the formula

\[
D_q(|x|;\Omega,t) := \frac{\text{Tr} \{\rho E^-(x,t) \cdot E^+(x,t)\}}{4\pi \int_{4\pi} d\Omega \text{Tr} \{\rho E^-(x,t) \cdot E^+(x,t)\}}, \tag{29}
\]

whenever the denominator of (29) is not identically zero. If the denominator is zero, then the directivity of the QSS is not defined.

**Remark 2.** The expression \(\text{Tr} \{\rho E^-(x,t) \cdot E^+(x,t)\}\) can be evaluated in either the Heisenberg or interaction picture. In the Heisenberg picture, the field operators evolve according to the full Hamiltonian, while the state \(\rho\) remains time-independent. On the other hand, in the interaction picture, we evolve \(E(x,t)\) based on the free Hamiltonian as given by (15), while \(\rho\) becomes time-dependent through evolution under the interaction Hamiltonian. However, as photon detection theory is predominantly formulated in the interaction picture, we choose to employ the interaction picture for specific calculations.

**Remark 3.** When comparing (27) with (29), it becomes evident that our quantum directivity definition aligns most naturally with the time domain, whereas the classical antenna’s definition pertains to the frequency domain. Although not inherently conceptually significant, this disparity underscores certain fundamental structural distinctions between classical
and quantum radiation. Nevertheless, if we operate within the Heisenberg picture and apply the unimpeded or free evolution of the field operators as described in (15), the contrast between these two expressions loses its significance, given that (15) indeed exhibits a structure remarkably akin to the phasor representation found in classical quantities.

The rationale behind Definition 2 stems from the fact that the quantity featured in the numerator of (29) signifies the photon detection rate at the position \( \mathbf{x} \). Referring to (28), this essentially represents an average of a photon number density operator. Consequently, in order to formulate an ideal reference isotropic radiator at the position \([\mathbf{x}]\), we integrate the numerator across all solid angles \(4\pi\) and then normalize by dividing it by \(4\pi\) to yield the isotropic radiation density. It is worth noting that the constant factors \(2\epsilon_0/\hbar c\) appearing in (26) mutually cancels out in this process. Additional comments and analysis of this definition can be found below in Secs. V and VI. We next turn to the important question of localizability.

### IV. The Nonlocalizability of the Directivity of Electromagnetic Quantum Systems

We proceed to define the nonlocalizability of a local number operator. The most natural and commonly employed approach is to establish this property based on the commutativity of nonoverlapping support domains. This methodology is inspired by the fundamental principle in quantum physics—especially quantum field theory—that two experimentally measurable observables are considered independent if they commute [40].

**Definition 3.** (Nonlocalizability) Consider a given local operators. Let \( V \subset \mathbb{R}^3 \) and \( V' \subset \mathbb{R}^3 \) be two subsets. Then if the following condition holds

\[
\exists V, V' \subset \mathbb{R}^3 \mid V \cap V' = \emptyset \implies [N(V), N(V')] \neq 0,
\]

we say that \( N(V) \) is nonlocalizable. A localizable operator is a local operator that is never nonlocalizable.

Considering the widely recognized absence of a position-space wavefunction for photons [53] and the inadequate nature of several proposed alternatives aimed at approximating localizable number density operators\(^7\), it’s reasonable to expect that the quantum antenna directivity construction provided in Definition 2 would similarly result in nonlocal characteristics. As will be demonstrated below, this is indeed the case as the subsequent theorem illustrates.

**Theorem 2.** The operator (26) is nonlocalizable.

**Proof.** Without loss of generality, we work in the Schrödinger picture and set \(2\epsilon_0/\hbar c = 1\), allowing us to rewrite (26) as

\[
N(V) = \int_{\mathbf{x} \in V} d^3x \ \mathbf{E}^{(-)}(\mathbf{x}) \cdot \mathbf{E}^{(+)}(\mathbf{x}).
\]

To further simplify the presentation, we deploy the shorthand notation \( E_i := E_i^{(+)}(\mathbf{x}), E'_i := E_i^{(+)}(\mathbf{x}') \), where \( \mathbf{E}^{(+)} = \sum_{i=1}^{3} E_i \hat{n}_i. \) From the bi-linearity of the commutator, we get

\[
[N(V), N(V')] = \int_{V} d^3x \int_{V'} d^3x' \sum_{i=1}^{3} \sum_{j=1}^{3} \left[ E_i^{\dagger} E_i, E'_j^{\dagger} E'_j \right]
\]

Utilizing the standard commutator identity

\[
\]

we can evaluate

\[
\left[ E_i^{\dagger} E_i, E'_j^{\dagger} E'_j \right] = E_i^{\dagger} \left[ E_i, E'_j^{\dagger} \right] E'_j + E'_j \left[ E'_j, E_i^{\dagger} \right] E_i,
\]

where (7) was used to nullify the second and third terms in the RHS of (33). Next, with the help of (5), the commutators in the RHS of (34) can be evaluated, leading to

\[
\left[ E_i^{\dagger} E_i, E'_j^{\dagger} E'_j \right] = \left( E_i^{\dagger} E_j' - E_j' E_i^{\dagger} \right) (-\nabla^2)^{1/2} \delta_{ij}^+ (\mathbf{x} - \mathbf{x}'),
\]

where we have used the fact that \( \delta_{ij}^+ (\mathbf{x} - \mathbf{x}') = \delta_{ij}^+ (\mathbf{x}' - \mathbf{x}).\)

Substituting (35) into (32), we arrive at

\[
\left[ N(V), N(V') \right] = \sum_{ij=1}^{3} \int_{V} d^3x \int_{V'} d^3x' \left[ (-\nabla^2)^{1/2} \delta_{ij}^+ (\mathbf{x} - \mathbf{x}') \right] E_{ij}(\mathbf{x}, \mathbf{x}'),
\]

where the operator \( E_{ij}(\mathbf{x}, \mathbf{x}') \) is defined by

\[
E_{ij}(\mathbf{x}, \mathbf{x}') := E_i^{(-)}(\mathbf{x}) E_j^{(+)}(\mathbf{x}') - E_i^{(-)}(\mathbf{x}') E_j^{(+)}(\mathbf{x}).
\]

We will now show that when \( V \cap V' = \emptyset \), the commutator \([N(V), N(V')]\) does not vanish. First, we note that in quantum field theory the dependence of field operators on space generally comes in the form of linear combinations of the form \(\hat{E}_{ij} \hat{E}_{ij}(\mathbf{x}, \mathbf{x}')\), where \(\hat{E}_{ij}\) is an operator with no spatial dependence and \(\hat{E}_{ij}(\mathbf{x}, \mathbf{x}')\) is a complex-valued spatial function [40], [67]. Consequently, we will replace \( E_{ij}(\mathbf{x}, \mathbf{x}') \) by \(\hat{E}_{ij} \hat{E}_{ij}(\mathbf{x}, \mathbf{x}')\) even though the former is in general a linear combination of terms with the latter’s form. For example, it is clear from field expansions such as (2) that the operator \(\hat{E}_{ij}(\mathbf{x}, \mathbf{x}')\) may be written as a sum of space-independent operators of the form \(a_{\alpha} a_{\alpha} \), \(aa_{\alpha} \), and so on, multiplied by exponential functions of the form \(\exp[ik \cdot (\mathbf{x} - \mathbf{x}')]\), etc. Based on this understanding, using (35), and with a slight abuse of

\(^7\)Cf. Sec. III and references cited therein.

\(^8\)Recall that as in (5), the operator \((-\nabla^2)^{1/2}\) is applied to the \(x\)-function.
notation, we find the following:

\[ [N(V), N(V')] \]

\[
\sim \sum_{i,j=1}^{3} \int_{V} d^3x \int_{V'} d^3x' \left[ (-\nabla^2)^{1/2} \delta_{ij}^L(x - x') \right] \hat{E}_{ij} \hat{E}_{ij}(x, x')
\]

\[
= \sum_{i,j=1}^{3} \hat{E}_{ij} \int_{V} d^3x' \int_{V} d^3x \hat{E}_{ij}(x, x') \left( (-\nabla^2)^{1/2} \delta_{ij}^L(x - x') \right)
\]

\[
= \sum_{i,j=1}^{3} \hat{E}_{ij} \int_{V} d^3x' \int_{V} d^3x \delta_{ij} \cdot \delta(x-x') \cdot \hat{E}_{ij}(x, x')
\]

\[
= \frac{3}{4\pi} \sum_{i,j=1}^{3} \int_{V} d^3x \int_{V} d^3x' \left[ \left( -\nabla^2 \right)^{1/2} \nabla_i \nabla_j \frac{1}{|x-x'|} \hat{E}_{ij}(x, x') \right]
\]

where the function \( \hat{E}' \) is defined by \( \hat{E}'_{ij}(x, x') := (-\nabla^2)^{1/2} \hat{E}_{ij}(x, x') \). In the second equality in (38) we interchanged the order of the integrals. The third and last equalities in (38) were derived by applying the integration by parts formula (12) to the \(|x-x'|\)-integral and the inverse delta function expansion in space (9), respectively.

We finally note that the first term in the last equality in (38) vanishes since by assumption the volumes \( V \) and \( V' \) are nonoverlapping. On the other hand, the second term containing the second-order derivative \( \nabla_i \nabla_j \frac{1}{|x-x'|} \) does not vanish in fact, it decays slowly, with decay rate of \( R^{-3} \), \( R := |x-x'| \), or faster. Therefore, the commutator \( [N(V), N(V')] \) does not vanish on some nonzero function \( F(x) \).

We can improve on Theorem 2 by providing an explicit calculation of the decay rate of the local field density operator commutators.

**Theorem 3.** Let \( V, V' \subset \mathbb{R}^3 \) be disjoint regions in space. Then the Commutator \( [N(V), N(V')] \) decays as \( |x-x'|^{-4} \).

**Proof.** The commutator \( [N(V), N(V')] \) can be expanded as

\[ [N(V), N(V')] \]

\[
\sim \sum_{i,j=1}^{3} \int_{V} d^3x \int_{V'} d^3x' \left[ (-\nabla^2)^{1/2} \delta_{ij}^L(x - x') \right] \hat{E}_{ij} \hat{E}_{ij}(x, x')
\]

\[
= \frac{3}{4\pi} \sum_{i,j=1}^{3} \int_{V} d^3x \int_{V} d^3x' \left[ \left( -\nabla^2 \right)^{1/2} \nabla_i \nabla_j \frac{1}{|x-x'|} \hat{E}_{ij}(x, x') \right]
\]

Here, we have employed (9) and subsequently applied the outcome derived in (38). In the latter, we determined that the initial term on the right-hand side of (9) holds no impact on the commutator, particularly when \( V \) and \( V' \) remain separate entities.

We now explicitly evaluate the integrand of (39) as follows:

\[
(-\nabla^2)^{1/2} \nabla_i \nabla_j \frac{1}{|x-x'|}
\]

\[
= \int \frac{d^3k}{(2\pi)^3} k \cdot \nabla_x \left\{ \frac{1}{|x-x'|} \right\} e^{ik \cdot x}
\]

\[
= \int \frac{d^3k}{(2\pi)^3} k \cdot k \cdot \nabla_x \left\{ \frac{1}{|x|} \right\} e^{ik \cdot x} e^{-ik \cdot x'}
\]

\[
= -4\pi \int \frac{d^3k}{(2\pi)^3} k \cdot k \cdot \nabla_x \left\{ \frac{1}{|x|} \right\} e^{ik \cdot x} e^{-ik \cdot x'}
\]

\[
= 4\pi \nabla_i \nabla_j \int \frac{d^3k}{(2\pi)^3} \frac{k}{|k|^2} e^{ik \cdot (x-x')}
\]

\[
= 2\pi \nabla_i \nabla_j \frac{1}{|x-x'|^2}
\]

For deriving the first equality, we utilized (10) and (11) after expanding them in the Fourier domain. For the second equality of (40) we used \( F \{ \nabla_x f(x) \} = ik \cdot F \{ f(x) \} \). For the third equality we applied the shift property of the Fourier transform. For the fourth equality, the Fourier transform pair \( F_x \{ \frac{1}{|x-x'|} \} = 4\pi/k^2 \) was used. In the fifth equality we again applied \( F_x \{ \nabla_x f(k) \} = ik \cdot F \{ f(k) \} \). Finally, to obtain the last equality we employed the Fourier transform pair \( F^{-1} \{ k^{-1} \} = 2\pi^2/|k|^2 \). Based on equations (39) and (40), we can infer that for \( x \neq x' \), the commutator \( [N(V), N(V')] \) asymptotically decays following a \( |x-x'|^{-4} \) trend.

\[ \square \]

**V. Discussion and Examples**

**A. The Single-Mode QSS**

Let’s initially consider the highly specialized case where only one mode is emitted by the QSS. Its steady state density operator just prior to interaction with the detector is denoted as \( \rho \). From equation (15), the positive-frequency component is given by \( \mathbb{E}(+)(x, t) = e^{i\omega t}/\hbar \omega k/2\pi e^{i\omega t} V a \exp[i(k \cdot x - \omega t)] \), where \( e \) is the unit polarization vector of the single mode radiation field whose annihilation operator is \( a \). A straightforward calculation of the directivity based on (29) yields \( \mathcal{D}_q(|x|, \Omega) = \text{Tr} \{ \rho a \cdot a \} / \frac{1}{4\pi} \int_{4\pi} \text{d}\Omega \text{Tr} \{ \rho a \cdot a \} = 1 \), which is independent of position \( x \) and time \( t \). In simpler terms, irrespective of the specific values of \( k \) and the mode’s polarization \( s \), as prescribed in Definition 2, the single-mode QSS always demonstrates perfect isotropy, with the same value of directivity not only along all directions, but also at various radial positions.\(^{11}\) This outcome is indeed expected, as depicted in Figure 1(b), where the spacetime directivity
holds a complementary relationship with the momentum space directivity depicted in Figure 1(a). In the context of the latter scenario, the directivity of a single mode naturally peaks in the directions aligned with the mode’s wavevector $k$. This contrast accentuates the inverse nature of spacetime and momentum space directivities. For a single-mode photon, the momentum is sharply defined, leading to completely indeterminate photon position. Consequently, the radiator’s directivity in space becomes perfectly isotropic. Therefore, from the viewpoint of arriving at distinctive radiation properties, single-mode QSSs are trivial and not interesting. The minimum complexity necessary to achieve directive behavior is that of a two-mode system, which we will explore further next.

### B. The Two-Mode QSS

Let us consider now the next level of complexity when two modes are radiated instead of one. Let their corresponding polarizations and wave numbers be $\hat{e}_{1/2}$ and $k_{1/2}$, respectively. The field is given by the linear sum $E^{(+)}(x,t) = \hat{e}_1 i \sqrt{\hbar \omega_{k_1}/2 \varepsilon_0 V_1} \exp(ik_1 \cdot x - i\omega_{k_1} t) + \hat{e}_2 i \sqrt{\hbar \omega_{k_2}/2 \varepsilon_0 V_2} \exp(ik_2 \cdot x - i\omega_{k_2} t)$, where $\hat{e}_{1/2}$ and $\alpha_{1/2}$ are the complex orthonormal polarization vectors and annihilation operators of the first and second modes, respectively, and we used the dispersion relation $\omega = ck$; $k := |k|$. To evaluate the quantum directivity of this two-mode system, we first compute the numerator of (29) as follows:

$$\text{Tr} \{ \rho E^{-}(x,t) \cdot E^{+}(x,t) \}$$

$$= \frac{\hbar c k_1}{2 \varepsilon_0 V_1} \text{Tr} \{ \rho a_1^\dagger a_1 \} + \frac{\hbar c k_2}{2 \varepsilon_0 V_2} \text{Tr} \{ \rho a_2^\dagger a_2 \}$$

$$+ \hat{e}_1^* \cdot \hat{e}_2 \frac{\hbar c \sqrt{\varepsilon_1 k_1 \varepsilon_2 k_2}}{2 \varepsilon_0 V} \text{Tr} \{ \rho a_1^\dagger a_2 \} e^{i(k_2 - k_1) \cdot x - i\omega_{k_2 - k_1} t}$$

$$+ \hat{e}_1^* \cdot \hat{e}_2 \frac{\hbar c \sqrt{\varepsilon_1 k_1 \varepsilon_2 k_2}}{2 \varepsilon_0 V} \text{Tr} \{ \rho a_2^\dagger a_1 \} e^{-i(k_2 - k_1) \cdot x + i\omega_{k_2 - k_1} t}.$$  \hspace{1cm} (41)

Next, the position vector is expanded in spherical coordinates as

$$x = |x| \hat{x}(\Omega), \quad \hat{x}(\Omega) = \hat{x}_1 \cos \varphi \sin \theta + \hat{x}_2 \sin \varphi \sin \theta + \hat{x}_3 \cos \theta.$$  \hspace{1cm} (42)

Substituting (42) into (41) and integrating over all solid angles $\Omega$, the ideal isotropic radiator reference, the denominator of (29), becomes

$$\frac{1}{2 \varepsilon_0 V} \Gamma(|x|)$$

$$= k_1 \text{Tr} \{ \rho a_1^\dagger a_1 \} + k_2 \text{Tr} \{ \rho a_2^\dagger a_2 \}$$

$$+ \frac{1}{4\pi} \sqrt{k_1 k_2 \alpha_{12}} e^{-i(k_2 - k_1) t} \int d\Omega e^{i(k_2 - k_1) \cdot |x| \cdot \hat{x}(\Omega)}$$

$$+ \frac{1}{4\pi} \sqrt{k_1 k_2 \alpha_{21}} e^{i(k_2 - k_1) t} \int d\Omega e^{-i(k_2 - k_1) \cdot |x| \cdot \hat{x}(\Omega)}$$

$$= k_1 \alpha_{11} + k_2 \alpha_{22}$$

$$+ 2 \sqrt{k_1 k_2} \text{Re} \left\{ \alpha_{12} e^{-i(k_2 - k_1) t} \right\} \frac{\sin(|x||k_1 - k_2|)}{|x||k_1 - k_2|}. $$  \hspace{1cm} (43)

In this context, the c-numbers $\alpha_{ij}$ are defined by $\alpha_{ij} := \hat{e}_i^* \cdot \hat{e}_j \text{Tr} \{ \rho a_i^\dagger a_j \}$ and are real for $i = j$, where in the latter case they indicate the number of photons in the $i$th mode within the radiation state $\rho$. For obtaining the last equality in (43) we used the identity $[71]

$$\int_{4\pi} d\Omega_k e^{i\mathbf{k} \cdot \mathbf{x}} = \int_{4\pi} d\Omega_\infty e^{i\mathbf{k} \cdot \mathbf{x}} = 4\pi \frac{\sin(|x||k|)}{|x||k|}. $$ \hspace{1cm} (44)

Finally, putting (41) and (43) into (29) yields the following formula for the radiating two-mode quantum system:

$$D_q(|x|, \Omega, t) =$$

$$\sum_{i=1}^{2} k_i \alpha_{ii} + 2(k_1 k_2)^{1/2} \text{Re} \left\{ \alpha_{12} e^{-i(k_2 - k_1) t} \right\} \frac{\sin(|x||k_1 - k_2|)}{|x||k_1 - k_2|}.$$  \hspace{1cm} (45)

Writing in the numerator of (45), we have made use of the symmetry relation $\alpha_{12} = \alpha_{21}^*$, which follows from the fact that $\rho$ is a hermitian operator.

A couple of observations on the two-mode QSS’s directivity formula (45) are in order. First, we note that for $k_1 = k_2$, $D_q(|x|, \Omega, t)$ becomes independent of space and time, i.e., perfectly isotropic with directivity equal to 1, which is self consistent with what we have already established in Sec. V-A on single-mode QSSs. Second, for $k_1 \neq k_2$ but $k_1 = k_2$, the directivity is a function of $\mathbf{x}$ but is independent of time. Third, from the limit $\lim_{|x| \to \infty} \sin(|x||k_1 - k_2|)/|x||k_1 - k_2| = 0$, we can deduce the maximum attainable directivity in the far-field zone:

$$\max_{\Omega \in 4\pi, t \in \mathbb{R}} \lim_{|x| \to \infty} D_q(|x|, \Omega, t) = 1 + \frac{2 \sqrt{k_1 k_2} |\alpha_{12}|}{\sum_{i=1}^{2} k_i \alpha_{ii}},$$  \hspace{1cm} (46)

which is valid when $|x| \gg \max\{2\pi/k_1, 2\pi/k_2\}$. Without loss of generality we may assume that the real numbers $\alpha_{11}$, $\alpha_{22}$, and $|\alpha_{12}|$ are in the interval $[0, 1]$. For example, in the special case when $\alpha_{11} = \alpha_{22}$, the maximal directivity becomes $1 + (|\alpha_{12}| / (\alpha_{11}) \sqrt{k_1 k_2} / (k_1 + k_2)/2)$. In other words, the two-mode directivity is maximized by those modes chosen such that the ratio between the geometric means of the wavenumbers, $\sqrt{k_1 k_2}$, to the arithmetic mean, $(k_1 + k_2)/2$, is maximized. Usually, this optimization problem must be solved under some constraints on the allowable values of $k_1$ and $k_2$.

### C. The Continuous-Mode QSS

As explained in Section V-B, the foundational arrangement requisite for crafting a directive radiation pattern is a two-mode configuration. To ascend the complexity ladder, we consider the arbitrary general scenario wherein the count of modes is unrestricted and can be substantial or even infinite. In this context, pairs of modes are incrementally incorporated, and their addition is orchestrated by the global quantum state $\rho$, which assumes the role of a guiding coordinator.

To complete this presentation, we now give the general treatment for continuous-mode radiation. Using the complete
general field operator expansion (15), the numerator of (29) can be put in the the following form:

\[
\text{Tr} \left\{ \rho \mathbf{E}^{(-)}(x, t) \cdot \mathbf{E}^{(+)}(x, t) \right\} = \frac{\hbar c}{2\pi V} \sum_{k} \sum_{k'} Q(k, k') e^{i(k' - k) \cdot x - i\epsilon(k' - k)t},
\]

where

\[
Q(k, k') := \sum_{ss'} \left( \hat{e}_{k_s}^* \cdot \hat{e}_{k_{s'}} \right) \sqrt{k_1 k_2} \text{Tr} \left\{ \rho a_{k_s} a_{k_{s'}} \right\}.
\]

The expression (47) can be reformulated by defining the \( k \)-indexed array \( \mathbf{g}[k; x, t] := [\exp(ik \cdot x - i\epsilon k t)] \), after which (47) can be rewritten as

\[
\text{Tr} \left\{ \rho \mathbf{E}^{(-)}(x, t) \cdot \mathbf{E}^{(+)}(x, t) \right\} = \frac{\hbar c}{2\pi V} \sum_{kk'} \mathbf{g}^*(k; x, t)Q(k, k')g(k'; x, t).
\]

It is straightforward to show that since \( \rho \) is a Hermitian operator then the matrix \( Q \) satisfies the symmetry condition \( Q(k, k') = Q^*(k', k) \). Therefore, \( Q(k, k') \) is a Hermitian matrix and (49) is a Hermitian form.

Next, the denominator of (29) may be evaluated using the same method used in Sec. V-B, yielding:

\[
\frac{1}{4\pi} \int d\Omega \text{Tr} \left\{ \rho \mathbf{E}^{(-)}(x, t) \cdot \mathbf{E}^{(+)}(x, t) \right\} = \frac{\hbar c}{2\pi V} \sum_{k} \sum_{k'} Q(k, k') e^{-i\epsilon(k' - k)t} \frac{\sin(|x||k - k'|)}{|x||k - k'|},
\]

where the identity (44) was utilized. Consequently inserting (47) and (50) into (29), the final quantum directivity expression becomes

\[
D_q(|x|, \Omega, t) = \frac{\sum_{kk'} \mathbf{g}^*(k; x, t)Q(k, k')g(k'; x, t)}{\sum_{kk'} Q(k, k') \frac{\sin(|x||k - k'|)}{|x||k - k'|} e^{-i\epsilon(k' - k)t}}, \quad (50)
\]

We can also take the limit of \( V \to \infty \), transitioning to continuous-mode representation via the recipe \( \int_{\mathbb{R}^3} dk \frac{d^2k'}{(2\pi)^3} \leftrightarrow (1/V) \sum_k \), leading to

\[
D_q(|x|, \Omega, t) = \frac{\int d^2k \int d^2k' \mathbf{g}^*(k; x, t)Q(k, k')g(k'; x, t)}{\int d^2k \int d^2k' \frac{\sin(|x||k - k'|)}{|x||k - k'|} e^{-i\epsilon(k' - k)t}}. \quad (51)
\]

The discrete expression (50) has the advantage of being amenable to matrix analysis results pertinent to optimization [72], which is helpful for finding maximum (and minimum) directivity values [73]. On the other hand, the continuous form (51) can be useful for deriving closed form expressions since integrals are easier to evaluate than sums.

**VI. CONCLUDING REMARKS**

An advantage of the directivity theory expounded within this paper is its capability to furnish a precise and rigorous characterization of a QSS. Indeed, a QSS finds its explicit definition as the minimal set of data required for the computation of the directivity (29). In this context, we delineate a QSS as a triad encompassing: 1) a collection of wave vectors \( \mathbf{k} \), each associated with 2) a pair of polarization degrees, and 3) a Hermitian matrix \( Q(k, k') \) established by (48). Hence, the quantum state \( \rho \) becomes assimilated within the Hermitian matrix \( Q(k, k') \). Notably, \( \rho \) may exhibit time dependence akin to the interaction picture, but remains devoid of positional dependency. Hence, the quantum state of the QSS emerges as a global measure applicable across the entirety of the system. Conversely, when a localized insight into the system’s behavior at specific positions is sought, the requisite information can be derived from \( Q \) through the utilization of the formulas (50) or (51).

Our investigation has revealed that the quantum directivity formula (29), rooted in photon detection theory, exhibits a nonlocal nature. To accommodate this characteristic, we have introduced a rigorous framework for characterizing nonlocalizable density operators, thereby enabling the exploration of how photons emitted by QSSs manifest localization in space. Our inquiry demonstrates that, akin to analogous constructs, the full localization of photons remains elusive. Nevertheless, explicit computations have demonstrated a noteworthy pattern: for two measurements conducted at spatial points \( \mathbf{x} \) and \( \mathbf{x}' \), the commutator of these measurements diminishes as \( |\mathbf{x} - \mathbf{x}'|^2 \). Hence, in quantum information processing endeavors necessitating the simultaneous measurement of photonic fields at spatially disparate locales — examples being quantum Multiple-Input Multiple-Output (MIMO) communication systems and quantum multi-access links [74]–[78] — it becomes imperative to account for the precise commutator value. In practical scenarios such as the design of optimal receivers for quantum MIMO channels, the commutator of photon counting rates emerges as an external constraint pivotal to the core of digital receiver design optimization [76]. Conversely, when the spatial separation between measurement sites expands considerably, one has the discretion to overlook the nonlocal attributes of the QSS. In such instances, our theory furnishes quantitative mechanisms to assess the error incurred by this decision.

Illustrative examples illustrating the proposed quantum directivity formula have unveiled a series of noteworthy structural features. It has been observed that single-mode QSSs inherently manifest perfect isotropy across space. Introducing a higher degree of complexity, the two-mode configuration surfaces as the minimal structure capable of rendering sophisticated directivity characteristics. Subsequently, we progressed to encompass the general scenario, accommodating an arbitrary number of modes. Detailed expressions were derived, spanning both discrete and continuous formulations. Evidently, the presented theory facilitates efficient computations of maximal directivity, particularly when the number of modes remains modest. However, in instances involving a substantial count of modes, we advocate recourse to the theorems governing the maximization of Hermitian forms within matrix analysis. This recommendation stems from the realization that our directivity formulation can be cast in such forms, offering a strategic approach for large-scale mode scenarios.
REFERENCES


