THE EXACT FORMULA FOR THE ELLIPSE PERIMETER

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Abstract

This is the geometric proof paper of the exact formula for the ellipse’s perimeter
ABSTRACT

I has been working on the paper “The New Developed Theorems Based on The Relation Between The Circle and The Ellipses” which will be shortly published it. During this time, I came up a method to calculate the perimeter for the ellipse’s perimeter, which surprisingly there is no exist one. All of the formulas are only approximation. The purpose of this paper is to establish the exact ellipse perimeter formula.

INTRODUCTION

Since Kepler discover the ellipse orbit in the stars. Through the study of the ellipse geometry. We came up with many formulas for the area of the ellipse and others, but there is no exact formula for the ellipse’s perimeter. There are several approximation formulas, but there is not much explanation and proof on how that formula was created. Even the formula of Sir. Srinivasa Ramanujan which is the best and has most accurate result. While working on my paper “The New Developed Theorems Based on The Relation Between The Circle and The Ellipses” which will be shortly published it. I came up a way how to calculate the perimeter of an ellipse based on basic geometric construction.

Key Words: Major axis, Minor axis, Semi-major axis, Semi-minor axis, Projected angle, Transformation angle, Equator tangent line, Isosceles Triangle, Pythagoras Theorem.

THE ELLIPSE PERIMETER EXACT FORMULA.

The ellipse with Semi major axis equals a and Semi minor axis equals b. Then its perimeter P is calculated by the following formula

\[ P = 4a \sqrt{1 + \left( \frac{b}{a} \right)^2 \left( \frac{2}{\pi} \cos^{-1} \left( \frac{b}{a} \right) \right)^2 - 1} \]

Proof

1) Proving the intersected curve line of the plan cut through the cylinder is an ellipse.

Let’s have the tube cylinder tangent to the equator of an inscribed sphere. And M is the equator plane cut through the equator tangent line of the sphere and at bisector of the cylinder as in figure 1a.
Let $N$ is an inclined plane cuts through the cylinder and the sphere passing through the centroid $O$ of the sphere at an angle $\theta$. Figure 1b

![Figure 1](image_url)

$OL = AO'$
$BL = OO'$

$\implies$ The two right triangle angle $\triangle BOL$ and $\triangle OAO'$ are congruence

$AO = OB = a$
$CO = OD = OL = r$
$\cos(\Theta) = \cos(\angle BOL) = r/a$

Figure 1c illustrate the intersect closed loop curve line of the plane $N$ and the tube cylinder. Let $E$ is a point on this intersected closed loop curve line. And $K$ is the projected point of $E$ onto the equator tangent line of the sphere (Figure 1c).

$E(x',y')$ Cartesian coordinate of $E$ on the plane $N$
$K(x,y)$ Cartesian coordinate of $K$ on the plane $M$

$y = y'$
$x = x' \cos(\Theta)$
$OK = OL = r$
$EK = x' \sin(\Theta)$ \hspace{1cm} (1)

Polar parameter equation for $E$ would be
It's the polar parameter equation of the ellipse.

It proves that intersected curve line of the plan cut through the cylinder is an ellipse with

Semi-major axis = a
Semi-minor axis = b = r (The radius of the tube cylinder)

2) The perimeter of the ellipse.

Back to figure (1b), after dividing the cylinder with with the plane N we have two identical objects as in figure 2a below.
If you put these two parts against each other as in figure 2b and rotate them against the side then they will match to each other as a mirroring image because their dimensions are the same.

If we disconnect the 3D shape at A and B at the base of these two objects, and spreading flat down to transform them onto a 2D plane, then we will have a isosceles triangles ABA' and BAB'as in figure 3. The reason for the AB and BA' are straight line because the height equation of EK is a straight line equation (1). If the sector AB is curved then if this sector is convex on one object then it has to be concave on the other subject or vice versa then they won’t be identical. So AB and BA’ have to be straight line.

Figure 3

Simplifying this transformation could be presented as figure 4a to 4b below
Then the perimeter of the ellipse equivalent to the sum of side length AB and BA'. So the perimeter of the ellipse P could be expressed as.

\[ P = AB + BA' = 2AB \]

Regard to triangle ABA' in figure 2b.

\[ BH = h \]

\[ \overline{AB}^2 = \overline{AH}^2 + \overline{BH}^2 \quad (3) \quad \text{Pythagoras theorem on } \triangle AHB \text{ in fig. 2b} \]

\[ \overline{AH} = b \pi \quad \text{the base's circumference.} \]

\[ \Rightarrow P = 2\sqrt{b^2 \pi^2 + h^2} \quad (4) \]

It's checked out that if the angle theta equals zero then h equals zero. So the semi major axis and the semi minor axis are equal the radius of the cylinder.

\[ P = 2b \times \pi \]

In this case the ellipse's a circle with the radius equals the radius of the tube cylinder.

\[ H^2 = 4a^2 - 4b^2 \]

Substitute into equation (4) above we have

\[
P = 2\sqrt{b^2 \pi^2 + 4a^2 - 4b^2} \\
= 2\sqrt{4a^2 + b^2 (\pi^2 - 4)} \\
= 2\sqrt{4a^2 (1 + \frac{b^2}{a^2} (\frac{\pi^2}{4} - 1))} \\
P = 4a \sqrt{1 + (\frac{b}{a})^2 ((\frac{\pi}{2})^2 - 1)}
\]

Again it's checked out for a special case when both semi-major and minor axis are the same then a = b

\[ P = 2a \times \pi \quad \text{It's the circle's circumference} \]

As the semi minor axis approaching zero as long as it is greater than zero then the equation does show the perimeter of the ellipse also approaching 4a. "b" could not equals zero because that would not be ellipse anymore. But for the sake of argument if the circumference is considered as a
complete loop, where it is the distance where you start at one point and return back to the same point, then it’s still work. P is 4a even it is a straight line..

CONCLUSION

With the above basic geometric construction, I proved that the exact formula for the ellipse perimeter is.

\[ P = 4a \sqrt{1 + \left(\frac{b}{a}\right)^2 \left(\frac{c^2}{a^2} - 1\right)} \]

Notes: All of the figure drawing does not reflect the accuracy of the image dimension. They are all estimated to illustrate the calculation concept.

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