Padâ© a coustoporoelasticity for 3D wave propagation in prestressed porous rocks with inelastic deformations

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Abstract
In this paper, we extend the velocity-stress acoustoporoelastic formulation from 2D to 3D, and Padâ© acoustoelasticity to Padâ© acoustoporoelasticity. Applications to experimental data with porous rocks differentiate the acoustoelasticity, acoustoporoelasticity, and Padâ© acoustoporoelasticity in accuracy. The SSG-FD numerical method was used to simulate the propagation of elastic waves in both 3D acoustoporoelastic and Padâ© acoustoporoelastic media. By comparing the theoretical and calculated wave velocities, the presented numerical scheme is verified using plane-wave theoretical solutions.
Padé acoustoporoelasticity for 3D wave propagation in prestressed porous rocks with inelastic deformations

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ABSTRACT

Insights into wave propagation in prestressed porous rocks have great interesting in geophysical applications, such as remote monitoring in-situ stresses. Wave-induced small dynamic fields superposed onto statically deformed objects can be addressed traditionally by acoustoporoelastic theory that extends the classical acoustoelasticity of solids to porous media by incorporating Biot’s theory. Stress-induced deformations in porous rocks are of a progressively scaling feature with increasing prestress, undergoing linear elastic, hyperelastic (nonlinearly elastic), and inelastic deformations prior to mechanical failure. Conventional acoustoporoelastic theory is based on the Taylor expansion for the cubic strain-energy function with linear strains under finite-magnitude prestress. The theory with third-order elastic constants only accounts for stress-induced hyperelasticity, insufficient to handle inelastic deformations with nonlinear strains of compliant microstructures. We replace the Taylor expansion by the Padé approximation to the strain energy function, leading to Padé acoustoporoelastic equations for inelastic deformations under large-magnitude prestress. Theoretical results from plane-wave analyses agree well with the laboratory measurements of fluid-saturated Portland sandstones under confining and uniaxial prestresses. Finite-difference simulations are implemented to solve the first-order velocity-stress formulation of Padé acoustoporoelastic equations for elastic wave propagation in prestressed porous media under isotropic (confining) and anisotropic (uniaxial and pure-shear) prestresses. The resulting wavefield snapshots show the propagation of fast-P and slow-P and S waves in acoustoporoelastic media, illustrating stress-induced velocity orthotropies, strongly related to the direction of prestress. Comparisons with conventional acoustoporoelastic simulations provide a framework to estimate stress-
induced inelastic strains from seismic responses in velocity and anisotropy.

INTRODUCTION

Acoustic velocities in porous rocks are sensitive to prestress. The nonlinear stress dependence of elastic properties is strongly related to finite deformations in compliant pores under large-magnitude prestress (Walsh, 1965; Nur and Simmons, 1969). This issue has been extensively addressed based on various microcrack models by assuming closure of microcracks or compression of grain contacts (e.g., Cheng and Toksöz, 1979; Zimmerman et al., 1986; Johnson and McCall, 1994; Shapiro, 2003). As stressed by Winkler and McGowan (2004), a more general description for stress-induced velocity variations is based on the theory of acoustoelasticity with the Piola-Kirchhoff stress equations of motion. It extends the classical theory of elasticity to stress-induced hyperelasticity by superposing wave-induced small disturbances onto elastically deforming objects. Because of great interesting in seismic exploration of prestressed oil/gas reservoirs, acoustoelastic methods have been introduced into the field of geophysics to understand the stress-induced variations in anisotropy (Sayers, 1988), velocity (Winkler and Liu, 1996), and scattering attenuation (Guo et al., 2009), and further applied to well-logging data for the remote monitoring of in-situ stresses (Sinha and Kostek, 1996; Cao et al., 2004; Chen et al., 2009; Lei et al., 2012).

Stress-induced deformations in porous rocks typically present as a progressively scaling feature with increasing prestress, resulting in linear elastic, hyperelastic (nonlinearly elastic), and inelastic deformations prior to mechanical failure. Traditional 2oeCs (bulk and shear moduli) in the realm of linear elasticity are insufficient to describe stress-induced nonlinear deformations. The theory of acoustoelasticity (Pao et al., 1985) takes into account the stress-induced elastic nonlinearity by incorporating a cubic strain-energy function with linear strains (Sinha and Plona, 2001) under finite-magnitude prestress. This limited nonlinearity can be described by effective third-order elastic constants (3oeCs), sometimes called $A$, $B$, and $C$ (Green, 1973). The acoustoelasticity in solids has been extended as acoustoporoelasticity for wave propagation in porous rocks by incorporating Biot’s theory (e.g., Grinfeld and Norris, 1996; Donskoy et al., 1997; Guo, 2008). Ba et al. (2013) extend the applicability of the acoustoporoelastic approach to fluid-saturated porous media, but the predicted velocities do not
agree with the measurements at high pore pressures, where the velocity values show an exponential
behavior as a function of differential pressure, possibly explained by the closure of microcracks. Based
on the experimental measurements of Winkler and McGowan (2004), the acoustoporoelastic 3oeCs can
interpret stress-induced velocity changes for dry rocks, but fail to describe experiments for water-
saturated rocks.

Sinha and Plona (2001) approach inelastic deformations by incorporating plastic strains into
acoustoelastic equations. The dual-porosity model (Shapiro, 2003) has been incorporated into the
classical acoustoelastic 2oeCs (Fu and Fu, 2018) and 3oeCs (Fu et al., 2020; Ling et al., 2021) for
describing the nonlinear stress dependency of compliant pores. The dual-porosity effective elastic
constants explain the strong sensitivity of compliant pores to effective stresses and thus accurately
describe the nonlinearity of fluid-saturated rocks based on the model-dependent methods by assuming
closure of microcracks. It should be stressed that both the classical theories of acoustoelasticity and
acoustoporoelasticity are based on the Taylor expansion of the strain energy function. Replacing the
Taylor expansion, Fu and Fu (2017) use the Padé expansion to approximate the 2D strain energy function
for large-amplitude strains under large prestress. The resulting Padé acoustoelasticity can predict the
strong inelasticity close to mechanical failure where the Padé coefficients, $a$ and $b$, characterize the
microstructural dependence of elastic constants. In this study, we will extend the Padé expansion from
2D acoustoelasticity to 3D acoustoporoelasticity.

Great progress has been made in both the theoretical and experimental aspects of acoustoelasticity
and acoustoporoelasticity, but dedicated numerical simulations are rarely reported. Yang et al. (2022a,
2022b, 2023) perform FD numerical simulations for elastic wave propagation in acoustoelastic and
acoustoporoelastic media under confining, uniaxial, and pure shear prestresses, which demonstrates the
significant impact of prestressing conditions on seismic responses in velocity and anisotropy. In this
study, we apply the standard staggered-grid finite-difference (SSG-FD) to 3D Padé acoustoelasticity and
acoustoporoelasticity equations for elastic wave propagation in fluid-saturated porous media subject to
confining, uniaxial, and pure shear prestresses. It provides further insights into the sensitivity of velocity
and anisotropy to large prestress.

We first briefly introduce acoustoelasticity (Pao et al., 1985) and Padé acoustoelasticity (Fu and Fu,
Yang et al., 2022b) to 3D cases by incorporating the 3D acoustoelastic 3oeCs into the velocity-stress formulations of 3D poroelasticity equations. Velocity-stress formulation of 3D Padé acoustoporoelasticity equations is formulated by incorporating the 3D acoustoelastic 3oeCs into the velocity-stress formulations of 3D poroelasticity equations. Applications to experimental data with porous rocks differentiate the acoustoelasticity, acoustoporoelasticity, and Padé acoustoporoelasticity in accuracy. The SSG-FD numerical method is based on eighth-order (for the space derivatives) and second-order (for the time derivatives) FD approximations. The presented numerical scheme is validated using the plane-wave theoretical solution through the comparison of theoretical and calculated wave velocities. Two states of loading prestress, isotropic (confining) and anisotropic (uniaxial and pure shear) prestress, are investigated to model the propagation of elastic waves in both 3D acoustoporoelastic and Padé acoustoporoelastic media, which illustrates the prestress-induced anisotropy (orthotropy) of velocities.

THEORETICAL BACKGROUND

3D velocity-stress acoustoporoelasticity equations

The first-order velocity-stress form of transversely isotropic poroelasticity equations for the 3D case can be written as (Carcione, 2001)

\[
\begin{align*}
\rho \dot{v}_1 &= \rho_m (\tau_{11,1} + \tau_{12,2} + \tau_{13,3}) + \rho_f p_1 + \rho_f b_1 q_1 \\
\rho \dot{v}_2 &= \rho_m (\tau_{12,1} + \tau_{22,2} + \tau_{23,3}) + \rho_f p_2 + \rho_f b_2 q_2 \\
\rho \dot{v}_3 &= \rho_m (\tau_{13,1} + \tau_{23,2} + \tau_{33,3}) + \rho_f p_3 + \rho_f b_3 q_3 \\
\rho \dot{q}_1 &= -\rho_f (\tau_{11,1} + \tau_{12,2} + \tau_{13,3}) - \rho_b p_1 - \rho_b b_1 q_1 \\
\rho \dot{q}_2 &= -\rho_f (\tau_{12,1} + \tau_{22,2} + \tau_{23,3}) - \rho_b p_2 - \rho_b b_2 q_2 \\
\rho \dot{q}_3 &= -\rho_f (\tau_{13,1} + \tau_{23,2} + \tau_{33,3}) - \rho_b p_3 - \rho_b b_3 q_3
\end{align*}
\]

where the particle velocities \( v_i = \dot{u}_i \) and \( q_i = \dot{w}_i \) with \( u_i \) and \( w_i \) are the solid displacements and relative
fluid displacements, respectively. \( \tau_{ij} \) is the components of the total stress tensor. \( p \) represents the pore pressure. The bulk density \( \rho_b = \phi \rho_f + (1 - \phi) \rho_s \) where \( \rho_f \) and \( \rho_s \) are the fluid and solid densities, respectively. The effective fluid density \( \rho_m = v \rho_f / \phi \) with \( v \) and \( \phi \) being the tortuosity and porosity, respectively. \( \rho = \rho_m \rho_b - \rho_f \rho_f \). The viscosity coefficient \( b_v = \eta / \kappa \) with \( \eta \) and \( \kappa \) being the viscosity and permeability, respectively. \( c_{ij} \) is the 2oeCs tensor of the dry rock. The poroelastic effective stress coefficients \( \alpha_i \) and the poroelastic solid-fluid coupling modulus \( M \) have the following form,

\[
\begin{align*}
\alpha_1 &= 1 - \frac{(c_{11} + c_{12} + c_{13})}{3K_s} \\
\alpha_2 &= 1 - \frac{(c_{12} + c_{22} + c_{23})}{3K_s} \\
\alpha_3 &= 1 - \frac{(c_{13} + c_{23} + c_{33})}{3K_s} \\
\alpha_4 &= -\frac{(c_{14} + c_{24} + c_{34})}{3K_s} \\
\alpha_5 &= -\frac{(c_{15} + c_{25} + c_{35})}{3K_s} \\
\alpha_6 &= -\frac{(c_{16} + c_{26} + c_{36})}{3K_s} \\
M &= \frac{K_s}{\left(1 - \frac{K_s}{K_f}\right)} \\
\bar{R} &= \frac{1}{9} [c_{11} + c_{12} + c_{13} + 2(c_{12} + c_{13} + c_{23})] \\
\end{align*}
\]

where \( K_s \) and \( K_f \) are the bulk moduli of solid and fluid phases, respectively.

The matrix form (Carcione, 2001) of constitutive equations for general anisotropic poroelastic media in 3D can be explicitly expressed as (de la Puente et al., 2008)

\[
\begin{bmatrix}
\tau_{11} \\
\tau_{22} \\
\tau_{33} \\
\tau_{23} \\
\tau_{13} \\
\tau_{12} \\
-p
\end{bmatrix} =
\begin{bmatrix}
c_{11}^u & c_{12}^u & c_{13}^u & c_{14}^u & c_{15}^u & c_{16}^u & M\alpha_1 \\
c_{12}^u & c_{22}^u & c_{23}^u & c_{24}^u & c_{25}^u & c_{26}^u & M\alpha_2 \\
c_{13}^u & c_{23}^u & c_{33}^u & c_{34}^u & c_{35}^u & c_{36}^u & M\alpha_3 \\
c_{14}^u & c_{24}^u & c_{34}^u & c_{44}^u & c_{45}^u & c_{46}^u & M\alpha_4 \\
c_{15}^u & c_{25}^u & c_{35}^u & c_{45}^u & c_{55}^u & c_{56}^u & M\alpha_5 \\
c_{16}^u & c_{26}^u & c_{36}^u & c_{46}^u & c_{56}^u & c_{66}^u & M\alpha_6 \\
M\alpha_1 M\alpha_2 M\alpha_3 M\alpha_4 M\alpha_5 M\alpha_6
\end{bmatrix}
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{33} \\
e_{23} \\
e_{13} \\
e_{12} \\
-\varsigma
\end{bmatrix},
\]

where the components of the undrained stiffness tensor \( c_{ij}^u = c_{ij} + M\alpha_i \alpha_j \). \( \varsigma = -\nabla \cdot (\phi w_l) \) is the spatial variation of the fluid content. For the draining condition, the elastic constant \( c_{ij} \) can be replaced by the 3D acoustoelastic stiffness matrix \( A_{ij} \) in equation (A-9). Thus, the tensorial constitutive equation
for 3D anisotropic acoustoporoelastic media can be expressed explicitly as

\[
\begin{bmatrix}
\tau_{11} \\
\tau_{22} \\
\tau_{33} \\
\tau_{23} \\
\tau_{13} \\
\tau_{12} \\
-p
\end{bmatrix} =
\begin{bmatrix}
\tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} & \tilde{A}_{14} & \tilde{A}_{15} & \tilde{A}_{16} & \tilde{M}\tilde{\alpha}_1 \\
\tilde{A}_{12} & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} & \tilde{A}_{25} & \tilde{A}_{26} & \tilde{M}\tilde{\alpha}_2 \\
\tilde{A}_{13} & \tilde{A}_{23} & \tilde{A}_{33} & \tilde{A}_{34} & \tilde{A}_{35} & \tilde{A}_{36} & \tilde{M}\tilde{\alpha}_3 \\
\tilde{A}_{14} & \tilde{A}_{24} & \tilde{A}_{34} & \tilde{A}_{44} & \tilde{A}_{45} & \tilde{A}_{46} & \tilde{M}\tilde{\alpha}_4 \\
\tilde{A}_{15} & \tilde{A}_{25} & \tilde{A}_{35} & \tilde{A}_{45} & \tilde{A}_{55} & \tilde{A}_{56} & \tilde{M}\tilde{\alpha}_5 \\
\tilde{A}_{16} & \tilde{A}_{26} & \tilde{A}_{36} & \tilde{A}_{46} & \tilde{A}_{56} & \tilde{A}_{66} & \tilde{M}\tilde{\alpha}_6 \\
\tilde{M}\tilde{\alpha}_1 & \tilde{M}\tilde{\alpha}_2 & \tilde{M}\tilde{\alpha}_3 & \tilde{M}\tilde{\alpha}_4 & \tilde{M}\tilde{\alpha}_5 & \tilde{M}\tilde{\alpha}_6 & \tilde{M}
\end{bmatrix}
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{33} \\
e_{23} \\
e_{13} \\
e_{12} \\
2\epsilon_{23}
\end{bmatrix},
\] (4)

where the undrained stiffness tensor of acoustoporoelasticity \( \tilde{A}_{ij} = A_{ij} + \tilde{M}\tilde{\alpha}_i\tilde{\alpha}_j \). The acoustoporoelastic effective stress coefficients \( \tilde{\alpha}_i \) and the poroelastic fluid-solid coupling modulus \( \tilde{M} \) have the following form,

\[
\begin{align*}
\tilde{\alpha}_1 &= 1 - \frac{(A_{11} + A_{12} + A_{13})}{3K_s} \\
\tilde{\alpha}_2 &= 1 - \frac{(A_{12} + A_{22} + A_{23})}{3K_s} \\
\tilde{\alpha}_3 &= 1 - \frac{(A_{13} + A_{23} + A_{33})}{3K_s} \\
\tilde{\alpha}_4 &= -\frac{(A_{14} + A_{24} + A_{34})}{3K_s} \\
\tilde{\alpha}_5 &= -\frac{(A_{15} + A_{25} + A_{35})}{3K_s} \\
\tilde{\alpha}_6 &= -\frac{(A_{16} + A_{26} + A_{36})}{3K_s} \\
\tilde{M} &= \left(1 - \frac{\tilde{K}^u}{K_s}\right) - \phi \left(1 - \frac{K_s}{K_f}\right)
\end{align*}
\] (5)

\( \tilde{K}^u = \frac{1}{9}[A_{11} + A_{12} + A_{13} + 2(A_{12} + A_{13} + A_{23})] \)

Similar to equation (1), the first-order velocity-stress formulations for acoustoporoelasticity can be written explicitly as
\[
\begin{align*}
\rho \ddot{v}_1 &= \rho_m (\tau_{11,1} + \tau_{12,2} + \tau_{13,3}) + \rho_f p_1 + \rho_f b_v q_1, \\
\rho \ddot{v}_2 &= \rho_m (\tau_{12,1} + \tau_{22,2} + \tau_{23,3}) + \rho_f p_2 + \rho_f b_v q_2, \\
\rho \ddot{v}_3 &= \rho_m (\tau_{13,1} + \tau_{23,2} + \tau_{33,3}) + \rho_f p_3 + \rho_f b_v q_3, \\
\dot{p}_1 &= -\rho_f (\tau_{11,1} + \tau_{12,2} + \tau_{13,3}) - \rho_b p_1 - \rho_b b_v q_1, \\
\dot{p}_2 &= -\rho_f (\tau_{12,1} + \tau_{22,2} + \tau_{23,3}) - \rho_b p_2 - \rho_b b_v q_2, \\
\dot{p}_3 &= -\rho_f (\tau_{13,1} + \tau_{23,2} + \tau_{33,3}) - \rho_b p_3 - \rho_b b_v q_3 \\
\end{align*}
\]

\[\tau_{11} = \tilde{A}_{11} v_{1,1} + \tilde{A}_{12} v_{2,2} + \tilde{A}_{13} v_{3,3} + \tilde{a}_1 \tilde{M} (q_{1,1} + q_{2,2} + q_{3,3}),\]
\[\tau_{22} = \tilde{A}_{12} v_{1,1} + \tilde{A}_{22} v_{2,2} + \tilde{A}_{23} v_{3,3} + \tilde{a}_2 \tilde{M} (q_{1,1} + q_{2,2} + q_{3,3}),\]
\[\tau_{33} = \tilde{A}_{13} v_{1,1} + \tilde{A}_{23} v_{2,2} + \tilde{A}_{33} v_{3,3} + \tilde{a}_3 \tilde{M} (q_{1,1} + q_{2,2} + q_{3,3}),\]
\[\dot{v}_{12} = \tilde{A}_{44} (v_{1,2} + v_{2,1}),\]
\[\dot{v}_{13} = \tilde{A}_{55} (v_{1,3} + v_{3,1}),\]
\[\dot{v}_{23} = \tilde{A}_{66} (v_{2,3} + v_{3,2}),\]
\[\dot{p} = -\tilde{M} (\tilde{a}_1 v_{1,1} + \tilde{a}_2 v_{2,2} + \tilde{a}_3 v_{3,3} + q_{1,1} + q_{2,2} + q_{3,3}).\]

We see that the first-order velocity-stress acoustoporoelasticity equations can be derived from the velocity-stress anisotropic poroelasticity equations by replacing the poroelastic stiffness matrix of the 2D Padé acoustoelasticity, prestressed background static deformation and wave-induced small-amplitude dynamic deformation. The former refers to the draining process where the background matrix and porous fluids experience the same strain. The latter is usually regarded as an undrained instantaneous process because of the high frequency of elastic waves (Fu et al., 2018). The prestress loading can be regarded to act on the background matrix only, yielding the change in elastic moduli.

### 3D velocity-stress Padé acoustoporoelasticity equations

As shown in Appendix A, Fu and Fu (2017) propose the 2D Padé acoustoelasticity for wave propagation in prestressed solids. We formulate the 3D Padé acoustoelastic stiffness matrix as shown in equation (A-11). We can use stiffness-matrix replacement to formulate the 3D Padé acoustoporoelasticity for wave propagation in porous media under large-magnitude prestress. Unlike traditional acoustoporoelasticity, The Padé acoustoporoelasticity accounts for inelastic deformations due to the closure of compliant pores where the Padé coefficients, \(a\) and \(b\), are closely related to the structure of compliant pores (Fu and Fu, 2017).

The first-order velocity-stress formulation of 3D Padé acoustoporoelasticity equations can be derived from the velocity-stress anisotropic poroelasticity equations by replacing the 3D poroelastic stiffness...
matrix (equation (3)) of the 2oeCs with the 3D Padé acoustoelastic stiffness matrix $A_{ij}^p$ (equation (A-11)) of the 3oeCs. Thus, the tensorial constitutive equation for 3D anisotropic Padé acoustoporoelastic media can be expressed explicitly as

$$
\begin{bmatrix}
\tau_{11} \\
\tau_{22} \\
\tau_{33} \\
\tau_{12} \\
\tau_{13} \\
\tau_{23}
\end{bmatrix}
= \begin{bmatrix}
\tilde{A}_{11}^p & \tilde{A}_{12}^p & \tilde{A}_{13}^p & \tilde{A}_{14}^p & \tilde{A}_{15}^p & \tilde{A}_{16}^p & \tilde{M}^p \tilde{a}_1^p \\
\tilde{A}_{12}^p & \tilde{A}_{22}^p & \tilde{A}_{23}^p & \tilde{A}_{24}^p & \tilde{A}_{25}^p & \tilde{A}_{26}^p & \tilde{M}^p \tilde{a}_2^p \\
\tilde{A}_{13}^p & \tilde{A}_{23}^p & \tilde{A}_{33}^p & \tilde{A}_{34}^p & \tilde{A}_{35}^p & \tilde{A}_{36}^p & \tilde{M}^p \tilde{a}_3^p \\
\tilde{A}_{14}^p & \tilde{A}_{24}^p & \tilde{A}_{34}^p & \tilde{A}_{44}^p & \tilde{A}_{45}^p & \tilde{A}_{46}^p & \tilde{M}^p \tilde{a}_4^p \\
\tilde{A}_{15}^p & \tilde{A}_{25}^p & \tilde{A}_{35}^p & \tilde{A}_{45}^p & \tilde{A}_{55}^p & \tilde{A}_{56}^p & \tilde{M}^p \tilde{a}_5^p \\
\tilde{A}_{16}^p & \tilde{A}_{26}^p & \tilde{A}_{36}^p & \tilde{A}_{46}^p & \tilde{A}_{56}^p & \tilde{A}_{66}^p & \tilde{M}^p \tilde{a}_6^p \\
\tilde{M}^p \tilde{a}_1^p & \tilde{M}^p \tilde{a}_2^p & \tilde{M}^p \tilde{a}_3^p & \tilde{M}^p \tilde{a}_4^p & \tilde{M}^p \tilde{a}_5^p & \tilde{M}^p \tilde{a}_6^p & \tilde{M}^p
\end{bmatrix}
\begin{bmatrix}
e_{11} \\
e_{22} \\
e_{33} \\
e_{12} \\
e_{13} \\
e_{23}
\end{bmatrix}
+ 2e_{23},
$$

(7)

where the undrained stiffness tensor of Padé acoustoporoelasticity $\tilde{A}_{ij}^p = A_{ij}^p + \tilde{M}^p \tilde{a}_i^p \tilde{a}_j^p$. The Padé acoustoporoelastic effective stress coefficients $\tilde{a}_i^p$ and the poroelastic fluid-solid coupling modulus $\tilde{M}^p$ have the following form:

$$
\tilde{a}_1^p = 1 - \frac{(A_{11}^p + A_{12}^p + A_{13}^p)}{3K_s}
$$

$$
\tilde{a}_2^p = 1 - \frac{(A_{21}^p + A_{22}^p + A_{23}^p)}{3K_s}
$$

$$
\tilde{a}_3^p = 1 - \frac{(A_{31}^p + A_{32}^p + A_{33}^p)}{3K_s}
$$

$$
\tilde{a}_4^p = -\frac{(A_{41}^p + A_{42}^p + A_{43}^p)}{3K_s}
$$

$$
\tilde{a}_5^p = -\frac{(A_{51}^p + A_{52}^p + A_{53}^p)}{3K_s}
$$

$$
\tilde{a}_6^p = -\frac{(A_{61}^p + A_{62}^p + A_{63}^p)}{3K_s}
$$

$$
\tilde{M}^p = \frac{1}{\phi} \left( \frac{1}{\tilde{K}_s} + \phi \left( \frac{1}{\tilde{K}_f} \right) \right)
$$

$$
\tilde{K}^p = \frac{1}{9} \left[ A_{11}^p + A_{12}^p + A_{13}^p + 2(A_{12}^p + A_{13}^p + A_{23}^p) \right]
$$

(8)

Similar to equation (6), the first-order velocity-stress formulations for Padé acoustoporoelasticity can be written explicitly as
\[
\begin{align*}
\rho \ddot{v}_1 &= \rho_m (\tau_{11,1} + \tau_{12,2} + \tau_{13,3}) + \rho_f p_1 + \rho_f b_v q_1 \\
\rho \ddot{v}_2 &= \rho_m (\tau_{12,1} + \tau_{22,2} + \tau_{23,3}) + \rho_f p_2 + \rho_f b_v q_2 \\
\rho \ddot{v}_3 &= \rho_m (\tau_{13,1} + \tau_{23,2} + \tau_{33,3}) + \rho_f p_3 + \rho_f b_v q_3 \\
\rho \dot{q}_1 &= -\rho_f (\tau_{11,1} + \tau_{12,2} + \tau_{13,3}) - \rho_b p_1 - \rho_b b_v q_1 \\
\rho \dot{q}_2 &= -\rho_f (\tau_{12,1} + \tau_{22,2} + \tau_{23,3}) - \rho_b p_2 - \rho_b b_v q_2 \\
\rho \dot{q}_3 &= -\rho_f (\tau_{13,1} + \tau_{23,2} + \tau_{33,3}) - \rho_b p_3 - \rho_b b_v q_3
\end{align*}
\]  
\tag{9}

\[\begin{align*}
\dot{t}_{11} &= \tilde{A}_{11}^p v_{1,1} + \tilde{A}_{12}^p v_{2,2} + \tilde{A}_{13}^p v_{3,3} + \tilde{M}^p \tilde{a}_1^p (q_{1,1} + q_{2,2} + q_{3,3}) \\
\dot{t}_{22} &= \tilde{A}_{12}^p v_{1,1} + \tilde{A}_{22}^p v_{2,2} + \tilde{A}_{23}^p v_{3,3} + \tilde{M}^p \tilde{a}_2^p (q_{1,1} + q_{2,2} + q_{3,3}) \\
\dot{t}_{33} &= \tilde{A}_{13}^p v_{1,1} + \tilde{A}_{23}^p v_{2,2} + \tilde{A}_{33}^p v_{3,3} + \tilde{M}^p \tilde{a}_3^p (q_{1,1} + q_{2,2} + q_{3,3}) \\
\dot{t}_{12} &= \tilde{A}_{44}^p (v_{1,2} + v_{2,1}) \\
\dot{t}_{13} &= \tilde{A}_{55}^p (v_{1,3} + v_{3,1}) \\
\dot{t}_{23} &= \tilde{A}_{66}^p (v_{2,3} + v_{3,2}) \\
\dot{p} &= -\tilde{M}^p (\tilde{a}_1^p v_{1,1} + \tilde{a}_2^p v_{2,2} + \tilde{a}_3^p v_{3,3} + q_{1,1} + q_{2,2} + q_{3,3})
\end{align*}\]

Applications to experimental data with porous rocks

As indicated by Carcione (2001), for homogeneous plane waves, we can obtain the matrix form of the Christoffel equation,

\[
(R^{-1} \cdot \Gamma - v^2 I_6) \cdot V = 0,
\]  
\tag{10}

where \(R\) is a matrix related to the density and angular frequency, \(V\) is the particle velocity vector, \(I_6\) is the sixth-order unit matrix, and \(v = \omega/k\) is the complex velocity with \(\omega\) and \(k\) being the angular frequency and complex wavenumber, respectively. \(\Gamma\) is the Christoffel matrix, which can be written as

\[
\Gamma = L \cdot \tilde{A}^p \cdot L^T,
\]  
\tag{11}

where \(L\) is a matrix composed of the direction cosines along the propagation direction. \(\tilde{A}^p\) is the Padé acoustoporoelastic stiffness matrix (equation (7)). A more detailed description of \(R, V, L\) can be found in Carcione (2001). Let the determinant be zero, we have

\[
\det(R^{-1} \cdot \Gamma - v^2 I_6) = 0.
\]  
\tag{12}

The six eigenvalues can be obtained by solving equation (12), two of them are zero, and the remaining four eigenvalues represent the velocities of the fast P, slow P, and two S waves, respectively,
\[
\begin{align*}
\nu_{\text{Fast}-P}^2 &= \frac{m_i \alpha_{11}^p + \rho \bar{M}^p - 2 \rho_f \bar{M}^p \alpha_1^p + \sqrt{K}}{2(m_i \rho - \rho_f^2)}, \\
\nu_{\text{Slow}-P}^2 &= \frac{m_i \alpha_{11}^p + \rho \bar{M}^p - 2 \rho_f \bar{M}^p \alpha_1^p - \sqrt{K}}{2(m_i \rho - \rho_f^2)}, \\
\nu_{SV}^2 &= \frac{m_i \alpha_{55}^p}{m_i \rho - \rho_f^2}, \\
\nu_{SH}^2 &= \frac{m_i \alpha_{66}^p}{m_i \rho - \rho_f^2}.
\end{align*}
\]

where \( m_i = \nu \rho_f / \phi \) with \( \nu \) is the tortuosity, and

\[
K = \left( m_i \alpha_{11}^p \right)^2 - 2 m_i \alpha_{11}^p \rho \bar{M}^p + 4 \alpha_{11}^p \rho_f^2 \bar{M}^p - 4 m_i \alpha_{11}^p \rho_f \bar{M}^p \alpha_1^p + \left( \rho \bar{M}^p \right)^2 - 4 \rho \bar{M}^p \left( \bar{M}^p \right)^2 \alpha_1^p + 4 \rho m_i \bar{M}^p \left( \alpha_1^p \right)^2
\]

Based on the experimental data of fluid-saturated Portland sandstone under confining and uniaxial prestress (Winkler and Liu, 1996), we validate the theories of acoustoelasticity, acoustoporoelasticity, and Padé acoustoporoelasticity by comparing theoretical predictions and experimental results. The properties of Portland sandstone are listed in Table 1. Figure 1 compares the theoretical values of velocities with experimental velocities of P and S waves as a function of effective stress under isotropic (confining) prestress, whereas Figure 2 compares the theoretical and experimental velocities of P, SH, and SV waves under anisotropic (uniaxial) prestress. The experimental velocities (solid circles) are from Winkler and Liu (1996). We see that the three theories show quite different accuracies in prediction compared to experimental results.

In the range of small effective stresses, the velocities predicted by the three theories agree well with the experimental results under isotropic (confining) prestress possibly because of the closure of compliant pores and the isotropic behavior of stress-induced velocity variations. However, the acoustoelastic prediction shows large errors under anisotropic (uniaxial) prestress possibly because of the anisotropic closure of compliant pores. Considerable differences in prediction between the three theories occur in the range of large effective stresses. The strong nonlinearity due to the effect of compliant pores renders both the acoustoelastic and acoustoporoelastic predictions unapplicable particularly under anisotropic (uniaxial) prestress. The theory of Padé acoustoporoelasticity, as expected, gives a good prediction where the Padé coefficients \((a \text{ and } b)\)
To characterize the microstructural dependence of elastic constants. As indicated in Table 1, the 3oeCs ($A$, $B$, $C$) of Portland sandstone are (-2378, -35, 238) GPa obtained by the best fit of the experimental data at small effective stresses (less than 20 MPa) with the closure of compliant pores. The corresponding Padé coefficients can be obtained by the best fit of all experimental data, as $(a, b) = (-50, -24)$.

**Figure 1.** Comparison of the theoretical (lines) and experimental (dots) velocities of fluid-saturated Portland sandstone as a function of isotropic (confining) prestress for P (a) and S (b) waves.

**Figure 2.** Comparison of the theoretical (lines) and experimental (dots) velocities of fluid-saturated Portland sandstone as a function of anisotropic (uniaxial) prestress for P (a), SH (b), and SV (c) waves.

### NUMERICAL METHODOLOGY

**Standard staggered-grid FD method**

The SSG-FD method has been widely used for numerical simulations of wave propagation where partial differential velocity-stress equations are discretized over the staggered grids. Figure 3 shows the arrangement of field components (e.g., stresses and particle velocity) and material parameters (e.g., density, pore pressure, and elastic constants) in an 3D SSG-FD grid cell.
Figure 3. Schematic diagram of an 3D SSG-FD grid cell with positions of normal stresses and material parameters in rhombus, solid and relative fluid velocities in the solid triangles, squares, and circles, and shear stresses in hollow triangles, squares, and circles, respectively.

According to the configuration relationship in Figure 3, the discrete format of velocity and stress in equation (9) can be written as
\[
\begin{align*}
&v_{x|n} = v_{x|n-1} - \frac{\rho_f}{\rho_b} q_{x|n-1} \left( e^{\frac{-\rho_b \Delta t}{\rho}} - 1 \right) + \frac{\Delta t}{\rho} \left[ \rho_m \left( D_x^+ \tau_{xx} |n^- + D_y^+ \tau_{xy} |n^- + D_z^+ \tau_{xz} |n^- \right) \right] + \rho_f D_x^+ p |n^- \\
&v_{y|n} = v_{y|n-1} - \frac{\rho_f}{\rho_b} q_{y|n-1} \left( e^{\frac{-\rho_b \Delta t}{\rho}} - 1 \right) + \frac{\Delta t}{\rho} \left[ \rho_m \left( D_x^+ \tau_{xy} |n^- + D_y^+ \tau_{yy} |n^- + D_z^+ \tau_{yz} |n^- \right) \right] + \rho_f D_y^+ p |n^- \\
&v_{z|n} = v_{z|n-1} - \frac{\rho_f}{\rho_b} q_{z|n-1} \left( e^{\frac{-\rho_b \Delta t}{\rho}} - 1 \right) + \frac{\Delta t}{\rho} \left[ \rho_m \left( D_x^+ \tau_{xz} |n^- + D_y^+ \tau_{yz} |n^- + D_z^+ \tau_{zz} |n^- \right) \right] + \rho_f D_z^+ p |n^- \\
&q_{x|n} = q_{x|n-1} e^{\frac{-\rho_b \Delta t}{\rho}} - \frac{\Delta t}{\rho} \left( \rho_f \left( D_x^+ \tau_{xx} |n^- + D_y^+ \tau_{xy} |n^- + D_z^+ \tau_{xz} |n^- \right) + \rho_b D_x^+ p |n^- \right) \\
&q_{y|n} = q_{y|n-1} e^{\frac{-\rho_b \Delta t}{\rho}} - \frac{\Delta t}{\rho} \left( \rho_f \left( D_x^+ \tau_{xy} |n^- + D_y^+ \tau_{yy} |n^- + D_z^+ \tau_{yz} |n^- \right) + \rho_b D_y^+ p |n^- \right) \\
&q_{z|n} = q_{z|n-1} e^{\frac{-\rho_b \Delta t}{\rho}} - \frac{\Delta t}{\rho} \left( \rho_f \left( D_x^+ \tau_{xz} |n^- + D_y^+ \tau_{yz} |n^- + D_z^+ \tau_{zz} |n^- \right) + \rho_b D_z^+ p |n^- \right)
\end{align*}
\]

where \( n^- \) and \( n^+ \) represent \((n - 1/2)\Delta t\) and \((n + 1/2)\Delta t\), respectively. \( D^+ \) and \( D^- \) are the generalized expressions of forward and backward FD operators, respectively, as follows,

\[
\begin{align*}
D^+ f(i) &= \frac{1}{\Delta h} \sum_{l=1}^{L} c_l \left( f(i + l) - f(i - (l - 1)) \right) \\
D^- f(i) &= \frac{1}{\Delta h} \sum_{l=1}^{L} c_l \left( f(i + (l - 1)) - f(i - l) \right)
\end{align*}
\]

where \( \Delta h \) is the space interval size, \( c_l \) is the staggered-grid difference coefficient, and \( L \) is the length of the spatial derivative difference operator with \( L = 5 \) in the paper.

**PML absorbing boundary**
We employ the classical (split) perfectly matched layer (PML) (Chew & Liu 1996) as the absorbing boundary where wavefields are split into three orthogonal directions with each updated using the PML formulation of velocity-stress Padé acoustoelastic equations. Taking the $x$-component of the particle's solid velocity as an example, its PML discrete form can be written as

$$v_x^n = v_x^1 + v_x^2 + v_x^3,$$  (17)

with

$$v_x^1 = \frac{1}{1 + \frac{\Lambda_x \Delta t}{2}} \begin{pmatrix} \left(1 + \frac{\Lambda_x \Delta t}{2}\right) v_x^{n-1} - \frac{\rho_f}{\rho_b} q_x^{n-1} \left(\frac{-\rho_b b_d \Delta t}{\rho} - 1\right) \\ + \frac{\Delta t}{\rho} \left[\rho_m D_x^{+} \tau_{xx}^{n-1} + \rho_f D_x^{+} p_x^{n-1}\right] \end{pmatrix},$$

$$v_x^2 = \frac{1}{1 + \frac{\Lambda_y \Delta t}{2}} \begin{pmatrix} \left(1 + \frac{\Lambda_y \Delta t}{2}\right) v_x^{n-1} - \frac{\rho_f}{\rho_b} q_y^{n-1} \left(\frac{-\rho_b b_d \Delta t}{\rho} - 1\right) \\ + \frac{\Delta t}{\rho} \left[\rho_m D_y^{+} \tau_{xy}^{n-1} + \rho_f D_y^{+} p_y^{n-1}\right] \end{pmatrix},$$

$$v_x^3 = \frac{1}{1 + \frac{\Lambda_z \Delta t}{2}} \begin{pmatrix} \left(1 + \frac{\Lambda_z \Delta t}{2}\right) v_x^{n-1} - \frac{\rho_f}{\rho_b} q_z^{n-1} \left(\frac{-\rho_b b_d \Delta t}{\rho} - 1\right) \\ + \frac{\Delta t}{\rho} \left[\rho_m D_z^{+} \tau_{xz}^{n-1} + \rho_f D_z^{+} p_z^{n-1}\right] \end{pmatrix},$$

where the subscript and superscript of the variable represent the space and time steps, respectively. $\Lambda_x, \Lambda_y, \Lambda_z$ are the damping parameters in each coordinate direction, respectively, with the damping coefficient $\Lambda = \left(3V_{\text{max}} d^2/2W_p^3\right) \ln(Q^{-1})$ where $V_{\text{max}}$ is the maximum velocity in all the directions, $d$ is the distance to the inner boundary, $W_p$ is the width of the PML boundary layer, and the reflection coefficient $Q = 10^{-6}$.

**Numerical stability**

The numerical instability in the SSG-FD method is caused by the approximation of continuous derivatives using high-order operators and by the time updates using Taylor polynomials, which is usually affected by the coupling relationship between the time interval and space grid spacing. To reduce numerical stability, we must render the time step smaller than the propagation time of waves between two adjacent grid points as follows for 3D cases (Chen et al. 2006),

$$\Delta t \leq \frac{\Delta h}{V_{\text{max}} G \sqrt{3}},$$  (19)
where $G$ is the sum of the absolute values of the Taylor coefficients. For the fourth-order SSG-FD operator, $G = \frac{27}{24} + \frac{1}{24} = \frac{7}{6}$ (Li et al. 2021).

The grid dispersion can be improved by constraining the value of $\Delta h$ according to the Nyquist-Shannon sampling theory,

$$\Delta h \leq \frac{\lambda_{\text{min}}}{Y} = \frac{V_{\text{min}}}{Yf_{\text{max}}},$$

(20)

where $\lambda_{\text{min}}$ and $V_{\text{min}}$ represent the minimum wavelength and velocity, respectively, $f_{\text{max}}$ is the maximum frequency, and $Y$ is the number of grid points for one wavelength. In general, the shortest wavelength should be sampled by at least 5~6 grid points (i.e., $Y = 5$~6).

**NUMERICAL EXAMPLES**

In this section, we use the SSG-FD method to solve Padé acoustoporoelastic equations in the first-order velocity-stress scheme. We simulate wavefield snapshots under isotropic (confining) and anisotropic (uniaxial and pure shear) prestresses. According to the stability condition of equations (19) and (20), we choose the grid spacing $\Delta h = 9$ mm and the time step $\Delta t = 5\times 10^{-4}$ ms. Numerical examples are presented using fluid-saturated Portland sandstone with the model size $(1269 \times 1269 \times 1269)$ mm. The detailed poroelastic and acoustoporoelastic properties for solid and fluid phases listed in Table 1. The source is located at the center of the model, loaded as a vertical force, with a time history as

$$s(t) = [1 - 2\pi^2 f_0^2 (t - t_0)^2]e^{-\pi^2 f_0^2 (t - t_0)^2},$$

(21)

where $t_0$ is the delay time, and the main frequency $f_0 = 25$ kHz.

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
<th>SI Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bulk modulus of the solid</td>
<td>$K_s$</td>
<td>37</td>
<td>GPa</td>
</tr>
<tr>
<td>Bulk modulus of the dry rock</td>
<td>$K_d$</td>
<td>9.7</td>
<td>GPa</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>$\mu$</td>
<td>7.3</td>
<td>GPa</td>
</tr>
<tr>
<td>Bulk modulus of the fluid</td>
<td>$K_f$</td>
<td>2.25</td>
<td>GPa</td>
</tr>
<tr>
<td>Mineral density</td>
<td>$\rho_s$</td>
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<td>kg/m$^3$</td>
</tr>
<tr>
<td>Fluid density</td>
<td>$\rho_f$</td>
<td>1000</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Viscosity coefficient of fluid in pores</td>
<td>$\eta$</td>
<td>$10^{-3}$</td>
<td>Pa\cdot s</td>
</tr>
<tr>
<td>Rock permeability</td>
<td>$\kappa$</td>
<td>$4.2 \times 10^{-4}$</td>
<td>D</td>
</tr>
</tbody>
</table>

Table 1. Properties of the water-saturated Portland sandstone from Winkler and Liu (1996) and Revil et al. (2015)
Porosity \( \phi \) 20% --
Tortuosity \( v \) 3 --
3oeCs
\begin{align*}
A & = -2378 \text{ GPa} \\
B & = -35 \text{ GPa} \\
C & = 238 \text{ GPa}
\end{align*}

**Padé acoustoporoelasticity simulations under isotropic (confining) prestress**

The stress field is isotropic under the confining prestress \( P \). The same stress and strain are distributed along all the principal axes. The principal strain component can be expressed as

\[
\begin{cases}
 e_{11} = e_{22} = e_{33} = -\frac{P}{3K_d} \\
 e_{12} = e_{13} = e_{23} = 0
\end{cases}
\]  
(22)

where \( K_d = \lambda + 2/3\mu - \beta^2 M \) is the bulk modulus of dry rock. Substituting equations (22) into (A-11) yields the corresponding Padé acoustoporoelastic stiffness matrix.

**Figure 4.** Padé acoustoporoelastic numerical simulations for various confining prestresses: 3D wavefield snapshots (left panel) of the solid vertical velocity at \( t = 0.15 \text{ ms} \) and intercepted wavefield slices along the \( x \)-, \( y \)-, and \( z \)-axes (right three panels), respectively.

**Figures 4 and 5** show the 3D wavefield snapshots at \( t = 0.15 \text{ ms} \) under different confining prestresses for the solid and relative-fluid vertical velocities, respectively. The corresponding wavefield slices at
each prestress are intercepted along the three coordinates, respectively. The stress-induced velocities of fast P, S, and slow P waves increase isotropically with increasing prestresses. We see stronger P- and S-wave amplitudes in the solid phase than those in the fluid phase, whereas the slow P wave becomes rather strong in the fluid phase due to its nature -- out-of-phase motion of the solid phase for the fluid phase.

Figure 5. Padé acoustoporoelastic numerical simulations for various confining prestresses: 3D wavefield snapshots (left panel) of the relative-fluid vertical velocity at $t = 0.15$ ms and intercepted wavefield slices along the $x$, $y$, and $z$-axes (right three panels), respectively.

Figure 6 compares the fast P-wave seismograms in the solid phase simulated by the theories of acoustoporoelasticity and Padé acoustoporoelasticity, respectively, under different confining prestresses. We see different traveltimes between them under the same prestress. The acoustoporoelastic wavefronts propagate faster than the Padé acoustoporoelastic wavefronts. Figure 7 compares the predicted and simulated velocities of fast P, S, and slow P waves, as a function of various confining prestresses. Figure 8 compares the predicted and simulated velocities of these waves in all azimuthal propagation directions under the same confining prestress (20 MPa). We see that the theoretically predicted and numerically simulated velocities agree well, which validates the proposed numerical simulations at least in traveltimes.
**Figure 6.** Comparison of the fast P-wave seismograms in the solid phase (x-component) simulated by the theories of acoustoporoelasticity and Padé acoustoporoelasticity, respectively, under different confining prestresses.

**Figure 7.** Comparison of the theoretical and simulated velocities of fast P, S, and slow P waves as a function of various confining stresses.

**Figure 8.** Comparison of the theoretical and simulated velocities of fast P, S, and slow P waves in all azimuthal propagation directions under the same confining prestress (20 MPa).
Padé acoustoporoelasticity simulations under anisotropic (uniaxial) prestress

The uniaxial prestress results in an anisotropic stress field. The stress-induced velocity anisotropy is of orthotropy strongly related to the orientation of prestresses. We consider the uniaxial prestress $P$ loaded along the $z$-axis, which reduces the length of the $z$-axis, but elongates the lengths of the $x$-axis and $y$-axis. The principal strain components, in this case, can be expressed as

$$
\begin{align*}
    e_{11} &= e_{22} = \frac{P\lambda}{2\mu(3\lambda + 2\mu)} \\
    e_{33} &= -\frac{P(\lambda + \mu)}{\mu(3\lambda + 2\mu)} \\
    e_{12} &= e_{13} = e_{23} = 0
\end{align*}
$$

(23)

Substituting equations (23) into (A-11) yields the corresponding Padé acoustoporoelastic stiffness matrix.

![3D wavefield snapshots](image)

Figure 9. Padé acoustoporoelastic numerical simulations for various uniaxial prestresses: 3D wavefield snapshots (left panel) of the solid vertical velocity at $t = 0.15$ ms and intercepted wavefield slices along the $x$-, $y$-, and $z$-axes (right three panels), respectively.

Figures 9 and 10 show the 3D wavefield snapshots at $t = 0.15$ ms under various uniaxial prestresses for the solid and relative-fluid vertical velocities, respectively. The corresponding wavefield slices at each prestress are intercepted along the three coordinates, respectively. We see that the stress-induced...
velocity anisotropy makes the fast qP, slow qP, and qS wavefronts deform in both the x- and y-directions. The compression along the z-axis increases the wave velocity along the z-direction, while the resulting tension along the x- and y-axes reduces the wave velocity, thus making these wavefronts form ellipses on the xOz and yOz planes with the z-axis as the major axis. However, these wavefronts on the xOy plane present isotropic circles because of the same sign and magnitude of strains along the x- and y-axes.

With increasing uniaxial prestress, the azimuthal anisotropy of these waves becomes more and more obvious, consistent with the mechanical behavior of uniaxial stress. Particularly at P = 150 MPa, the phenomenon of S-wave splitting can be clearly seen due to the enhancement of anisotropy. The relative amplitude changes of these waves in the solid and relative fluid velocities are similar to those under the confining prestress.

**Figure 10.** Padé acoustoporoelastic numerical simulations for various uniaxial prestresses: 3D wavefield snapshots (left panel) of the relative-fluid vertical velocity at t = 0.15 ms and intercepted wavefield slices along the x-, y-, and z-axes (right three panels), respectively.

**Padé acoustoporoelasticity simulations under anisotropic (pure shear) prestress**

In this case, the z-axis is compressed while the x- and y-axes are elongated, but with the same absolute value of strains for both these axes, resulting in a pure shear deformation. It is a tension-
compression bidirectional stress state. The principal strain components under the pure shear prestress can be written as

\[
\begin{cases}
-e_{11} = -e_{22} = e_{33} = -\frac{p}{2\mu} \\
e_{12} = e_{13} = e_{23} = 0
\end{cases}
\] (24)

Substituting equations (24) into (A-11) yields the corresponding Padé acoustoporoelastic stiffness matrix.

Figure 11. Padé acoustoporoelastic numerical simulations for various pure shear prestresses: 3D wavefield snapshots (left panel) of the solid vertical velocity at \( t = 0.15 \) ms and intercepted wavefield slices along the \( x \)-, \( y \)-, and \( z \)-axes (right three panels), respectively.

Figures 11 and 12 show the 3D wavefield snapshots at \( t = 0.15 \) ms under various pure shear prestresses for the solid and relative-fluid vertical velocities, respectively. The corresponding wavefield slices at each prestress are intercepted along the three coordinates, respectively. Similar to the case of uniaxial prestress, the stress-induced velocity anisotropy is of orthotropy and the circular wavefront becomes more elliptical with increasing pure shear prestresses. The difference is that the absolute values of strains are equal along the three principal axes, implying that the stress-induced velocity anisotropy is more sensitive to pure shear prestress changes. The velocities of fast \( qP \), slow \( qP \), and \( qS \) waves increase much more in the \( z \)-axis direction compared with the uniaxial case.
Figure 1. Padé acoustoporoelastic numerical simulations for various pure shear prestresses: 3D wavefield snapshots (left panel) of the relative-fluid vertical velocity at \( t = 0.15 \) ms and intercepted wavefield slices along the \( x \)-, \( y \)-, and \( z \)-axes (right three panels), respectively.

As indicated in the chapter of theoretical background, the Padé acoustoporoelasticity accounts for stronger nonlinearity than acoustoporoelasticity due to the Padé coefficients that characterize the microstructural dependence of elastic constants. As shown in Figure 2 for the comparison of Padé acoustoporoelasticity and acoustoporoelasticity, the strong nonlinearity due to compliant pores reduces the level of stress-induced velocity variations. Figure 13 compares the waveforms of the solid vertical velocity between Padé and conventional acoustoporoelastic simulations under three different types of prestress at \( P = 40 \) MPa. We see that the diversities of wave travel times between these two different simulations are consistent with the theoretical predictions. Both the Padé and conventional acoustoporoelastic simulations under the uniaxial and pure shear prestresses demonstrate obvious orthotropic features. Particularly, under the pure shear prestress with the tension-compression bidirectional stress state, the acoustoporoelastic wavefronts appear distorted changes due to the effect of excessive anisotropy, whereas the Padé acoustoporoelastic wavefronts seem normal.

Figure 13. Comparison of Padé and conventional acoustoporoelastic simulations for the solid vertical velocity under different prestress types at \( P = 40 \) MPa: (a) Confining prestress; (b) Uniaxial prestress; (c) Pure shear prestress.
CONCLUSION

Prestressed porous rocks with compliant microstructures generally undergo linear elastic, hyperelastic (nonlinearly elastic), and inelastic deformations prior to mechanical failure. Small-amplitude wave propagation in such statically deformed rocks can be described traditionally by acoustoporoelastic theory that extends the classical acoustoelasticity of solids to porous media by incorporating Biot’s theory. The theory based on the cubic strain-energy function with third-order elastic constants only accounts for stress-induced hyperelasticity with linear strains under finite-magnitude prestress. Replacing the Taylor expansion in the derivation of acoustoporoelastic equations, the Padé approximation can be applied to the strain energy function to account for inelastic deformations under large-magnitude prestress. The resulting Padé acoustoporoelasticity can be alternatively derived from anisotropic poroelasticity equations by replacing the poroelastic stiffness matrix with the available Padé acoustoelastic stiffness matrix consisting of second-order and third-order elastic constants.

Theoretical expressions for wave speed via prestress can be obtained by plane-wave analyses in the absence (acoustoporoelasticity) and presence (Padé acoustoporoelasticity) of inelastic strains. The corresponding third-order elastic constants and Padé coefficients can be obtained by the best fitting to the linear and nonlinear segments of experimental data, respectively. Applications to experimental data of fluid-saturated Portland sandstone under confining and uniaxial prestresses differentiate acoustoporoelasticity and Padé acoustoporoelasticity in accuracy. Theoretical predictions by the Padé acoustoporoelasticity agree more accurately with ultrasonic measurements, especially for higher effective stresses. The standard staggered-grid finite-difference method with a perfectly matched layer absorbing boundary is used to solve the first-order velocity-stress formulation of 3D Padé acoustoporoelastic equations for wave propagation in prestressed porous media under isotropic (confining) and anisotropic (uniaxial and pure-shear) prestresses. We validate our numerical scheme by the plane-wave theoretical solution. Comparisons between theoretical and calculated (numerically simulated) wave velocities are conducted for fast P wave, slow P wave, and S wave as a function of hydrostatic prestress.

Numerical simulations for 3D wavefield snapshots show the significant impact of prestressing conditions on the velocity and attenuation of waves. The stress-induced velocity anisotropy is of
orthotropy strongly related to the direction of prestress. In the case of isotropic (confining) prestress, the
stress and strain are the same in all directions, and the stress-induced velocity changes are isotropic. The
fast-P and S waves are stronger in the solid phase and become weaker in the fluid phase, whereas the
slow-P wave becomes rather strong in the fluid phase due to its nature: out-of-phase motion of the solid
phase for the fluid phase. For anisotropic (uniaxial and pure-shear) prestress, the stress-induced velocity
anisotropy makes the fast qP, slow qP, and qS wavefronts ellipses on the xOz and yOz planes with the
z-axis as the major axis. With increasing uniaxial prestress, the azimuthal anisotropy of these waves
becomes more and more obvious, and particularly at $P = 150$ MPa, the phenomenon of S-wave splitting
can be clearly seen due to the enhancement of anisotropy. Both the Padé and conventional
acoustoporoelastic simulations under anisotropic prestress demonstrate similar orthotropic features, but
the acoustoporoelastic wavefronts appear distorted changes due to the effect of excessive anisotropy,
whereas the Padé acoustoporoelastic wavefronts seem normal. The plane-wave analyses and numerical
simulations conducted in this study provide a potential technique for estimating stress-induced inelastic
strains in the propagating media from seismic responses in velocity and anisotropy.

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APPENDIX A

PADÉ ACOUSTOElasticity Equations

We approximate the strain energy function $W$ using Padé as (Fu and Fu, 2017)

$$W = \frac{\lambda + \frac{2\mu}{2} I_1^2 - 2\mu I_2}{1 + al_1} + \frac{l + \frac{2m}{3} I_3^3 - 2mI_1I_2 + nl_3}{1 + bl_4},$$  \hspace{1cm} (A-1)

where $I_1, I_2, I_3$ are the strain invariants of the medium. $\lambda$ and $\mu$ are the lame constant. $l, m, n$ are the
3oeCs of the medium. The factor $1/(1 + al_1)$ and $1/(1 + bl_4)$ introduce large strain perturbations on
the linear and nonlinear strain energies, respectively. $a$ and $b$ are the introduced Padé coefficients, which
cannot be directly expressed by conventional elastic constants, and may be related to the inelastic
deformation in the compliant pores. We can rewrite equation A-1 as

\[ W = \frac{1}{2} c_{ijkl} e_{ij} e_{kl} + \frac{1}{6} c_{ijklmn} e_{ij} e_{kl} e_{mn}, \quad (A-2) \]

where \( c_{ijkl} \) is the 2oeCs tensor and \( c_{ijklmn} \) is the 3oeCs tensor. The Lagrangian strain tensor \( e_{ij} = \frac{1}{2} (u_{ij} + u_{ij}) \), and the relationship with the strain invariants \( I_1, I_2, I_3 \) of the medium is

\[
\begin{align*}
I_1 &= e_{11} + e_{22} + e_{33} \\
I_2 &= \begin{vmatrix}
    e_{11} & e_{12} \\
    e_{22} & e_{23}
\end{vmatrix} + \begin{vmatrix}
    e_{11} & e_{13} \\
    e_{23} & e_{33}
\end{vmatrix} + \begin{vmatrix}
    e_{22} & e_{23} \\
    e_{33} & e_{33}
\end{vmatrix} \\
I_3 &= \begin{vmatrix}
    e_{11} & e_{12} & e_{13} \\
    e_{22} & e_{23} & e_{33}
\end{vmatrix}. 
\end{align*}
\]

(A-3)

According to the derivation method in Pao et al. (1985), we can get Padé acoustoelasticity equation in the initial coordinate system:

\[ A_{ijkl} \frac{\partial^2 u_K}{\partial X_i \partial X_l} = \rho \ddot{u}_i, \quad (A-4) \]

with

\[ A_{ijkl} = \frac{c_{ijkl}}{1 + a L_1} \delta_{ik} + c_{ijkl}, \quad (A-5) \]

where \( u(X,t) \) is the displacement of the medium in the initial coordinate system, \( X \) represents the position vector of the particle under a certain loading, \( C_{ijkl} \) is called equivalent stiffness, depends on material constants and initial displacement field, and has symmetry.

\[
C_{ijkl} = \frac{c_{ijkl}}{1 + a L_1} + \frac{c_{ijklmn}}{1 + b L_1} e_{NN} + \frac{c_{ijklmn}}{1 + a L_1} \partial X_i \partial X_l + \frac{c_{ijklmn}}{1 + a L_1} \partial X_j \partial X_l + \frac{c_{ijklmn}}{1 + a L_1} \partial X_k \partial X_l + \frac{c_{ijklmn}}{1 + a L_1} \partial X_l \partial X_m, \quad (A-6)
\]

where \( e_{NN} = e_{11} + e_{22} + e_{33} \) is the infinitesimal strain. We use Voigt's notation (replace 11 by 1, 22 by 2, 33 by 3, 23 by 4, 31 by 5, and 12 by 6) to contract the indices of \( c_{ijkl} \) and \( c_{ijklmn} \) (\( I, J, K, L, M, N \)

= 1, 2, and 3) to \( c_{ij} \) and \( c_{ijk} \) (\( i, j, k = 1, 2, ..., 6 \)). For an isotropic medium, \( c_{ij} \) can be expressed by the 2oeCs \( \lambda \) and \( \mu \).

\[
\begin{cases}
    c_{11} = c_{22} = c_{33} = \lambda + 2\mu \\
    c_{12} = c_{21} = c_{13} = c_{31} = c_{23} = c_{32} = \lambda \\
    c_{44} = c_{55} = c_{66} = \mu
\end{cases}
\]

(A-7)
Different scholars have used different 3oeCs. According to the conversion relationship between different 3oeCs, $c_{ijk}$ can be expressed by the 3oeCs $(A, B, C)$ as

\[
\begin{align*}
    c_{111} &= c_{222} = c_{333} = 2A + 6B + 2C \\
    c_{144} &= c_{255} = c_{366} = B \\
    c_{112} &= c_{223} = c_{113} = c_{122} = c_{233} = 2B + 2C \\
    c_{155} &= c_{244} = c_{344} = c_{166} = c_{266} = c_{355} = \frac{1}{2}A + B . \\
    c_{123} &= 2C \\
    c_{456} &= \frac{1}{4}A
\end{align*}
\]  \( (A-8) \)

The 3D acoustoeelastic stiffness matrix can be written as \( (\text{Yang et al. 2023}) \)

\[
\begin{align*}
    A_{11} &= \lambda + 2\mu + (3\lambda + 6\mu + 2A + 6B + 2C)e_{11} + (2B + 2C - \lambda - 2\mu)(e_{22} + e_{33}) \\
    A_{12} &= \lambda + (\lambda + 2B + 2C)(e_{11} + e_{22}) + (\lambda + 2C)e_{33} \\
    A_{13} &= \lambda + (\lambda + 2B + 2C)(e_{11} + e_{33}) + (\lambda + 2C)e_{22} \\
    A_{14} &= 2\lambda e_{23} + 2Be_{23}, A_{15} = (2\lambda + 4\mu)e_{13} + (A + 2B)e_{13} \\
    A_{16} &= (2\lambda + 4\mu)e_{12} + (A + 2B)e_{12}, A_{26} = (2\lambda + 4\mu)e_{12} + (A + 2B)e_{12} \\
    A_{22} &= \lambda + 2\mu + (3\lambda + 6\mu + 2A + 6B + 2C)e_{22} + (2B + 2C - \lambda - 2\mu)(e_{11} + e_{33}) \\
    A_{23} &= \lambda + (\lambda + 2C)e_{11} + (\lambda + 2B + 2C)(e_{22} + e_{33}) \\
    A_{24} &= (2\lambda + 4\mu)e_{23} + (A + 2B)e_{23}, A_{25} = 2\lambda e_{13} + 2Be_{13} \\
    A_{33} &= \lambda + 2\mu + (3\lambda + 6\mu + 2A + 6B + 2C)e_{33} + (2B + 2C - \lambda - 2\mu)(e_{11} + e_{22}) \\
    A_{34} &= (2\lambda + 4\mu)e_{33} + (A + 2B)e_{33}, A_{35} = (2\lambda + 4\mu)e_{13} + (A + 2B)e_{13}
\end{align*}
\]  \( (A-9) \)

In the 3D case, we take $A_{11}$ as an example to expand, then substitute equations A-6, A-7 and A-8 into A-5 to get

\[
A_{11}^P = \frac{1}{1 + a l_1} \left( c_{11}e_{11} + c_{22}e_{22} + c_{33}e_{33} \right) + \frac{c_{11}}{1 + a l_1} e_{11} + \frac{c_{111}}{1 + b l_1} e_{11} + \frac{c_{112}}{1 + b l_1} e_{22} + \frac{c_{113}}{1 + b l_1} e_{33} + \frac{4}{1 + a l_1} c_{11}
\]

\[
= \frac{\lambda + 2\mu}{1 + a l_1} + \left( \frac{5\lambda + 10\mu}{1 + a l_1} + \frac{2A + 6B + 2C}{1 + b l_1} \right) e_{11} + \left( \frac{\lambda}{1 + a l_1} + \frac{2B + 2C}{1 + b l_1} \right) e_{22} + e_{33}
\]

\( (A-10) \)

where $l_1 = e_{11} + e_{22} + e_{33}$. Similarly, we can obtain the stiffness matrix based on Padé approximation under prestressed loading:
\[
A_{11}^1 = \frac{\lambda + 2\mu}{1 + a_i} + \left(\frac{5\lambda + 10\mu}{1 + a_i} + \frac{2A + 6B + 2C}{1 + b_i}\right) e_{11} + \left(\frac{\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) (e_{22} + e_{33})
\]
\[
A_{11}^2 = \frac{\lambda}{1 + a_i} + \left(\frac{2\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{11} + \left(\frac{\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{23}
\]
\[
A_{11}^3 = \frac{2\lambda}{1 + a_i} + \left(\frac{2\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{11} + \left(\frac{\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{23}
\]
\[
A_{11}^4 = \frac{2\lambda + 2\mu}{1 + a_i} + \left(\frac{2\lambda + 2\mu}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{11} + \left(\frac{\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) (e_{22} + e_{33})
\]
\[
A_{11}^5 = \frac{2\lambda + 2\mu}{1 + a_i} + \left(\frac{2\lambda + 2\mu}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{11} + \left(\frac{\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{23}
\]
\[
A_{11}^6 = \frac{2\lambda + 2\mu}{1 + a_i} + \left(\frac{2\lambda + 2\mu}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{11} + \left(\frac{\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{23}
\]
\[
A_{11}^7 = \frac{2\lambda + 2\mu}{1 + a_i} + \left(\frac{2\lambda + 2\mu}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{11} + \left(\frac{\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{23}
\]
\[
A_{11}^8 = \frac{2\lambda + 2\mu}{1 + a_i} + \left(\frac{2\lambda + 2\mu}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{11} + \left(\frac{\lambda}{1 + a_i} + \frac{2B + 2C}{1 + b_i}\right) e_{23}
\]

\[
(A - 11)
\]

Conflict of interest

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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