Impedance Matching by Tuner Stubs (Single, Double and Triple) Analytical Method

Slobodan Babic 1 and Cevdet Akyel 1

1Affiliation not available

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Abstract

In electronics, microwave and RF engineering, the impedance matching is the practice of designing the input impedance of an electrical load or the output impedance of its corresponding signal source to maximize the power transfer or minimize signal reflection from the load. In many cases, the given load needs to be impedance-matched to a transmission line of given characteristic impedance. Usually, it is 50 $\Omega$. In this approach we will work with the different characteristic impedance.
IMPEDEANCE MATCHING BY TUNER STUBS
(SINGLE, DOUBLE AND TRIPLE)
ANALYTICAL METHOD
INTRODUCTION

In electronics, microwave and RF engineering, the impedance matching is the practice of designing the input impedance of an electrical load or the output impedance of its corresponding signal source to maximize the power transfer or minimize signal reflection from the load. In many cases, the given load needs to be impedance-matched to a transmission line of given characteristic impedance. Usually, it is 50 Ω. In this approach we will work with the different characteristic impedance.

In general, for the matching, the tuning stubs are widely used to match any complex load to a transmission line. Tuning stubs are segments of open-ended or short-ended transmission lines that are used for distributed impedance matching with the line at the appropriate distances from the load and with their appropriate lengths. Both open and short stubs can be used for impedance matching. From the transmission line theory, we know that open and short stubs have a pure imaginary input impedance. In view of circuit topology, tuning stubs can be connected to the transmission line circuit in series or parallel configurations. In view of circuit complexity, there are single-stub, double-stubs and triple-stubs matching network designs.

In the case of the single stub the distance from the load and its length are required for given the load impedance as well as the characteristic impedances of the line and the stub. In many cases these impedences are equal but in this approach they are different. Single stubs can match any load impedance to a transmission line, but the problem is a variable distance between the stab and the load.

To overcome the drawbacks of the single-stub matching technique, the double-stub matching technique is employed. This is way the double stabs are preferable because they are inserted at predetermined locations. Thus, for given distance between stubs and the given positions from the first stub and the load the stubs’ lengths are required. Also, the load impedance is given. These stubs can be connected in parallel or in series with the line as the single stubs. A disadvantage of double stubs is that they cannot match all loads.

To overcome this problem, triple stubs are used. With them all loads can be matched. The distances between the second and the third stub, the second and the first stub are given as well as the distance between the first stub and the load. The stub lengths are required. Even though there is no unique solution for triple stub matching we think that the presented method could be a useful contribution on this subject. Also, we found the unique solution in the triple stub matching under some conditions. As it was mentioned we will work with the different impedances everywhere either in the transmission line or in the tuner.
stubs. The short-circuited stubs are preferred to open circuited stubs because the latter radiate some energy at high frequencies. All combinations either for short stubs or for open stubs can be treated with this approach. We propose this approach as the educational tool for the students and the people which work in this domain (engineers, physicists) because they can make their own MATLAB or MATHEMATICA codes using the formulas given in this work.

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SINGLE AND DOUBLE TUNER STUB MATCHING IN PARALLEL (ANALYTICAL APPROACH-PART I)

1. SINGLE SHORT OR OPEN STUB IN PARALLEL

The load impedance \( Z_L = A + jB, \ (A, B \in R) \) on the transmission line with the characteristic impedance \( Z_0 \) and without losses is to be matched by a single short stub tuner in parallel. The stub is distanced from the load by \( d_1 \), and its length is \( l_1 \). The characteristic impedance of the stub \( Z_S = kZ_0 \). Find the stub’s position from the load and its length to have the perfect matching. We can note the characteristic impedances are different. In the following calculations we will work with the normalized impedances or admittances.

1.1. DETERMINATION OF \( d_1 \)

Let us introduce,

\[
\begin{align*}
    a &= \tan(\beta d_1), a \in R \\
    b &= \tan(\beta l_1), b \in R \\
    y_L &= \frac{1}{z_L} = p_L + jq_L
\end{align*}
\]

where,

\[
\begin{align*}
    p_L &= \frac{A}{Y_0(A^2 + B^2)}, \quad q_L = \frac{-B}{(A^2 + B^2)Y_0}
\end{align*}
\]

To have the matching between \( A-A' \), the following condition must be satisfied,

\[
1 = \frac{1}{z_{A-A'}} = \frac{1}{z_{S0}} + \frac{1}{z_{(d_1)}}, \quad 1 = y_{(d_1)} + y_S
\]  \( (1) \)

where,

\[
\begin{align*}
    y_S &= jB_S, \quad B_S(c-c) = -\frac{1}{kb} \left(\frac{S}{C}\right) \quad \text{and} \quad B_S(c-o) = \frac{b}{k'} \left(\frac{O}{C}\right) \\
    y_{(d_1)} &= \frac{y_L + j \tan(\beta d_1)}{1 + j y_L \tan(\beta d_1)} = g_L + jb_L
\end{align*}
\]
\[ g_L = \frac{p_L (1 + a^2)}{(1 - a q_L)^2 + a^2 p_L^2} \]

\[ b_L = \frac{a (1 - p_L^2 - q_L^2) + q_L (1 - a^2)}{(1 + a q_L)^2 + a^2 p_L^2} \]

![Diagram of a Single Stub (S/C or O/C)](image)

**Fig. 1. Single stub (S/C or O/C)**

\[ 1 = [g_L + j b_L] + j B_{S0} = g_L + j [b_L + B_S] \]  \hspace{1cm} (2)

The condition (2) gives,

\[ g_L = \frac{p_L (1 + a^2)}{(1 - a q_L)^2 + a^2 p_L^2} = 1 \]  \hspace{1cm} (3)

and

\[ b_L + B_S = 0 \text{ or } B_S = -b_L \]  \hspace{1cm} (4)

From (3)

\[ a_{1,2} = \frac{-B Z_0 \pm \sqrt{D}}{Z_0(Z_0 - A)} \]  \hspace{1cm} (5)

where,

\[ D = A Z_0 [B^2 + (A - Z_0)^2] > 0 \forall A, B, Z_0 \]
Now, the distance between the stab and the load is

\[ d_{1}^{(1,2)} = \frac{\lambda}{2\pi} \tan(a_{1,2}) \]  

(6)

### 1.2. DETERMINATION OF \( l_{1} \)

From (4) one has two possibilities,

**a) SHORT STUB (S/C)**

\[ B_{s} = -\frac{1}{kb} = -b_{L} \]

that gives,

\[ b_{1,2} = \frac{1}{kb_{L}(a_{1,2})} \]

\[ l_{1}^{(1,2)} = \frac{\lambda}{2\pi} \tan(b_{1,2}) \]  

(7)

**b) OPEN STUB (O/C)**

\[ B_{s} = \frac{b}{k} = -b_{L} \]

that gives,

\[ b_{1,2} = -kb_{L}(a_{1,2}) \]

\[ l_{1}^{(1,2)} = \frac{\lambda}{2\pi} \tan(b_{1,2}) \]  

(8)

### 1.3. SPECIAL CASES

**1.3.1. SHORT STUB, (S/C)**

\[ A = Z_{0} \]

\[ a_{1} = -\frac{B}{2A}, \quad a_{2} = -\infty \]

\[ b_{1} = -\frac{A}{kB}, \quad b_{2} = \frac{A}{kB} \]

\[ d_{1}^{(1)} = \frac{\lambda}{2\pi} \tan(a_{1}) ; \quad d_{1}^{(2)} = \frac{\lambda}{4} \]
\[ l_1^{(1,2)} = \frac{\lambda}{2\pi} \tan(b_{1,2}) \]

\[ \lambda \text{ is the wavelength.} \]

1.3.2. OPEN STUB, (O/C)

\[ A = Z_0 \]

\[ a_1 = -\frac{B}{2A}, \quad a_2 = -\infty \]

\[ b_1 = \frac{B}{A} k, \quad b_2 = -\frac{B}{A} k \]

\[ d_1^{(1)} = \frac{\lambda}{2\pi} \tan(a_1) \lambda; \quad d_1^{(2)} = \frac{\lambda}{4} \]

\[ l_1^{(1,2)} = \frac{\lambda}{2\pi} \tan(b_{1,2}) \]

From the previous solutions one can see two solutions. Which one to choose? It is recommended to choose the one with the shortest lengths of the transmission line. Shorter transmission lines provide smaller and slightly cheaper matching networks. Moreover, there is a more fundamental reason why we select the solution with the shortest lines because the matching bandwidth is larger. Thus,

\[ d_1 = \min(d_1^{(1)}, d_1^{(2)}) \]

and

\[ l_1 = \min(l_1^{(1)}, l_1^{(2)}) \]

1.4. DETERMINATION OF THE SWR AT ANY SECTION OF THE TRANSMISSION LINE

(a) BETWEEN THE LOAD AND THE FIRST STUB, b) AFTER THE FIRST STUB

a) Before the single stub (close to the load)

\[ \Gamma_{BLoad} = \frac{Z_L - Z_0}{Z_L + Z_0} \]

\[ | \Gamma_{BLoad} | = \frac{| Z_L - Z_0 |}{| Z_L + Z_0 |} = \frac{\sqrt{|A - Z_0|^2 + B^2}}{\sqrt{|A + Z_0|^2 + B^2}} \]
\[ \text{SWR}_{\text{Load}} = \frac{1 + |\Gamma_B|}{1 + |\Gamma_B|} \]

\[ \text{SWR}_{\text{Load}} = \frac{\sqrt{[A + Z_0]^2 + B^2} + \sqrt{[A - Z_0]^2 + B^2}}{\sqrt{[A + Z_0]^2 + B^2} - \sqrt{[A - Z_0]^2 + B^2}} \]  

(9)

b) After the single stub (close to generator)

S/C

\[ y_L = p_L + jq_L \]

\[ p_L = \frac{A}{Y_0(A^2 + B^2)}, \quad q = \frac{-B}{Y_0(A^2 + B^2)} \]

The total admittance just at the junction A-A',

\[ y_{Tj} = y_{d1} + y_s = y_L + j\tan(\beta d) \left( \frac{1}{1 + jy_L\tan(\beta l)} \right) - j \frac{1}{\tan(\beta l)k} \]

\[ y_{Tj} = \left\{ \frac{p_L(1 + a^2)}{(1 - a q_L)^2 + p_L^2 a^2} + j\left[ \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 - a q_L)^2 + p_L^2 a^2} - \frac{1}{bk} \right] \right\} \]

\[ y_{Tj} = P_L + jQ_L \]

\[ p_L = \frac{p_L(1 + a^2)}{(1 - a q_L)^2 + p_L^2 a^2} \]

\[ Q_L\left( \frac{S}{C} \right) = \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 - a q_L)^2 + p_L^2 a^2} - \frac{1}{bk} \]

\[ \Gamma_{\text{Stub}} = \frac{1 - y_{Tj}}{1 + y_{Tj}} \]

\[ |\Gamma_{\text{Stub}}\left( \frac{S}{C} \right)| = \sqrt{\frac{(1 - P_L)^2 + Q_L^2\left( \frac{S}{C} \right)}{(1 + P_L)^2 + Q_L^2\left( \frac{S}{C} \right)}} \]
\[ SWR_{AStub(O,C)} = \frac{1 + |\Gamma_{AStub(O,C)}|}{1 + |\Gamma_{AStub(O,C)}|} \]

If \( a = b = 0 \) (the case just after the load),

\[ |\Gamma_{ALoad}| = \frac{\sqrt{(1 - p_L)^2 + q_L^2}}{\sqrt{(1 + p_L)^2 + q_L^2}} \]

\[ SWR_{ALoad} = \frac{1 + |\Gamma_{ALoad}|}{1 + |\Gamma_{ALoad}|} \]

These expressions for \( |\Gamma_{ALoad}| \) and \( SWR_{ALoad} \) give the same results as those given in a)

**O/C**

\[ y_{Tj} = P_L + jQ_L \]

\[ P_L = \frac{p_L(1 + a^2)}{(1 - aq_L)^2 + p_L^2a^2} \]

\[ Q_L(O,C) = \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 - aq_L)^2 + p_L^2a^2} + \frac{b}{k} \]

\[ |\Gamma_{AStub(O,C)}| = \frac{\sqrt{(1 - P_L)^2 + Q_L^2(O,C)}}{\sqrt{(1 + P_L)^2 + Q_L^2(O,C)}} \]

\[ SWR_{AStub(O,C)} = \frac{1 + |\Gamma_{AStub(O,C)}|}{1 + |\Gamma_{AStub(O,C)}|} \]

**2. EXAMPLES.**

**2.1)** Match a load impedance \( Z_L = (100 + j80)\Omega \) to a line with characteristic impedance \( Z_0 = 75\Omega \) using a single-stub tuner. Find one solution using an open-circuited stub and another using a short-circuited stub for \( Z_S = 75\Omega \). Stub is in parallel with the line. Find SWR before and after the stub.
Solution:

0/C

The Smith chart:

\[ d_1^{(1)} = 0.229 \lambda \quad l_1^{(1)} = 0.377 \lambda \]

\[ d_1^{(2)} = 0.407 \lambda \quad l_1^{(2)} = 0.123 \lambda \]

From this work:

\[ d_1^{(1)} = 0.2276 \lambda \quad l_1^{(1)} = 0.3776 \lambda \]

\[ d_1^{(2)} = 0.4059 \lambda \quad l_1^{(2)} = 0.1224 \lambda \]

S/C

The Smith chart:

\[ d_1^{(1)} = 0.229 \lambda \quad l_1^{(1)} = 0.127 \lambda \]

\[ d_1^{(2)} = 0.407 \lambda \quad l_1^{(2)} = 0.373 \lambda \]

From this work:

\[ d_1^{(1)} = 0.2276 \lambda \quad l_1^{(1)} = 0.127 \lambda \]

\[ d_1^{(2)} = 0.4059 \lambda \quad l_1^{(2)} = 0.3724 \lambda \]

Verification of the SWR before and after the single stub

Near the load:

\[ \Gamma_{BS} = \frac{Z_L - Z_0}{Z_L + Z_0}, \quad |\Gamma_{BS}| = 0.4356 \]

\[ SWR_{BS} = \frac{1+|\Gamma_{BS}|}{1-|\Gamma_{BS}|} = \frac{1 + 0.4356}{1 - 0.4356} = 2.5435 \]

After the single stub near the generator:

\[ z_{Tj} = \frac{Z_{d1}Z_S}{Z_{d1} + Z_S} = (1.0005 + j 3.68 \cdot 10^{-7}) \Omega \]
\[ |\Gamma_{AS}| = 2.8472 \cdot 10^{-4} \]

\[ SWR_{AS} = 1.0006 \]

Thus, there is not the reflected wave, and the calculations are confirmed.

2.2) A shunt single stub tuner is used to match a load impedance to a 50 \( \Omega \) transmission line at 1 GHz. The load consists of a series circuit composed of a 25 \( \Omega \) resistor and a 3.979 (\( nH \)) inductor. A single stub is in parallel with the line.

a) Find the required length and position, in wavelengths, of a short-circuited stub made from a section of the same 50 (\( \Omega \)) line.

b) Repeat (a) if the short-circuited stub is made of a section of a line that has a characteristic impedance of 75 \( \Omega \).

Solution:

\[ Z_L = R + j\omega L = R + j2\pi fL = (25 + j25)(\Omega), Z_0 = 50 (\Omega), k = 1 \]

a) The Smith chart:

\[ d_1^{(1)} = 0, \quad l_1^{(1)} = 0.375\lambda \]

\[ d_1^{(2)} = 0.324\lambda, \quad l_1^{(2)} = 0.125\lambda \]

From this work:

\[ d_1^{(1)} = 0, \quad l_1^{(1)} = 0.375\lambda \]

\[ d_1^{(2)} = 0.3238\lambda, \quad l_1^{(2)} = 0.125\lambda \]

b) The Smith chart:

\[ d_1^{(1)} = 0, \quad l_1^{(1)} = 0.406\lambda \]

\[ d_1^{(2)} = 0.324\lambda, \quad l_1^{(2)} = 0.094\lambda \]

From this work (\( k = 1.5 \)):

\[ d_1^{(1)} = 0, \quad d_2^{(1)} = 0.4064\lambda \]

\[ d_1^{(2)} = 0.3238\lambda, \quad d_2^{(2)} = 0.0936\lambda \]
2.3) Design a single stub matching network to match a load of \( Z_L = (15 + j10)\Omega \) to the 50 \( \Omega \) lossless line. The open stub is in parallel with the line.

Solution:

The Smith chart:

\[ d_1^{(1)} = 0.044\lambda, \quad l_1^{(1)} = 0.147\lambda \]

\[ d_1^{(2)} = 0.387\lambda, \quad l_1^{(2)} = 0.353\lambda \]

From this work:

\[ d_1^{(1)} = 0.0440\lambda, \quad l_1^{(1)} = 0.1473\lambda \]

\[ d_1^{(2)} = 0.3874\lambda, \quad l_1^{(2)} = 0.3527\lambda \]

2.4) Design a single stub matching network to match a load of \( Z_L = (50 + j50)\Omega \) to the 50 \( \Omega \) lossless line. The short stub is in parallel with the line, (Special case- singular case).

Solution:

The Smith chart:

\[ d_1^{(1)} = 0.25\lambda, \quad l_1^{(1)} = 0.125\lambda \]

\[ d_1^{(2)} = 0.427\lambda, \quad l_1^{(2)} = 0.375\lambda \]

From this work:

\[ d_1^{(1)} = 0.2500\lambda, \quad l_1^{(1)} = 0.1250\lambda \]

\[ d_1^{(2)} = 0.4262\lambda, \quad l_1^{(2)} = 0.3750\lambda \]

2.5) For a load impedance \( Z_L = (60 - j80)\Omega \), design two single-stub (short circuit) shunt tuning networks to match this load to a 50 \( \Omega \) lossless line. If the load is matched at 2 GHz and that the load consists of a resistor and capacitor in series (60\( \Omega \), 0.995\( pF \)), plot the reflection coefficient magnitude from 1 to 3 GHz for each solution.

Solution:
The Smith chart,

I) 
\[ d_1^{(1)} = 0.110\lambda, \quad l_1^{(1)} = 0.095\lambda \]

II) 
\[ d_1^{(2)} = 0.260\lambda, \quad l_1^{(2)} = 0.405\lambda \]

From this work:
\[ d_1^{(1)} = 0.1104\lambda, \quad l_1^{(1)} = 0.0950\lambda \]
\[ d_1^{(2)} = 0.2594\lambda, \quad l_1^{(2)} = 0.4050\lambda \]

Obviously (Fig. 2) that solution I) has a significantly better bandwidth than solution II). This is because both \(d\) and \(l\) are shorter for solution I), which reduces the frequency variation of the match.

2.6) An aerial of \((300 - j300)\Omega\) is to be matched with \(600\Omega\) line. The matching is to be done by means of low loss \(600\Omega\) short single stub line.
a) Find the position and the length of the stub line.
b) Find the SWR before and after the stub for both solutions at a).

Solution:
a) 
\[ d_1^{(1)} = 0 \lambda, \quad l_1^{(1)} = 0.125 \lambda \]
\[ d_1^{(2)} = 0.1762 \lambda, \quad l_1^{(2)} = 0.3750 \lambda \]
b)

b1) For the first couple:
\[ |\Gamma_{BeforeS}| = 0.4472 \]
\[ SWR_{BeforeS} = 2.6180 \]

For the second couple:
\[ |\Gamma_{BeforeS}| = 0.4472 \]
\[ SWR_{bBeforeS} = 2.6180 \]

b2) For the first couple:
\[ |\Gamma_{AfterS}| = 0 \]
\[ SWR_{AfterS} = 1 \]

For the second couple:
\[ |\Gamma_{AfterS}| = 5.7541 \cdot 10^{-5} \]
\[ SWR_{AfterS} = 1.0001 \]

Thus, there is not the reflected wave for both cases so that the calculations are confirmed.
3. DOUBLE SHORT OR OPEN STUBS IN PARALLEL

The disadvantage to single-stub tuning is that it is not easy to vary the distance $d_1$ between the load and the stub. Generally new elements can only be connected at the ends of the line and not in between. This difficulty of not having a variable length line can be overcome by using two short, circuited stubs a fixed length apart, as shown in Figure 3. This fixed length is usually $d_2 = 0.375\lambda$. A match is made by adjusting the length of the stubs $l_1$ and $l_2$. One problem with the double-stub tuner is that not all loads can be matched for a given stub spacing.

The load impedance $Z_L = A + jB$, $(A, B \in R)$ on the transmission line with the characteristic impedance $Z_0$ and without losses is to be matched by a double-stub tuner in parallel. The first stub is distanced from the load by $d_1$ and the distance between them is $d_2$. Their characteristic impedences are respectively $Z_{S1} = k_1 Z_0$ and $Z_{S2} = k_2 Z_0$. Find the lengths $l_1$ and $l_2$.

Using the same procedures such as for the single stub configuration, we have,

$$a = \tan(\beta d_1), a \in R$$

$$b = \tan(\beta l_1), a \in R$$
\[ y_L = \frac{1}{z_L} = p_L + jq_L \]

where,
\[ p_L = \frac{A}{Y_0(A^2 + B^2)}, \quad q_L = \frac{-B}{(A^2 + B^2)Y_0} \]

\[ y_{(d_1)} = \frac{y_L + j \tan(\beta d_1)}{1 + jy_L \tan(\beta d_1)} = g_L + jb_L \]

where,
\[ g_L = \frac{p_L(1 + a^2)}{(1 - aq_L)^2 + a^2p_L^2} \]
\[ b_L = \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 - aq_L)^2 + a^2p_L^2} \]

\[ y_{s1} = jB_{s1} \]
\[ y_{(B-B')} = y_{(d_1)} + jB_{s1} = [g_L + j(b_L + B_{s1})] \]

Let us introduce,
\[
m = \tan(\beta d_2), \; m \in R
\]
\[
n = \tan(\beta l_2), \; n \in R
\]

\[ y_{(d_2)} = \frac{y_{(B-B')} + j \tan(\beta d_2)}{1 + jy_{(B-B')} \tan(\beta d_2)} = \frac{g_L + j(b_L + B_{s1} + m)}{1 + jm[g_L + j(b_L + B_{s1})]} \]

\[ y_{s2} = jB_{s2} \]
\[ y_{(A-A')} = y_{(d_2)} + y_{s2} \]

The conditions to have perfect matching are:

\[ Re[y_{(A-A')}'] = \frac{g_L(1 + m^2)}{[1 - m(b_L + B_{s1})]^2 + m^2g_L^2} = 1 \] (12)

\[ Im[y_{(A-A')}'] = -B_{s2} = \frac{m(1 - g_L^2 - (b_L + B_{s1})^2) + (b_L + B_{s1})(1 - m^2)}{[1 - m(b_L + B_{s1})]^2 + m^2g_L^2} \] (13)
From (11) we obtain,

\[ g_L^2m^2 - g_L(1 + m^2) + [1 - (b_L + B_{S1})m]^2 = 0 \] (14)

that gives,

\[ B_{S1}^{(1,2)} = \frac{1 - mB_L \mp \sqrt{g_L[(1 + m^2) - m^2g_L]}}{m} \] (15)

From (15) it is possible to find the region (FORBIDDEN REGION) where the matching is not possible,

\[ [(1 + m^2) - m^2g_L] < 0 \text{ (FORBIDDEN REGION)} \]

Thus, the limits for \( g_L \),

\[ 0 \leq g_L \leq \frac{1 + m^2}{m^2} = \frac{1}{[\sin(\beta d_2)]^2} \] (16)

From (14) and (15) we have,

\[ B_{S2}^{(1,2)} = \frac{1}{m} \left[ 1 \mp \frac{1 + m^2 - m^2g_L}{g_L} \right] \] (17)

### 3.1. DETERMINATION OF \( l_1 \) AND \( l_2 \)

a) SHORT STUBS (S/C)

\[
B_{S1}^{(1,2)} = -\frac{1}{k_1 b_{S1(c-c)}^{(1,2)}}, \quad B_{S2}^{(1,2)} = -\frac{1}{k_2 n_{S2(c-c)}^{(1,2)}}
\]

That gives,

\[
b_{S1(c-c)}^{(1,2)} = \frac{1}{k_1 mB_L - 1 \pm \sqrt{G_L[(1 + m^2) - m^2G_L]}} \] (18)

\[
n_{S2(c-c)}^{(1,2)} = \frac{m}{k_2 \left[ \mp \frac{1 + m^2 - m^2G_L}{G_L} - 1 \right]} \] (19)
b) OPEN STUBS (O/C)

\[ B_{s1}^{(1,2)} = \frac{b_{s1(c-o)}}{k_1}, \quad B_{s2}^{(1,2)} = \frac{n_{s2(c-o)}}{k_2} \]

That gives,

\[ b_{s1(c-o)}^{(1,2)} = k_1 \frac{1 - mB_L + \sqrt{G_L[(1 + m^2) - m^2G_L]}}{m} \]

\[ n_{s2(c-o)}^{(1,2)} = \frac{k_2}{m} \left[ 1 + \sqrt{\frac{1 + m^2 - m^2G_L}{G_L}} \right] \]

Finally,

\[ l_{1(c-o)}^{(1,2)} = \frac{\lambda}{2\pi} \tan^{-1} \left( b_{s1(c-o)}^{(1,2)} \right) \]

\[ l_{2(c-o)}^{(1,2)} = \frac{\lambda}{2\pi} \tan^{-1} \left( n_{s1(c-o)}^{(1,2)} \right) \]

\[ l_{1(c-o)}^{(1,2)} = \frac{\lambda}{2\pi} \tan^{-1} \left( b_{s1(c-o)}^{(1,2)} \right) \]

\[ l_{2(c-o)}^{(1,2)} = \frac{\lambda}{2\pi} \tan^{-1} \left( n_{s1(c-o)}^{(1,2)} \right) \]

It is recommended to choose the shortest lengths of the stubs (smaller and slightly matching networks as well as the matching bandwidth is larger). Thus,

\[ l_1 = \min (l_1^{(1)}, l_1^{(2)}) \]

\[ l_2 = \min (l_2^{(1)}, l_2^{(2)}) \]

SPECIAL CASES

3.2. \( d_1 = d_2 = \frac{\lambda}{4} \)

\[ a = \tan(\beta d_1) = \infty, \quad m = \tan(\beta d_2) = \infty \]
\[ g_L = \frac{p_L}{p_L^2 + q_L^2} \]

\[ b_L = \frac{q_L}{p_L^2 + q_L^2} \]

a) SHORT STUBS

\[ b_{CS1}^{(1,2)} = \frac{1}{k_1} \frac{1}{b_L \pm \sqrt{g[1 - g_L]}} \]

\[ n_{CS2}^{(1,2)} = \pm \frac{1}{k_2} \frac{g_L}{\sqrt{1 - g_L}} \]

b) OPEN STUBS

\[ b_{OS1}^{(1,2)} = k_1[-b_L \mp \sqrt{g_L[1 - g_L]}}] \]

\[ n_{OS2}^{(1,2)} = \mp k_2 \frac{1 - g_L}{g_L} \]

3.3. \( d_1 = \frac{\lambda}{4} \) and \( d_2 \neq \frac{\lambda}{4} \)

\[ a = \tan(\beta d_1) = \infty \]

\[ g_L = \frac{p_L}{p_L^2 + q_L^2} \]

\[ b_L = \frac{q_L}{p_L^2 + q_L^2} \]

c) SHORT STUBS

The same expressions for \( b_{CS1}^{(1,2)} \) and \( n_{CS2}^{(1,2)} \) as in the general case.

d) OPEN STUBS

The same expressions for \( b_{OS1}^{(1,2)} \) and \( n_{OS2}^{(1,2)} \) as in the general case.

3.4. \( d_1 \neq \frac{\lambda}{4} \) and \( d_2 = \frac{\lambda}{4} \)
\[ m = \tan(\beta d_2) = \infty \]

\[ g_L = \frac{p_L(1 + a^2)}{(1 + aq L)^2 + a^2 p_L^2} \]

\[ b_L = \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 + aq L)^2 + a^2 p_L^2} \]

\textbf{e) SHORT STUBS}

\[ b_{CS}^{(1,2)} = \frac{1}{k_1} \frac{1}{b_L \pm \sqrt{g_L[1 - g_L]}} \]

\[ n_{CS}^{(1,2)} = \pm \frac{1}{k_2} \sqrt{\frac{g_L}{1 - g_L}} \]

\textbf{a) OPEN STUBS}

\[ b_{OS}^{(1,2)} = k_1[-B_L \mp \sqrt{g_L[1 - g_L]}] \]

\[ n_{OS}^{(1,2)} = \mp k_2 \frac{1 - g_L}{g_L} \]

\[ l_{CS}^{(1,2)} = \frac{1}{\beta} \tan(b_{CS}^{(1,2)}) = \frac{\lambda}{2\pi} \tan(b_{CS}^{(1,2)}) \]

\[ l_{CS}^{(1,2)} = \frac{1}{\beta} \tan(n_{CS}^{(1,2)}) = \frac{\lambda}{2\pi} \tan(n_{CS}^{(1,2)}) \]

\[ l_{OS}^{(1,2)} = \frac{1}{\beta} \tan(b_{OS}^{(1,2)}) = \frac{\lambda}{2\pi} \tan(b_{OS}^{(1,2)}) \]

\[ l_{OS}^{(1,2)} = \frac{1}{\beta} \tan(n_{OS}^{(1,2)}) = \frac{\lambda}{2\pi} \tan(n_{OS}^{(1,2)}) \]

\textbf{4. THE MAXIMUM DISTANCE BETWEEN THE STUB TUNERS}

The load will be outside the Forbidden region if
\[ g_L \leq 1 + \frac{1}{m^2} = \frac{1}{\sin^2(\beta d_2)} = g_{\text{max}}(m) \]

where,

\[ m = \tan(\beta d_2) \]

I) \( g_L > 1 \)

\[ d_{2\text{max}(1)} = \frac{\lambda}{2\pi} \arcsin \left( \frac{1}{g_{\text{max}}(m)} \right) = \frac{\lambda}{2\pi} \arcsin \left( \frac{1}{g_L} \right) \]

II) \( g_L = 1 \)

\[ d_{2\text{max}(2)} = \frac{\lambda}{4} \]

III) \( 0 < g_L < 1 \)

From \( m = \tan(\beta d_2) \)

\[ d_2 = \frac{\lambda}{2\pi} \arctan(m) \]

\( d_2 \) must be positive so that,

\[ d_2 = \frac{\lambda}{2\pi} \left[ \pi - \arctan(|m|) \right], \text{ for } m < 0 \]

\[ d_2 = \frac{\lambda}{2\pi} \left[ \pi - \arctan(\varepsilon) \right], \text{ for } m \equiv -\varepsilon \]

\[ d_2 = \frac{\lambda}{2\pi} \arctan(m), \text{ for } m \geq 0 \]

where \( \varepsilon \) is exceedingly small constant (for example \( \varepsilon = 10^{-12} \)).

For \( m = 0 \)

\[ d_{2\text{max}(3)} = 0 \]

In this case the stub 2 is the same position as the stub 1 so that this is the case of the single stub.

For \( m > 0 \)

\[ d_{2\text{max}(4)} = 0.25\lambda \]
From the Fig.4, it is also obvious that $d_{2\text{max}}(5) = 0.25\lambda$.

$m < 0$

$$d_{2\text{max}}(6) = (0.5 - \varepsilon)\lambda$$

From the Fig.5 it is also obvious that $d_{2\text{max}}$ is remarkably close to $d_{2\text{max}} = 0.5\lambda$. 
Thus,

\[ d_{2\text{max}} = \max(d_{2\text{max}(i)}, i = 0,1,\ldots,6) = (0.5 - \varepsilon)\lambda \]

5. **The Minimum Length Between The Load And The First Stub In The Double Stub Matching**

The condition for which the load is outside the FORBIDDEN REGION:

\[ g_L \leq \frac{1 + m^2}{m^2} = \frac{1}{[\sin(\beta d_2)]^2} = \frac{1}{Q} \]

where,

\[ Q = \frac{m^2}{1 + m^2} < 1 \]

\[ g_L = \frac{p_L(1 + a^2)}{(1 - aq_L)^2 + a^2p_L^2} \leq \frac{1}{Q} \]

\[ p_L = \frac{A}{Y_0(A^2 + B^2)}, \quad q_L = \frac{-B}{(A^2 + B^2)Y_0} \]

or

\[ (1 - aq_L)^2 + a^2p_L^2 \geq Qp_L(1 + a^2) \]

The minimum length between the load and the first stub in double stub matching can be obtained from the equation:

\[ (1 - aq_L)^2 + a^2p_L^2 = Qp_L(1 + a^2) \]

\[ a^2[p_L^2 + q_L^2 - Qp_L] - 2q_la + 1 - Qp_L = 0 \]

\[ a_{1,2} = \frac{q_L \pm \sqrt{D}}{p_L^2 + q_L^2 - Qp_L} \tag{22} \]

\[ D = q_L^2 - (1 - Qp_L)[p_L^2 + q_L^2 - Qp_L] = Qp_L[p_L^2 + q_L^2 + 1 - Qp_L] - p_L^2 \]

or

\[ \Delta = p_L\{Q[p_L^2 + q_L^2 + 1 - Qp_L] - p_L\} \]

1) If \( p_L^2 + q_L^2 = Qp_L \)
\[ \Delta = p_L \{ Q - p_L \} \]

\[ a_1 = \infty, \quad d_{1(1)} = \frac{\lambda}{4} \]

\[ a_2 = \frac{1}{2q_L}, \quad d_{1(2)} = \frac{\lambda}{2\pi} \tan (a_2) \]

2) If \( \Delta \geq 0 \)

\[ \tan (\beta d_{1(1,2)}) = a_{1,2} = \frac{q_L \pm \sqrt{D}}{p_L^2 + q_L^2 - Qp_L} \]

\[ d_{1(1,2)} = \frac{\lambda}{2\pi} \tan (a_{1,2}) \]

We choose \( d_{\text{min}} = \min (d_{1(1)}, d_{1(2)}) \) \hspace{1cm} (23)

6. EXAMPLES

6.1) A load consists of a 4 (\( nH \)) inductor which has a series internal resistance of 19.2 (\( \Omega \)).

a) This load needs to be matched to a 50 (\( \Omega \)) lossless co-axial transmission line by means of a double stub matching network, consisting of two short-circuit stubs, spaced 0.375\( \lambda \) apart. The stub nearest to the load is 0.1\( \lambda \) away from it. Determine the possible combinations of stub lengths which are required to match the load to the line at the operating frequency of 1835 (\( MHz \)). You may assume all line sections and stubs are 50 (\( \Omega \)).

b) Repeat(a) if the circuit stubs are open.

Solution:

\[ Z_L = (19.2 + j46.17)\Omega, \quad k_1 = k_2 = 1 \]

\[ d_1 = 0.1 \lambda, \quad d_2 = 0.375 \lambda \]

a) SHORT STUBS

Smith chart:

\[ l_1^{(1)} = 0.096\lambda \quad l_2^{(1)} = 0.039\lambda \]

\[ l_1^{(2)} = 0.212\lambda \quad l_2^{(2)} = 0.425\lambda \]

From this work:
\[ l_1^{(1)} = 0.0959\lambda \quad l_2^{(1)} = 0.0393\lambda \]
\[ l_1^{(2)} = 0.2122\lambda \quad l_2^{(2)} = 0.4250\lambda \]

b) OPEN STUBS

Smith chart:
\[ l_1^{(1)} = 0.346\lambda \quad l_2^{(1)} = 0.289\lambda \]
\[ l_1^{(2)} = 0.462\lambda \quad l_2^{(2)} = 0.175\lambda \]

From this work:
\[ l_1^{(1)} = 0.3459\lambda \quad l_2^{(1)} = 0.2893\lambda \]
\[ l_1^{(2)} = 0.4622\lambda \quad l_2^{(2)} = 0.1750\lambda \]

6.2) The load impedance \( Z_L = (60 - j80)\) \(\Omega\) on a 50 \(\Omega\) line is to be matched by a double-stub tuner of 0.125\(\lambda\) spacing. What stub lengths \(l_1\) and \(l_2\) are necessary? Stubs are shorted with the same characteristic impedance as the line. \([d_1 = 0; \quad d_2 = 0.125\lambda]\). 

Solution:

\[ k_1 = k_2 = 1, \quad \text{SHORT STUBS} \]

Smith chart:
\[ l_1^{(1)} = 0.232\lambda \quad l_2^{(1)} = 0.1\lambda \]
\[ l_1^{(2)} = 0.396\lambda \quad l_2^{(2)} = 0.454\lambda \]

From this work:
\[ l_1^{(1)} = 0.2319\lambda, \quad l_2^{(1)} = 0.0998\lambda \]
\[ l_1^{(2)} = 0.3965\lambda, \quad l_2^{(1)} = 0.4542\lambda \]

From [1]:
\[ l_1^{(1)} = 0.2319\lambda, \quad l_2^{(1)} = 0.0998\lambda \]
\[ l_1^{(2)} = 0.3965\lambda, \quad l_2^{(1)} = 0.4542\lambda \]
Design a double-stub shunt tuner to match a load impedance \( Z_L = (60 - j 80) \Omega \) to a 50 \( \Omega \) line. The stubs are to be open-circuited stubs and are spaced \( \lambda/8 \) apart. If this load consists of a series resistor and capacitor \((60 \Omega, 0.995 \mu F)\), and the match frequency is 2 GHz, plot the reflection coefficient magnitude versus frequency from 1 to 3 GHz.

Solution:

Smith chart:

\[
l_1^{(1)} = 0.146 \lambda, \quad l_2^{(1)} = 0.204 \lambda
\]

\[
l_1^{(2)} = 0.483 \lambda, \quad l_2^{(2)} = 0.350 \lambda
\]

From this work: \( k_1 = k_2 = 1 \)

\[
l_1^{(1)} = 0.1465 \lambda, \quad l_2^{(1)} = 0.2042 \lambda
\]

\[
l_1^{(2)} = 0.4819 \lambda, \quad l_2^{(2)} = 0.3498 \lambda
\]

From [1]:

\[
l_1^{(1)} = 0.1465 \lambda, \quad l_2^{(1)} = 0.2042 \lambda
\]

\[
l_1^{(2)} = 0.4819 \lambda, \quad l_2^{(2)} = 0.3498 \lambda
\]

Fig. 6. Reflection coefficient magnitudes versus frequency for the tuning circuits.
The first solution has a much narrower bandwidth than the second (primed) solution because both stubs for the second solution are somewhat longer (and closer to $0.5\lambda$) than the stubs of the first solution (Fig 6).

6.4) The load impedance $Z_L = (50 + j50)\ \Omega$ on a 50 $\Omega$ line is to be matched by a double-stub tuner of $0.375\lambda$ spacing. What stub lengths $l_1$ and $l_2$ are necessary? Stubs are shorted with the same characteristic impedance as the line. [$d_1 = 0; \ d_2 = 0.375\lambda$].

Solution:

Smith Chart:

$l_1^{(1)} = 0.305\ \lambda$, $l_1^{(1)} = 0.349\ \lambda$

$l_1^{(2)} = 0.1\ \lambda$, $l_2^{(2)} = 0.056\ \lambda$

From this work:

$l_1^{(1)} = 0.3058\lambda$, $l_2^{(1)} = 0.3506\ \lambda$

$l_1^{(2)} = 0.1006\lambda$, $l_2^{(2)} = 0.0558\ \lambda$

6.5) The terminating impedance is $Z_L = (100 + j100)\ \Omega$ and the characteristic impedance $Z_0$ of the line and the stubs is 50 $\Omega$. The first stub is away $0.4\ \lambda$ from the load. The spacing between the two stubs is stub $3/8\ \lambda$. Determine the length of the short-circuited stubs when the match is achieved.

Solution:

Smith chart:

$l_1^{(1)} = 0.373\ \lambda$, $l_2^{(1)} = 0.337\ \lambda$

$l_1^{(2)} = 0.143\ \lambda$, $l_2^{(2)} = 0.058\ \lambda$

This work:

$l_1^{(1)} = 0.3720\ \lambda$, $l_2^{(1)} = 0.3403\ \lambda$

$l_1^{(2)} = 0.1410\ \lambda$, $l_2^{(2)} = 0.0577\ \lambda$

6.6) The terminating impedance is $Z_L = (50 + j100)\ \Omega$ and the characteristic impedance $Z_0$ of the line and the stubs is 100 $\Omega$. The first stub is away $0.25\ \lambda$ from the load. The spacing between the two stubs is stub $1/8\ \lambda$. If the length of
the second stub is 0.15 \lambda. Determine the other lengths of the short-circuited stubs when the match is achieved.

Solution:
This is the special case (a = \infty)
This work:
\[ l_1^{(1)} = 0.1364 \lambda, \quad l_2^{(1)} = 0.1494 \lambda \]
\[ l_1^{(2)} = 0.3636 \lambda, \quad l_2^{(2)} = 0.4442 \lambda \]

By the technique Smith Chart: \[ l_2^{(1)} = 0.15 \lambda. \]
This work gives: \[ l_2^{(1)} = 0.1494 \lambda. \]

6.7) For each of the following sets of values of and associated with the double-stub matching technique, determine whether it is possible to achieve a match between the line and the load:

a) \( z_{LN} = (0.3 + j0.4) \Omega, d_1 = 0; \quad d_2 = 0.375 \lambda]. \\
b) \( z_{LN} = 0.5 \Omega, d_1 = 0.125 \lambda; \quad d_2 = 0.375 \lambda]. \\
c) \( z_{LN} = (2.5 - j0.5) \Omega, d_1 = 0.25 \lambda; \quad d_2 = 5 \lambda/8]. \\

Solution:

a) Yes. it is possible to achieve a match between the line and the load because,
\[ [(1 + m^2) - m^2 g_L] = (1 + 1) - 1 \cdot 1.2 = 0.8 > 0 \]
The load is outside of the Forbidden Region.
\[ l_1^{(1)} = 0.4102 \lambda, \quad l_2^{(1)} = 0.2211 \lambda \]
\[ l_1^{(2)} = 0.1922 \lambda, \quad l_2^{(2)} = 0.0801 \lambda \]

b) Yes. It is possible to achieve a match between the line and the load because,
\[ [(1 + m^2) - m^2 g_L] = (1 + 1) - 1 \cdot 0.8 = 1.2 > 0 \]
The load is outside of the Forbidden Region.

\[ l_1^{(1)} = 0.3336 \lambda, \quad l_2^{(1)} = 0.2852 \lambda \]

\[ l_1^{(2)} = 0.0998 \lambda, \quad l_2^{(2)} = 0.0672 \lambda \]

c) No. It is not possible to achieve a match between the line and the load because,

\[ [(1 + m^2) - m^2 g_L] = (1 + 1) - 1 \cdot 2.5 = -0.5 < 0 \]

The load is inside of the Forbidden Region.

6.8) In an experiment to determine the unknown impedance of a microwave antenna, a slotted section of transmission line is first connected to a short-circuit load. In this case, adjacent voltage minima are found 23.6 cm and 35.4 cm away from the load. After replacing the short-circuit with antenna, the location of minima shifted to points 27.8 and 39.6 cm, respectively, away from the load with the SWR equal to 2.6. If the slotted transmission line is lossless of characteristic impedance \( Z_0 = 50(\Omega) \) and the velocity of propagation \( v = c \), determine the following:

a) Operating frequency,

b) Unknown impedance of microwave antenna,

c) It is required to match the antenna in part a section of transmission line of characteristic impedance \( Z_0 = 50(\Omega) \). Show that the double-stub tuner available in the laboratory and shown in Figure 4 is not suitable to achieve the required matching.

d) It is suggested that we insert a length \( d \) of a 50(\Omega) lossless transmission line between the antenna and the side A-A of the double stub tuner. Find the \textit{minimum} length of such a transmission-line section and the lengths \( l_1 \) and \( l_2 \) of the stubs required to achieve matching.

Solution:

a) Short-circuit

\[ d_{\min}^{(1)} = 23.6 \text{ cm} = 0.236 \text{ m} \]

\[ d_{\min}^{(2)} = 35.4 \text{ cm} = 0.354 \text{ m} \]

\[ d_{\min}^{(n)} = \frac{\lambda}{4\pi} [\theta_\rho + (2n + 1)\pi] \]
\[
d_{\text{min}}(2) - d_{\text{min}}(1) = \frac{\lambda}{4\pi} 2\pi = \frac{\lambda}{2}
\]
\[
\lambda = 2[d_{\text{min}}(2) - d_{\text{min}}(1)] = 0.236 \, m
\]
\[
f = \frac{c}{\lambda} = 1.2712 \, (GHz)
\]

b) Circuit with antenna

\[
d_{\text{min}}(1) = 27.8 \, cm = 0.278 \, m
\]
\[
d_{\text{min}}(2) = 39.6 \, cm = 0.396 \, m
\]
\[
d_{\text{min}}(n) = \frac{\lambda}{4\pi} [\theta_\rho + (2n + 1)\pi]
\]
\[
d_{\text{min}}(1) = 0.278 = \frac{\lambda}{4\pi} [\theta_\rho + 3\pi]
\]
\[
\theta_\rho = 4.712\pi - 3\pi = 1.712\pi = 2\pi - 0.288\pi
\]
\[
\text{SWR} = \frac{1 + |\rho|}{1 - |\rho|}
\]
\[
|\Gamma| = \frac{\text{SWR} - 1}{\text{SWR} + 1} = \frac{1.6}{3.6} = 0.444
\]
\[
\Gamma = |\Gamma| e^{j\theta_\rho} = 0.444 e^{j(2\pi - 0.288\pi)} = 0.444 e^{-j0.288\pi}
\]
\[
\Gamma = 0.274 - j0.349
\]
\[
Z_{\text{antenna}} = Z_0 \frac{1 + \Gamma}{1 - \Gamma} = 50 \frac{1.274 - j0.349}{0.727 + j0.349} = (60 - j55)\Omega = Z_L
\]

c) With this impedance and stub location either on side A-A and B-B the matching can not be achieved because the loid is in the FORBIDDEN REGION.

d) We will put between the antenna and A-A the transmission line of a 50(\Omega), with the length \( l = x\lambda \) so that the distance from A-A to the first stab is \( d_1 = 0.11\lambda + l = (0.11 + x)\lambda \), and \( a = \tan[\beta(0.11 + x)\lambda] \).
Using (21) and (22) where the minimum length $d_1$ between the first stab and the load was calculated we obtain,

$$a_1 = -26.4506$$
$$a_2 = 0.6926$$

I)

$$\tan[\beta(0.11 + x)\lambda] = -26.4506$$
$$0.11 + x = 0.2560142$$

$$x = 0.1460142 \text{ or } d = 0.1460142\lambda, \quad d_{1(1)} = 0.2560142\lambda$$

II)

$$\tan[\beta(0.11 + x)\lambda] = 0.6926$$
$$0.11 + x = 0.096407$$

$$x = -0.013593 \text{ or } d = -0.013593\lambda \quad (\text{reject})$$

We choose,

$$x = 0.1460142 \quad \text{or} \quad d = 0.1460142\lambda, \quad d_{1(1)} = 0.2560142\lambda$$

By Smith chart:

$$d = 0.15\lambda$$

The lengths of the stabs by Smith chart are,

$$l_{1(1)}^{(1)} = 0.4\lambda, \quad l_{1(1)}^{(1)} = 0.303\lambda$$

By this work,

$$l_{1(1)}^{(1)} = l_{1(2)}^{(1)} = 0.401\lambda, \quad l_{2(1)}^{(1)} = l_{2(2)}^{(1)} = 0.300\lambda$$

6.9) A load of $(30 - j40)\Omega$ is connected to a $50\Omega$ line. The stubs are $\lambda/8$ apart and the stub 1 is $0.3\lambda$ from the load.
a) Find the lengths of the stubs required for the smallest SWR between stubs.
b) Also find the value of SWR between the load and the first stub, and between stubs.
c) Verify the SWR of the total matching after the second stub which must be 1.

Solution:

a) Smith chart:
\[ l_1^{(1)} = 0.34 \lambda, \quad l_2^{(1)} = 0.129 \lambda \]

This work:
\[ l_1^{(1)} = 0.3400 \lambda, \quad l_2^{(1)} = 0.1270 \lambda \]
\[ l_1^{(2)} = 0.4333 \lambda, \quad l_2^{(2)} = 0.4484 \lambda \]

b) SWR near the load before the first stub (for the first couple)
\[ |\Gamma_{BS1}| = 0.5 \]
\[ SWR_{BS1} = 3 \]

SWR between the stubs
\[ |\Gamma_{S1-S2}| = 0.4383 \]
\[ SWR_{S1-S2} = 2.5605 \]

c) SWR after the second stub the stubs
\[ |\Gamma_{AS2}| = 4.7277 \cdot 10^{-4} \]
\[ SWR_{AS2} = 1.0009 \]

Thus, there is not the reflected wave, so that the calculations are confirmed.

6.10) A double stub tuner is illustrated where the positions of the stubs on the transmission line are fixed but the short stub of the lengths of the \( l_1 \) and \( l_2 \) are variable. If the load is \( Z_L = (50 + j100)\Omega \) and the characteristic impedances \( Z_0 = 100\Omega \) is equal either of the line or the stubs. If \( d_1 = \lambda/4 \) and \( d_2 = \lambda/8 \). Determine the shortest lengths \( l_1 \) and \( l_2 \) to give no reflected wave at the point
A double stub is used to match a load of impedance $Z_L = (100 + j100)\Omega$ to a lossless transmission line with $Z_0 = 300\Omega$. The stubs are separated by $3\lambda/8$ and one of the stubs is located at the load.

a) Obtain the lengths of the two stubs if they are terminated by an open circuit. 
b) What is the value of the SWR in the main line and in the section between the two stubs? 
Verify the SWR after the second stub.

Solution:

a) O/C

$$l_1 = 0.1494\lambda \quad l_2 = 0.4364\lambda$$

$$l_1 = 0.4442\lambda \quad l_2 = 0.3399\lambda$$
b) For the second solution:

\[ |\Gamma_{BS2}| = 0.6192 \]

\[ SWR_{BS2} = \frac{1+|\Gamma_{BS2}|}{1-|\Gamma_{BS2}|} = 4.2521 \]

\[ Z_{Tj2} = 78.5220 + j97.6663 \]

c)

\[ |\Gamma_{AS2}| = 3.5795 \cdot 10^{-4} \]

\[ SWR_{AS2} = \frac{1+|\Gamma_{AS2}|}{1-|\Gamma_{AS2}|} = 1.0007 \]

Thus, there is not the reflected wave, so that the calculations are confirmed.

6.12) For the double-stub tuner, find the shortest values of \( l_1 \) and \( l_2 \) to match the load if \( Z_L = (100 + j50)\Omega \), \( Z_0 = Z_{S1} = Z_{S2} = 50\Omega \). The distances between the stubs and between the first stub are equal \( d_1 = d_2 = \lambda/8 \). The stubs are short circuits.

Solution:

The Smith chart:

\[ l_1 = 0.194\lambda, \quad l_2 = 0.15\lambda \]

\[ l_1 = 0.4\lambda, \quad l_2 = 0.442\lambda \]

The other possible design, where the respective lengths are found to be 0.4\( \lambda \) and 0.442\( \lambda \), is not recommended.

This work:

\[ l_1 = 0.1942\lambda, \quad l_2 = 0.1494\lambda \]

\[ l_1 = 0.3994\lambda, \quad l_2 = 0.4442\lambda \]

Thus, we choose also the first couple. All results are in the excellent agreement.

6.13) A lossless 100\( \Omega \) transmission line is to be matched with a \( (100 + j100)\Omega \) load using a double-stub tuner. Separation between the two stubs is \( \lambda/8 \) and the characteristic impedance is 100\( \Omega \). A load is connected right at the location
of the first stub. Determine the shortest possible lengths of the two stubs to obtain the matched condition and find the SWR between the two stubs.

Solution:

From this work: \( k_1 = k_2 = 1 \)

\[
l_1 = 0.3399\lambda, \quad l_2 = 0.1494\lambda
\]

\[
l_1 = 0.4364\lambda, \quad l_2 = 0.4442\lambda
\]

The first couple must be chosen.

\[
| \Gamma_{BetweenS1S2} | = 0.7454
\]

\[
SWR_{BS1S2} = \frac{1+| \Gamma_{BBetweenS1S2} |}{1-| \Gamma_{BetweenS1S2} |} = 6.8541
\]

The SWR after the second stub must be 1.

\[
Z_{TJS2} = (186.65 - j49.924)\Omega
\]

\[
| \Gamma_{AS2} | = 3.2507 \cdot 10^{-4}
\]

\[
SWR_{AS2} = \frac{1+| \Gamma_{AS2} |}{1-| \Gamma_{AS2} |} = 1.0007
\]

6.14) A lossless \( Z_0 = 75\Omega \) transmission line is to be matched with a \((150 + j15)\Omega\) load using a shunt-connected double-stub tuner. Separation between the two stubs is \( \lambda/8 \) and the characteristic impedance is \( Z_{S1} = Z_{S2} = 75\Omega \). The stub closest to the load (first stub) is \( \lambda/2 \) away from it.

a) Determine the shortest possible lengths of the two stubs to obtain the matched condition and find the SWR on each section of the transmission line.

b) Repeat (a) if \( Z_{S1} = Z_{S2} = 100\Omega \).

Solution:

a) \( k_1 = k_2 = 1 \)

\[
l_1 = 0.2793\lambda, \quad l_2 = 0.1482\lambda
\]
\[ l_1 = 0.4233\lambda, \quad l_2 = 0.4444\lambda \]

The first couple must be chosen.

\[ |\Gamma_{\text{Load}}| = 0.3392 \]

\[ \text{SWR}_{BS2} = \frac{1+|\Gamma_{\text{Load}}|}{1-|\Gamma_{\text{Load}}|} = 2.0266 \]

\[ Z_{TJS1} = 150.00 + j15.00 \]

\[ |\Gamma_{S1S2}| = 0.3485 \]

\[ \text{SWR}_{BS2} = \frac{1+|\Gamma_{S1S2}|}{1-|\Gamma_{S1S2}|} = 2.0696 \]

\[ Z_{TJS2} = (140.77 - j38.871)\Omega \]

\[ |\Gamma_{AS2}| = 3.0775 \cdot 10^{-4} \]

\[ \text{SWR}_{BS2} = \frac{1+|\Gamma_{AS2}|}{1-|\Gamma_{AS2}|} = 1.0006 \]

b) \( k_1 = k_2 = 1.333 \)

\[ l_1 = 0.2888\lambda, \quad l_2 = 0.1257\lambda \]

\[ l_1 = 0.4405\lambda, \quad l_2 = 0.4575\lambda \]

The first couple must be chosen.

\[ |\Gamma_{\text{Load}}| = 0.3392 \]

\[ \text{SWR}_{\text{Load}} = \frac{1+|\Gamma_{\text{Load}}|}{1-|\Gamma_{\text{Load}}|} = 2.0266 \]

\[ Z_{TJS1} = (150.00 + j15.00)\Omega \]

\[ |\Gamma_{S1S2}| = 0.3485 \]
\[ SWR_{BS2} = \frac{1 + |\Gamma_{S1S2}|}{1 - |\Gamma_{S1S2}|} = 2.0696 \]

\[ Z_{TJS2} = (140.72 - j\,38.955)\Omega \]

\[ |\Gamma_{AS2}| = 2.3194 \times 10^{-4} \]

\[ SWR_{AS2} = \frac{1 + |\Gamma_{AS2}|}{1 - |\Gamma_{AS2}|} = 1.0005 \]

6.15) A lossless \( Z_0 = 50\,\Omega \) transmission line is to be matched with a \((25 + j50)\Omega\) load using an open double-stub tuner. Separation between the two stubs is \(3\lambda/8\) and the characteristic impedance is \( Z_{S1} = Z_{S2} = 50\,\Omega \). The stub closest to the load (first stub) is \(0.2\lambda\) away from it.

a) Determine the shortest possible lengths of the two stubs to obtain the matched condition and find the SWR on each section of the transmission line.

b) Repeat (a) if \( Z_{S1} = 75\,\Omega, \ Z_{S2} = 100\,\Omega \).

Solution:

a) \( k_1 = k_2 = 1 \)

\[ l_1 = 0.3904\lambda, \quad l_2 = 0.1476\lambda \]

\[ l_1 = 0.3160\lambda, \quad l_2 = 0.2964\lambda \]

The first couple must be chosen.

\[ |\Gamma_{A\text{Load}}| = 0.6202 \]

\[ SWR_{A\text{Load}} = \frac{1 + |\Gamma_{A\text{Load}}|}{1 - |\Gamma_{A\text{Load}}|} = 4.2656 \]

\[ Z_{TJS1} = (25 + j50)\Omega \]

\[ |\Gamma_{S1S2}| = 0.5547 \]

\[ SWR_{BS2} = \frac{1 + |\Gamma_{S1S2}|}{1 - |\Gamma_{S1S2}|} = 3.4918 \]
\[ Z_{TJS2} = 90.1135 + j80.0173 \]

\[ |\Gamma_{AS2}| = 7.1621 \cdot 10^{-4} \]

\[ SWR_{AS2} = \frac{1 + |\Gamma_{AS2}|}{1 - |\Gamma_{AS2}|} = 1.0014 \]

b) \( k_1 = 1.5, \ k_2 = 2 \)

\[ l_1 = 0.3583 \lambda, \quad l_2 = 0.1929 \lambda \]

\[ l_1 = 0.2954 \lambda, \quad l_2 = 0.2737 \lambda \]

The first couple must be chosen.

\[ |\Gamma_{A\text{Load}}| = 0.6202 \]

\[ SWR_{BS2} = \frac{1 + |\Gamma_{A\text{Load}}|}{1 - |\Gamma_{A\text{Load}}|} = 4.2656 \]

\[ Z_{TJS1} = (25 + j50)\Omega \]

\[ |\Gamma_{S1S2}| = 0.5549 \]

\[ SWR_{BS2} = \frac{1 + |\Gamma_{S1S2}|}{1 - |\Gamma_{S1S2}|} = 3.4929 \]

\[ Z_{TJS2} = (89.9492 + j80.0367)\Omega \]

\[ |\Gamma_{AS2}| = 3.4674 \cdot 10^{-4} \]

\[ SWR_{AS2} = \frac{1 + |\Gamma_{AS2}|}{1 - |\Gamma_{AS2}|} = 1.0007 \]
7. SINGLE SHORT OR OPEN STUB IN SERIES

The load impedance \( Z_L = A + jB, \ (A, B \in R) \) (Fig 7.) on the transmission line with the characteristic impedance \( Z_0 \) and without losses is to be matched by a single short stub tuner in parallel. The stub is distanced from the load by \( d_1 \) and its length is \( l_1 \). The characteristic impedance of the stub \( Z_S = kZ_0 \). Find the stub’s position from the load and its length to have the perfect matching. We can note the characteristic impedances are different. In the following calculations we will work with the normalized impedances or admittances.

7.1. DETERMINATION OF \( d_1 \)

Let’s introduce,

\[
\begin{align*}
    a &= \tan(\beta d_1), \ a \in R \\
    b &= \tan(\beta l_1), \ b \in R \\
    z_L &= p_L + jq_L \\
    p_L &= \frac{A}{Z_0}, \quad q_L = \frac{B}{Z_0}
\end{align*}
\]
To have the matching between $A-A'$, the following condition must be satisfied,

$$1 = z_{A-A'} = z_{S0} + z_{(d_1)}, \quad 1 = z_{(d_1)} + z_S$$

where

$$z_S = jz_{S0}, \quad z_{S0(c-c)} = kb, \quad \left(\frac{S}{C}\right) \quad \text{and} \quad z_{S0(c-o)} = -\frac{1}{bk} \left(\frac{O}{C}\right)$$

$$z_{(d_1)} = \frac{z_L + j\tan(\beta d_1)}{1 + jz_L\tan(\beta d_1)} = r_L + jx_L$$

$$r_L = \frac{p_L(1 + a^2)}{(1 - aq_L)^2 + a^2p_L^2}$$

$$x_L = \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 - aq_L)^2 + a^2p_L^2}$$

$$1 = [r_L + jx_L] + jx_{S0} = r_L + j[x_L + x_{S0}]$$ (25)

The condition (2) gives,

$$r_L = \frac{p_L(1 + a^2)}{(1 - aq_L)^2 + a^2p_L^2} = 1$$ (26)

and

$$x_L + x_{S0} = 0 \quad \text{or} \quad x_{S0} = -x_L$$ (27)

From (26)

$$a_{1,2} = \frac{-BZ_0 \pm \sqrt{D}}{Z_0(Z_0 - A)}$$ (27)

where,

$$D = AZ_0[B^2 + (A - Z_0)^2] > 0 \quad \forall A, B, Z_0$$

Now, the distance between the stab and the load is
\[ d_1^{(1,2)} = \frac{\lambda}{2\pi} \tan(a_{1,2}) \]  

(28)

7.2. DETERMINATION OF \( l_1 \)

From (4) one has two possibilities,

c) SHORT STUB (S/C) 

\[ x_{S0} = kb = -x_L \]

That gives, 

\[ b_{1,2} = -kx_L(a_{1,2}) \]

\[ l_1^{(1,2)} = \frac{\lambda}{2\pi} \tan(b_{1,2}) \]  

(29)

d) OPEN STUB (O/C) 

\[ x_{S0} = -\frac{1}{kb} = -x_L \]

That gives, 

\[ b_{1,2} = \frac{1}{kb_L(a_{1,2})} \]

\[ l_1^{(1,2)} = \frac{\lambda}{2\pi} \tan(b_{1,2}) \]  

(30)

7.3. SPECIAL CASES

7.3.1. SHORT STUB (S/C)

a) \( A = Z_0 \)

\[ a_1 = \frac{2A}{B}, \quad a_2 = 0 \]

\[ b_1 = \frac{B}{kA}, \quad b_2 = -\frac{B}{kA} \]

and

\[ d_1^{(1)} = \frac{\lambda}{2\pi} \tan(a_1), \quad d_1^{(2)} = 0 \]
\[ l_1^{(1)} = \frac{\lambda}{2\pi} \tan(b_1), \quad l_1^{(2)} = \frac{\lambda}{2\pi} \tan(b_2) \]

b) \( A^2 + B^2 - AZ_0 = 0 \) et \( Z_0 > A \)

\[ a_1 = \infty, \quad a_2 = \frac{Z_0 - A}{B + \sqrt{A(Z_0 - A)}} \]

\[ b_1 = \frac{Z_0 B}{(A^2 + B^2)k} = \frac{B}{Ak}, \quad b_2 = \frac{1}{k} \frac{B Z_0 a_2^2 + a_2(A^2 + B^2 - Z_0^2) - B Z_0}{(Z_0 - a_2 B)^2 + A^2 a_2^2} \]

\[ d_1^{(1)} = \frac{\lambda}{4}, \quad d_1^{(2)} = \frac{\lambda}{2\pi} \tan(a_2) \]

\[ l_1^{(1)} = \frac{\lambda}{2\pi} \tan(b_1), \quad l_1^{(2)} = \frac{\lambda}{2\pi} \tan(b_2) \]

7.3.2. OPEN STUB (O/C)

c) \( A = Z_0 \)

\[ a_1 = \frac{2A}{B}, \quad a_2 = 0 \]

\[ b_1 = -\frac{Ak}{B}, \quad b_2 = \frac{Ak}{B} \]

and

\[ d_1^{(1)} = \frac{\lambda}{2\pi} \tan(a_1), \quad d_1^{(2)} = 0 \]

\[ d_2^{(1)} = \frac{\lambda}{2\pi} \tan(b_1), \quad d_2^{(2)} = \frac{\lambda}{2\pi} \tan(b_2) \]

d) \( A^2 + B^2 - AZ_0 = 0 \) et \( Z_0 > A \)

\[ a_1 = \infty, \quad a_2 = \frac{Z_0 - A}{B + \sqrt{A(Z_0 - A)}} \]

\[ b_1 = -\frac{(A^2 + B^2)k}{Z_0 B}, \quad b_2 = \frac{k[(Z_0 - a_2 B)^2 + A^2 a_2^2]}{-B Z_0 a_2^2 - a_2(A^2 + B^2 - AZ_0) + B Z_0} \]
\[d_1^{(1)} = \frac{\lambda}{4}, \quad d_1^{(2)} = \frac{\lambda}{2\pi} \tan(a_2)\]

\[l_1^{(1)} = \frac{\lambda}{2\pi} \tan(b_1), \quad l_1^{(2)} = \frac{\lambda}{2\pi} \tan(b_2)\]

From the previous solutions one can see two solutions. Which one to choose? It is recommended to choose the one with the shortest lengths of the transmission line. Shorter transmission lines provide smaller and slightly cheaper matching networks. Moreover, there is a more fundamental reason why we select the solution with the shortest lines because the matching bandwidth is larger.

Thus,

\[d_1 = \min(d_1^{(1)}, d_1^{(2)})\]

and

\[l_1 = \min(l_1^{(1)}, l_1^{(2)})\]

8. EXAMPLES.

8.1) Match a load impedance of \(Z_L = (100 + j80)\Omega\) to a line with characteristic impedance \(Z_0 = 75\Omega\) using a single stub tuner. Find one solution using an open-circuited stub and another using a short-circuited stub for \(Z_S = 75\Omega\). Stub is in series with the line. Find SWR before and after the stub.

Solution:

\[Z_L = A + jB = (100 + j80)\Omega; \quad A = 100\Omega, \quad B = 80\Omega, \quad Z_0 = 75\Omega, \quad k = 1\]

By the Smith chart,

\[S/C\]

\[d_1^{(1)} = 0.156\lambda, \quad l_1^{(1)} = 0.122\lambda\]

\[d_1^{(2)} = 0.477\lambda, \quad l_1^{(2)} = 0.377\lambda\]

\[O/C\]

\[d_1^{(1)} = 0.156\lambda, \quad l_1^{(1)} = 0.128\lambda\]

\[d_1^{(2)} = 0.478\lambda, \quad l_1^{(2)} = 0.373\lambda\]

This work:
S/C
\[ d_1^{(1)} = 0.1559\lambda, \quad l_1^{(1)} = 0.1224\lambda \]
\[ d_1^{(2)} = 0.4776\lambda, \quad l_1^{(2)} = 0.3776\lambda \]

O/C
\[ d_1^{(1)} = 0.1559\lambda, \quad l_1^{(1)} = 0.372\lambda \]
\[ d_1^{(2)} = 0.4776\lambda, \quad l_1^{(2)} = 0.1276\lambda \]

By [1] we obtain the same results.

**Verification of the SWR before and after the single stub**

a1) Near the load (short stub, the first couple):
\[ \Gamma_{BS} = \frac{Z_L - Z_0}{Z_L + Z_0}, \quad |\Gamma_{BS}| = 0.5648 \]
\[ SWR_{BS} = \frac{1 + |\Gamma_{BS}|}{1 - |\Gamma_{BS}|} = \frac{1 + 0.5648}{1 - 0.5648} = 3.5956 \]

After the single stub near the generator:
\[ Z_{Tj} = Z_{d1} + Z_S = 75(1.0001 - j 0.0000)\Omega \]
\[ |\Gamma_{AS}| = 4.9998 \cdot 10^{-5} \]
\[ SWR_{AS} = 1.0001 \]

a2) Near the load (open stub, the second couple):
\[ \Gamma_{BS} = \frac{Z_L - Z_0}{Z_L + Z_0}, \quad |\Gamma_{BS}| = 0.5648 \]
\[ SWR_{BS} = \frac{1 + |\Gamma_{BS}|}{1 - |\Gamma_{BS}|} = \frac{1 + 0.5648}{1 - 0.5648} = 3.5956 \]

After the single stub near the generator:
\[ Z_{Tj} = Z_{d1} + Z_S = 75(0.9995 - j 0.0003) \Omega \]

\[ | \Gamma_{AS} | = 2.5006 \cdot 10^{-4} \]

\[ SWR_{AS} = 0.9995 \]

Thus, there is not the reflected wave, and the calculations are confirmed.

8.2) Determine the single stub matching parameters \( (d \text{ and } l) \) required to match the load \( Z_L = (25 - j50) \Omega \) to a 50\( \Omega \) system.

a) Short and open stub in parallel,

b) Short and open stub in series

Solution:

a) The Smith chart:

\[ S/C \]

\[ d_1^{(1)} = 0.206 \lambda, \quad l_1^{(1)} = 0.410 \lambda \]

\[ d_1^{(2)} = 0.063 \lambda, \quad l_1^{(2)} = 0.090 \lambda \]

\[ O/C \]

\[ d_1^{(1)} = 0.206 \lambda, \quad l_1^{(1)} = 0.160 \lambda \]

\[ d_1^{(2)} = 0.063 \lambda, \quad l_1^{(2)} = 0.340 \lambda \]

This work:

\[ S/C \]

\[ d_1^{(1)} = 0.2067 \lambda, \quad l_1^{(1)} = 0.4102 \lambda \]

\[ d_1^{(2)} = 0.0631 \lambda, \quad l_1^{(2)} = 0.0898 \lambda \]

\[ O/C \]

\[ d_1^{(1)} = 0.2067 \lambda, \quad l_1^{(1)} = 0.1602 \lambda \]

\[ d_1^{(2)} = 0.0631 \lambda, \quad l_1^{(2)} = 0.3398 \lambda \]
These results are confirmed by [1].

b) The Smith chart:

\[ S/C \]

\[ d_1^{(1)} = 0.457\lambda, \; l_1^{(1)} = 0.160\lambda \]
\[ d_1^{(2)} = 0.313\lambda, \; l_1^{(2)} = 0.340\lambda \]

\[ O/C \]

\[ d_1^{(1)} = 0.457\lambda, \; l_1^{(1)} = 0.410\lambda \]
\[ d_1^{(2)} = 0.313\lambda, \; l_1^{(2)} = 0.090\lambda \]

This work:

\[ S/C \]

\[ d_1^{(1)} = 0.4567\lambda, \; l_1^{(1)} = 0.1602\lambda \]
\[ d_1^{(2)} = 0.3131\lambda, \; l_1^{(2)} = 0.3398\lambda \]

\[ O/C \]

\[ d_1^{(1)} = 0.4567\lambda, \; l_1^{(1)} = 0.4102\lambda \]
\[ d_1^{(2)} = 0.3131\lambda, \; l_1^{(2)} = 0.0898\lambda \]

These results are confirmed by [1].

8.3) Design a single stub matching network to match a load of \( Z_L = (50 + j50)\Omega \) to the 50 \( \Omega \) lossless line. The short stub is in series with the line, (Special case- singular case). Repeat it for the open stub.

Solution:

The Smith chart:

\[ S/C \]
\[d_1^{(1)} = 0.176\lambda, \quad l_1^{(1)} = 0.125\lambda\]
\[d_1^{(2)} = 0\lambda, \quad l_1^{(2)} = 0.375\lambda\]

O/C

\[d_1^{(1)} = 0.176\lambda, \quad l_1^{(1)} = 0.375\lambda\]
\[d_1^{(2)} = 0\lambda, \quad l_1^{(2)} = 0.125\lambda\]

From this work:

S/C

\[d_1^{(1)} = 0.1762\lambda, \quad l_1^{(1)} = 0.1250\lambda\]
\[d_1^{(2)} = 0\lambda, \quad l_1^{(2)} = 0.3750\lambda\]

O/C

\[d_1^{(1)} = 0.1762\lambda, \quad l_1^{(1)} = 0.3750\lambda\]
\[d_1^{(2)} = 0\lambda, \quad l_1^{(2)} = 0.1250\lambda\]

These results are confirmed by [1].

8.4) For a load impedance \(Z_L = (60 - j80)\Omega\), design two single-stub (short circuit) shunt tuning networks to match this load to a 50 \(\Omega\) lossless line. If the load is matched at 2 GHz and that the load consists of a resistor and capacitor in series (\(60\Omega, 0.995\)\(pF\)), plot the reflection coefficient magnitude from 1 to 3 GHz for each solution.

Solution:

The Smith chart:

I)
\[d_1^{(1)} = 0.110\lambda, \quad l_1^{(1)} = 0.095\lambda\]

II)
\[d_1^{(2)} = 0.260\lambda, \quad l_1^{(2)} = 0.405\lambda\]
From this work:

\[ d_1^{(1)} = 0.1104 \lambda \quad l_1^{(1)} = 0.0950 \lambda \]

\[ d_1^{(2)} = 0.2594 \lambda \quad l_1^{(2)} = 0.4050 \lambda \]

Fig. 8. Reflection coefficient magnitudes versus frequency for the tuning circuits

Obviously (Fig. 8) that solution I) has a significantly better bandwidth than solution II). This is because both \( d \) and \( l \) are shorter for solution I), which reduces the frequency variation of the match.

8.5) An aerial of \((300 - j300)\Omega\) is to be matched with 600\(\Omega\) line. The matching is to be done by means of low loss 600\(\Omega\) short single stub line.

c) Find the position and the length of the stub line.

d) Find the SWR before and after the stub for both solutions at a).

Solution:

a)

\[ d_1^{(1)} = 0 \lambda \quad l_1^{(1)} = 0.125 \lambda \]
\[ d_1^{(2)} = 0.1762 \lambda \quad l_1^{(2)} = 0.3750 \lambda \]

b)

b1) For the first couple:
\[ |\Gamma_{BS}| = 0.4472 \]
\[ SWR_{BS} = 2.6180 \]
For the second couple:
\[ |\Gamma_{BS}| = 0.4472 \]
\[ SWR_{BS} = 2.6180 \]

b2) For the first couple:
\[ |\Gamma_{AS}| = 0 \]
\[ SWR_{AS} = 1 \]
For the second couple:
\[ |\Gamma_{AS}| = 5.7541 \cdot 10^{-5} \]
\[ SWR_{AS} = 1.0001 \]

Thus, there is not the reflected wave for both cases so that the calculations are confirmed.
9. DOUBLE SHORT OR OPEN STUB IN SERIES

The disadvantage to single-stub tuning is that it is not easy to vary the distance \( d_1 \) between the load and the stub. Generally new elements can only be connected at the ends of the line and not in between. This difficulty of not having a variable length line can be overcome by using two short or open stubs a fixed length apart, as shown in Figure 9. This fixed length is usually \( d_2 = 0.375\lambda \). A match is made by adjusting the length of the stubs \( l_1 \) and \( l_2 \). One problem with the double-stub tuner is that not all loads can be matched for a given stub spacing.

The load impedance \( Z_L = A + jB, \ (A, B \in R) \) on the transmission line with the characteristic impedance \( Z_0 \) and without losses is to be matched by a double-stub tuner in series. The first stub is distanced from the load by \( d_1 \) and the distance between them is \( d_2 \). Their characteristic impedences are respectively \( Z_{S1} = k_1Z_0 \) and \( Z_{S2} = k_2Z_0 \). Find the lengths \( l_1 \) and \( l_2 \).

Using the same procedures such as for the single stub configuration, we have,

\[
a = \tan(\beta d_1), \ a \in R \\
b = \tan(\beta l_1), \ a \in R \\
z_L = p_L + jq_L
\]
where,
\[ p_L = \frac{A}{Z_0}, \quad q_L = \frac{B}{Z_0} \]

\[ z_{(d_1)} = \frac{z_L + j \tan(\beta d_1)}{1 + jz_L \tan(\beta d_1)} = r_L + jx_L \]

where
\[ r_L = \frac{p_L (1 + a^2)}{(1 - a^q_L)^2 + a^2 p_L^2} \]
\[ x_L = \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 - a^q_L)^2 + a^2 p_L^2} \]

\[ z_{S1} = jx_{S1} \]
\[ z_{(B-B')} = z_{(d_1)} + jx_{S1} = [r_L + j(x_L + x_{S1})] \]

Let us introduce,
\[ m = \tan(\beta d_2), m \in R \]
\[ n = \tan(\beta l_2), n \in R \]

\[ z_{(d_2)} = \frac{z_{(B-B')} + j \tan(\beta d_2)}{1 + jz_{(B-B')} \tan(\beta d_2)} = \frac{r_L + j(x_L + x_{S1} + m)}{1 + jm[r_L + j(x_L + x_{S1})]} \]

\[ z_{S2} = jx_{S2} \]
\[ z_{(A-A')} = z_{(d_2)} + z_{S2} \]

The conditions to have perfect matching are:
\[ \text{Re}[z_{(A-A')}] = \frac{r_L (1 + m^2)}{[1 - m(x_L + x_{S1})]^2 + m^2 r_L^2} = 1 \quad (31) \]

\[ \text{Im}[z_{(A-A')}] = -z_{S2} = \frac{m(1 - r_L^2 - (x_L + z_{S1})^2) + (x_L + x_{S1})(1 - m^2)}{[1 - m(x_L + x_{S1})]^2 + m^2 r_L^2} \quad (32) \]

From (31) we obtain,
\[ r_L^2 m^2 - r_L (1 + m^2) + [1 - (x_L + x_{S1})m]^2 = 0 \] (33)

that gives,

\[ x_{S1}^{(1,2)} = \frac{1 - mx_L \pm \sqrt{r_L[(1 + m^2) - m^2 r_L]}}{m} \] (34)

From (15) it is possible to find the region (FORBIDDEN REGION) where the matching is not possible,

\[ [(1 + m^2) - m^2 r_L] < 0 \] (FORBIDDEN REGION)

Thus, the limits for \( r_L \),

\[ 0 \leq r_L \leq \frac{1 + m^2}{m^2} = \frac{1}{[\sin (\beta d_2)]^2} \] (35)

From (33) and (34) we have,

\[ x_{S2}^{(1,2)} = \frac{1}{m} \left[ 1 \mp \sqrt{\frac{1 + m^2 - m^2 r_L}{r_L}} \right] \] (36)

9.1. DETERMINATION OF \( l_1 \) AND \( l_2 \)

c) SHORT STUB (S/C)

\[ Z_{S1}^{(1,2)} = - \frac{1}{k_1 b_{s1(c-c)}^{(1,2)}} \quad , \quad Z_{S2}^{(1,2)} = - \frac{1}{k_2 n_{s2(c-c)}^{(1,2)}} \]

That gives,

\[ b_{s1(c-c)}^{(1,2)} = \frac{1}{k_1 mz_L} - 1 \pm \sqrt{r_L[(1 + m^2) - m^2 r_L]} \] (37)

\[ n_{s2(c-c)}^{(1,2)} = \frac{m}{k_2 \left[ \pm \sqrt{\frac{1 + m^2 - m^2 r_L}{r_L}} - 1 \right]} \] (38)

d) OPEN STUB (O/C)
That gives,

\[ b_{s1(c-o)}^{(1,2)} = k_1 \left( 1 - mz_L + \sqrt{r_L \left[ (1 + m^2) - m^2r_L \right]} \right) \frac{m}{m} \]  

\[ n_{s2(c-o)}^{(1,2)} = \frac{k_2}{m} \left[ 1 \mp \sqrt{1 + m^2 - m^2r_L} \right] \frac{r_L}{r_L} \]  

Finally,

\[ l_{1(c-o)}^{(1,2)} = \frac{\lambda}{2\pi} \tan^{-1} \left( b_{s1(c-o)}^{(1,2)} \right) \]  

\[ l_{2(c-o)}^{(1,2)} = \frac{\lambda}{2\pi} \tan^{-1} \left( n_{s1(c-o)}^{(1,2)} \right) \]  

\[ l_{1(c-o)}^{(1,2)} = \frac{\lambda}{2\pi} \tan^{-1} \left( b_{s1(c-o)}^{(1,2)} \right) \]  

\[ l_{2(c-o)}^{(1,2)} = \frac{\lambda}{2\pi} \tan^{-1} \left( n_{s1(c-o)}^{(1,2)} \right) \]  

It is recommended to choose the shortest lengths of the stubs (smaller and slightly matching networks as well as the matching bandwidth is larger).

Thus,

\[ l_1 = \min (l_1^{(1)}, l_1^{(2)}) \]

and

\[ l_2 = \min (l_2^{(1)}, l_2^{(2)}) \]

**SPECIAL CASES**

9.2. \( d_1 = d_2 = \frac{\lambda}{4} \)

\[ a = \tan(\beta d_1) = \infty, \quad m = \tan(\beta d_2) = \infty \]
\[ r_L = \frac{p_L}{p_L^2 + q_L^2} \]
\[ x_L = \frac{q_L}{p_L^2 + q_L^2} \]

f) **SHORT STUBS**

\[ b^{(1,2)}_{CS1} = \frac{1}{k_1} \left( \frac{1}{x_L \pm \sqrt{r_L[1 - r_L]}} \right) \]
\[ n^{(1,2)}_{CS2} = \pm \frac{1}{k_2} \sqrt{r_L \frac{1}{1 - r_L}} \]

g) **OPEN STUBS**

\[ b^{(1,2)}_{OS1} = k_1 [-b_L \mp \sqrt{r_L[1 - r_L]}] \]
\[ n^{(1,2)}_{OS2} = \mp k_2 \sqrt{\frac{1 - r_L}{r_L}} \]

9.3. \( d_1 = \frac{\lambda}{4} \) and \( d_2 \neq \frac{\lambda}{4} \)

\[ a = \tan(\beta d_1) = \infty \]
\[ r_L = \frac{p_L}{p_L^2 + q_L^2} \]
\[ x_L = \frac{q_L}{p_L^2 + q_L^2} \]

h) **SHORT STUBS**

The same expressions for \( b^{(1,2)}_{CS1} \) and \( n^{(1,2)}_{CS2} \) as in the general case.

i) **OPEN STUBS**

The same expressions for \( b^{(1,2)}_{OS1} \) and \( n^{(1,2)}_{OS2} \) as in the general case.

9.4. \( d_1 \neq \frac{\lambda}{4} \) and \( d_2 = \frac{\lambda}{4} \)
\[ m = \tan(\beta d_2) = \infty \]

\[ r_L = \frac{p_L(1 + a^2)}{(1 + a q L)^2 + a^2 p_L^2} \]

\[ x_L = \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 + a q L)^2 + a^2 p_L^2} \]

**j) SHORT STUBS**

\[ b_{CS1}^{(1,2)} = \frac{1}{k_1} \frac{1}{x_L \pm \sqrt{r_L[1 - r]}} \]

\[ n_{CS2}^{(1,2)} = \pm \frac{1}{k_2} \sqrt{\frac{r_L}{1 - r_L}} \]

**b) OPEN STUBS:**

\[ b_{OS1}^{(1,2)} = k_1 [-B_L \mp \sqrt{r_L[1 - r_L]}] \]

\[ n_{OS2}^{(1,2)} = \mp k_2 \sqrt{\frac{1 - r_L}{r_L}} \]

\[ l_{CS1}^{(1,2)} = \frac{1}{\beta} \arctan\left(b_{CS1}^{(1,2)}\right) = \frac{\lambda}{2\pi} \arctan\left(b_{CS1}^{(1,2)}\right) \]

\[ l_{CS2}^{(1,2)} = \frac{1}{\beta} \arctan\left(n_{CS2}^{(1,2)}\right) = \frac{\lambda}{2\pi} \arctan\left(n_{CS2}^{(1,2)}\right) \]

\[ l_{OS1}^{(1,2)} = \frac{1}{\beta} \arctan\left(b_{OS1}^{(1,2)}\right) = \frac{\lambda}{2\pi} \arctan\left(b_{OS1}^{(1,2)}\right) \]

\[ l_{OS2}^{(1,2)} = \frac{1}{\beta} \arctan\left(n_{OS2}^{(1,2)}\right) = \frac{\lambda}{2\pi} \arctan\left(n_{OS2}^{(1,2)}\right) \]

**10. THE MAXIMUM DISTANCE BETWEEN THE STUB TUNERS**

The load will be outside the Forbidden region if
\[ r_L \leq 1 + \frac{1}{m^2} = \frac{1}{[\sin(\beta d_2)]^2} = r_{\text{max}}(m) \]

where

\[ m = \tan(\beta d_2) \]

IV) \( r_L > 1 \)

\[ d_{2\text{max}}(1) = \frac{\lambda}{2\pi} \sin \left( \frac{1}{r_{\text{max}}(m)} \right) = \frac{\lambda}{2\pi} \sin \left( \frac{1}{r_L} \right) \]

V) \( r_L = 1 \)

\[ d_{2\text{max}}(2) = \frac{\lambda}{4} \]

VI) \( 0 < r < 1 \)

From \( m = \tan(\beta d_2) \)

\[ d_2 = \frac{\lambda}{2\pi} \tan(m) \]

\( d_2 \) must be positive so that,

\[ d_2 = \frac{\lambda}{2\pi} [\pi - \tan(\lvert m \rvert)], \text{ for } m < 0 \]

\[ d_2 = \frac{\lambda}{2\pi} [\pi - \tan(\varepsilon)], \text{ for } m \approx -\varepsilon \]

\[ d_2 = \frac{\lambda}{2\pi} \tan(m), \text{ for } m \geq 0 \]

where \( \varepsilon \) is very small constant (for example \( \varepsilon = 10^{-12} \)).

For \( m = 0 \)

\[ d_{2\text{max}}(3) = 0 \]

In this case the stub 2 is the same position as the stub 1 so that this is the case of the single stub.

For \( m > 0 \)

\[ d_{2\text{max}}(4) = 0.25\lambda \]
From the Fig.10, it is also obvious that $d_{2\text{max}}(5) = 0.25\lambda$.

$m < 0$

$$d_{2\text{max}}(6) = (0.5 - \varepsilon)\lambda$$

From the Fig.11 it is also obvious that $d_{2\text{max}}$ is remarkably close to $d_{2\text{max}} = 0.5\lambda$.

Thus,
\[ d_{2\text{max}} = \max(d_{2\text{max}(i)}, i = 0,1,\ldots,6) = (0.5 - \varepsilon)\lambda \]

11. **The Minimum Length Between the Load and the First Stub in the Double Stub Matching**

The condition for which the load is outside the FORBIDDEN REGION:

\[
r_L \leq \frac{1 + m^2}{m^2} = \frac{1}{[\sin(\beta d_2)]^2} = \frac{1}{Q}
\]

where

\[
Q = \frac{m^2}{1 + m^2} < 1
\]

\[
r_L = \frac{p_L(1 + a^2)}{(1 - aq_L)^2 + a^2p_L^2} \leq \frac{1}{Q}
\]

\[
p_L = \frac{AZ_0}{(A^2 + B^2)}, \quad q_L = \frac{-BZ_0}{(A^2 + B^2)}
\]

or

\[
(1 - aq_L)^2 + a^2p_L^2 \geq Qp_L(1 + a^2)
\]

The minimum length between the load and the first stub in double stub matching can be obtained from the equation:

\[
(1 - aq_L)^2 + a^2p_L^2 = Qp_L(1 + a^2)
\]

\[
a^2[p_L^2 + q_L^2 - Qp_L] - 2q_L a + 1 - Qp_L = 0
\]

\[
a_{1,2} = \frac{q_L \pm \sqrt{D}}{p_L^2 + q_L^2 - Qp_L}
\]

\[
D = q_L^2 - (1 - Qp_L)[p_L^2 + q_L^2 - Qp_L] = Qp_L[p_L^2 + q_L^2 + 1 - Qp_L] - p_L^2
\]

or

\[
\Delta = p_L\{Q[p_L^2 + q_L^2 + 1 - Qp_L] - p_L\}
\]

3) If \(p_L^2 + q_L^2 = Qp_L\)

\[
\Delta = p_L\{Q - p_L\}
\]
\[ a_1 = \infty, \quad d_{1(1)} = \frac{\lambda}{4} \]
\[ a_2 = \frac{1}{2q_L}, \quad d_{1(2)} = \frac{\lambda}{2\pi}\tan(a_2) \]

4) If \( \Delta \geq 0 \)

\[
\tan(\beta d_{1(1,2)}) = a_{1,2} = \frac{q_L \pm \sqrt{D}}{p_L^2 + q_L^2 - Qp_L}
\]
\[ d_{1(1,2)} = \frac{\lambda}{2\pi}\tan(a_{1,2}) \]

We choose \( d_{\text{min}} = \min(d_{1(1)}, d_{1(2)}) \) \hspace{1cm} (42)

12. EXAMPLES.

12.1) Design a double stub matching network to match a load of \( \Gamma = \frac{2}{3} e^{j\frac{\pi}{2}} \)
load to the 50 \( \Omega \) lossless line. The two stubs are spaced 0.375 \( \lambda \) apart. The stub nearest to the load is 0.1 \( \lambda \) away from it. Carry out two designs: one using open circuit and one using short circuit stubs.

\[
\Gamma = \frac{2}{3} e^{j\frac{\pi}{2}} = \frac{2}{3} j \rightarrow z_L = \frac{1+\Gamma}{1-\Gamma} \quad Z_L = Z_0Z_L = Z_0\frac{1+\Gamma}{1-\Gamma}
\]
\[ Z_L = (19.23076923076923 + j 46.15384615384615)\Omega, \]
\[ z_L = Z_L/Z_0 = 0.3846153846153846 + j 0.9230769230769230 \]

FORBIDDEN REGION:

\[
r_L[(1 + m^2) - m^2r_L] = -0.3622327483733958 < 0 \hspace{1cm} (34)
\]

It is not possible to have the matching for this line because the impedance \( Z_L \) is in the FORBIDDEN REGION.

12.2) Design a double stub matching network to match a load of \( \Gamma = \frac{2}{3} e^{-j\frac{\pi}{2}} \)
load to the 50 \( \Omega \) lossless line. The two stubs are spaced 0.375 \( \lambda \) apart. The stub...
nearest to the load is 0.1 \lambda away from it. Carry out two designs: one using open circuit and one using short circuit stubs.

\[
\Gamma = \frac{2}{3} e^{-j\frac{\pi}{2}} = -\frac{2}{3} j \rightarrow Z_L = \frac{1+\Gamma}{1-\Gamma} \rightarrow Z_L = Z_0, \quad z_L = Z_0 \frac{1+\Gamma}{1-\Gamma}
\]

\[
Z_L = (19.23076923076923 - j 46.15384615384615) \Omega,
\]

\[
z_L = \frac{z_L}{Z_0} = 0.3846153846153846 - j 0.9230769230769230
\]

\[
r_L[(1 + m^2) - m^2 r_L] = 0.3676754292533759 > 0
\]  \hspace{1cm} (34)

The impedance is outside of the forbidden region.

**Short stubs**

First couple:

\[
l_1 = 0.4622499734697806 \lambda, \quad l_2 = 0.1749335457758014 \lambda
\]

Second couple:

\[
l_1 = 0.3458610767560822 \lambda, \quad l_2 = 0.2893620116984563 \lambda
\]

Let us verify the validity of obtained results using the SWR (first couple).

**SWR before the load**

\[
|\Gamma_{Before \ Load}| = 2/3
\]

\[
SWR_{Before \ Load} = 5
\]

**SWR between stubs**

\[
d_1 = 0.1 \lambda, \quad l_1 = 0.4622499734697806 \lambda
\]

\[
|\Gamma_{Between \ S1S2}| = 0.7000353604641002
\]

\[
SWR_{Between \ S1S2} = 5.667452547388138
\]
\textbf{SWR before the second stub}

\[ d_2 = 0.375\lambda, \quad l_2 = 0.1749335457758014\lambda \]

\[ Z_{BS2} = (10.24058065229624 - j19.68187715023507)\Omega \]

\[ |\Gamma_{BeforeS2}| = 1.310176068229350 \cdot 10^{-15} \]

\[ SWR_{BeforeS2} = 1.00000000000003 \]

Thus, there is not the reflected wave, so that the calculations are confirmed.

\textbf{Open stubs}

First couple:

\[ l_1 = \lambda, \quad l_2 = 0.\lambda \]

Second couple:

\[ l_1 = \lambda, \quad l_2 = \lambda \]

Let us verify the validity of obtained results using the SWR (second couple).

\textbf{SWR before the load}

\[ |\Gamma_{BeforeLoad}| = 2/3 \]

\[ SWR_{BeforeLoad} = 5 \]

\textbf{SWR between stubs}

\[ d_1 = 0.1\lambda, \quad l_1 = 0.4622499734697806\lambda \]

\[ |\Gamma_{BetweenS1S2}| = 0.7000353604641002 \]

\[ SWR_{BetweenS1S2} = 5.667452547388138 \]

\textbf{SWR before the second stub}
\[ d_2 = 0.375\lambda, \quad l_2 = 0.1749335457758014 \lambda \]

\[ Z_{BS2} = (10.24058065229624 - j19.68187715023507)\Omega \]

\[ |\Gamma_{BeforeS2}| = 1.310176068229350 \cdot 10^{-15} \]

\[ SWR_{BeforeS2} = 1.000000000000003 \]

Thus, there is not the reflected wave, so that the calculations are confirmed.

The smallest and the largest distances from the load for which the matching is possible:

\[ d_{1\text{min}} = 0.3348\lambda, \quad d_{1\text{max}} = 0.4152\lambda \]

The load impedance \( Z_L = (250 + j600)/13 \Omega \) on a 50 \( \Omega \) line is to be matched by a double-stub tuner of 0.375\( \lambda \) spacing. The stubs are in series. What stub lengths \( l_1 \) and \( l_2 \) are necessary if the first stub is 0.1\( \lambda \) away from the \( Z_L \)? Stubs are shorted. [\( d_1 = 0; \quad d_2 = 0.375\lambda \)].

From [1] the couple of lengths are:

\[ l_1 = 0.211\lambda, \quad l_2 = 0.426\lambda \]

\[ l_1 = 0.096\lambda, \quad l_2 = 0.036\lambda \]

From this work:

\[ l_1 = 0.211\lambda, \quad l_2 = 0.426\lambda \]

\[ l_1 = 0.096\lambda, \quad l_2 = 0.036\lambda \]

12.1) A load consists of a 4 \((nH)\) inductor which has a series internal resistance of 19.2(\( \Omega \)).

a) This load needs to be matched to a 50(\( \Omega \)) lossless co-axial transmission line by means of a double stub matching network, consisting of two short-circuit stubs, spaced 0.375\( \lambda \) apart. The stub nearest to the load is 0.1\( \lambda \) away from it. Determine the possible combinations of stub lengths which are required to match the load to the line at the operating frequency of 1835 (\( MHz \)). You may assume all line sections and stubs are 50 (\( \Omega \)).

b) Repeat(a) if the circuit stubs are open.

Solution:
\[ Z_L = (19.2 + j46.17)\Omega, \quad k_1 = k_2 = 1 \]
\[ d_1 = 0.1 \lambda, \quad d_2 = 0.375 \lambda \]

**c) SHORT STUBS**

Smith chart:
\[ \ell_1^{(1)} = 0.096 \lambda \quad \ell_2^{(1)} = 0.039 \lambda \]
\[ \ell_1^{(2)} = 0.212 \lambda \quad \ell_2^{(2)} = 0.425 \lambda \]

From this work:
\[ \ell_1^{(1)} = 0.0959 \lambda \quad \ell_2^{(1)} = 0.0393 \lambda \]
\[ \ell_1^{(2)} = 0.2122 \lambda \quad \ell_2^{(2)} = 0.4250 \lambda \]

**d) OPEN STUBS**

Smith chart:
\[ \ell_1^{(1)} = 0.346 \lambda \quad \ell_2^{(1)} = 0.289 \lambda \]
\[ \ell_1^{(2)} = 0.462 \lambda \quad \ell_2^{(2)} = 0.175 \lambda \]

From this work:
\[ \ell_1^{(1)} = 0.3459 \lambda \quad \ell_2^{(1)} = 0.2893 \lambda \]
\[ \ell_1^{(2)} = 0.4622 \lambda \quad \ell_2^{(2)} = 0.1750 \lambda \]

**12.2) The load impedance \( Z_L = (60 - j 80) \Omega \) on a 50 \( \Omega \) line is to be matched by a double-stub tuner of 0.125\( \lambda \) spacing. What stub lengths \( l_1 \) and \( l_2 \) are necessary? Stubs are shorted with the same characteristic impedance as the line. [\( d_1 = 0; \ d_2 = 0.125 \lambda \)].**

Solution:
\[ k_1 = k_2 = 1, \quad \text{SHORT STUBS} \]

Smith chart:
\[ \ell_1^{(1)} = 0.232 \lambda \quad \ell_2^{(1)} = 0.1 \lambda \]
Design a double-stub shunt tuner to match a load impedance $Z_L = (60 - j \ 80) \ \Omega$ to a 50 $\Omega$ line. The stubs are to be open-circuited stubs and are spaced $\lambda/8$ apart. If this load consists of a series resistor and capacitor ($60 \Omega, 0.995 \ pF$), and the match frequency is 2 GHz, plot the reflection coefficient magnitude versus frequency from 1 to 3 GHz.

Solution:

Smith chart:

\[ l_1^{(1)} = 0.146 \lambda, \quad l_2^{(1)} = 0.204 \lambda \]
\[ l_1^{(2)} = 0.483 \lambda, \quad l_2^{(2)} = 0.350 \lambda \]

From this work: $k_1 = k_2 = 1$

\[ l_1^{(1)} = 0.1465 \lambda, \quad l_2^{(1)} = 0.2042 \lambda \]
\[ l_1^{(2)} = 0.4819 \lambda, \quad l_2^{(2)} = 0.3498 \lambda \]

From [1]:

\[ l_1^{(1)} = 0.1465 \lambda, \quad l_2^{(1)} = 0.2042 \lambda \]
\[ l_1^{(2)} = 0.4819 \lambda, \quad l_2^{(2)} = 0.3498 \lambda \]
Fig. 12. Reflection coefficient magnitudes versus frequency for the tuning circuits.

The first solution has a much narrower bandwidth than the second (primed) solution because both stubs for the second solution are somewhat longer (and closer to 0.5λ) than the stubs of the first solution (Fig 12).

12.4) The load impedance $Z_L = (50 + j 50) \Omega$ on a 50 Ω line is to be matched by a double-stub tuner of 0.375λ spacing. What stub lengths $l_1$ and $l_2$ are necessary? Stubs are shorted with the same characteristic impedance as the line. [$d_1 = 0; \ d_2 = 0.375\lambda$].

Solution:

From Smith chart:

$$l_1^{(1)} = 0.305 \lambda, \ \ l_2^{(1)} = 0.349 \lambda$$

$$l_1^{(2)} = 0.1 \lambda, \ \ l_2^{(2)} = 0.056 \lambda$$

From this work:

$$l_1^{(1)} = 0.3058\lambda, \ \ l_2^{(1)} = 0.3506 \lambda$$

$$l_1^{(2)} = 0.1006\lambda, \ \ l_2^{(2)} = 0.0558\lambda$$

12.5) The terminating impedance is $Z_L = (100 + j100)\Omega$ and the characteristic impedance $Z_0$ of the line and the stubs is 50 Ω. The first stub is
away 0.4 λ from the load. The spacing between the two stubs is stub 3/8 λ. Determine the length of the short-circuited stubs when the match is achieved.

Solution:

Smith chart:

\[ l_1^{(1)} = 0.373 \lambda, \quad l_2^{(1)} = 0.337 \lambda \]

\[ l_1^{(2)} = 0.143 \lambda, \quad l_2^{(2)} = 0.058 \lambda \]

This work:

\[ l_1^{(1)} = 0.3720 \lambda, \quad l_2^{(1)} = 0.3403 \lambda \]

\[ l_1^{(2)} = 0.1410 \lambda, \quad l_2^{(2)} = 0.0577 \lambda \]

12.6) The terminating impedance is \( Z_L = (50 + j100) \Omega \) and the characteristic impedance \( Z_0 \) of the line and the stubs is 100 \( \Omega \). The first stub is away 0.25 \( \lambda \) from the load. The spacing between the two stubs is stub 1/8 \( \lambda \). If the length of the second stub is 0.15 \( \lambda \). Determine the other lengths of the short-circuited stubs when the match is achieved.

Solution:

This is the special case ( \( a = \infty \) )

This work:

\[ l_1^{(1)} = 0.1364 \lambda, \quad l_2^{(1)} = 0.1494 \lambda \]

\[ l_1^{(2)} = 0.3636 \lambda, \quad l_2^{(2)} = 0.4442 \lambda \]

By the technique Smith Chart: \( l_2^{(1)} = 0.15 \lambda. \)

This work gives: \( l_2^{(1)} = 0.1494 \lambda. \)

12.7) For each of the following sets of values of and associated with the double-stub matching technique, determine whether it is possible to achieve a match between the line and the load:

a) \( z_{LN} = (0.3 + j0.4) \Omega, d_1 = 0; \quad d_2 = 0.375 \lambda \).
b) \( z_{LN} = 0.5 \Omega, d_1 = 0.125 \lambda; \ d_2 = 0.375 \lambda \].

c) \( z_{LN} = (2.5 - j0.5) \Omega, d_1 = 0.25 \lambda; \ d_2 = 5\lambda/8 \].

Solution:

d) Yes. it is possible to achieve a match between the line and the load because,

\[
[(1 + m^2) - m^2 g_L] = (1 + 1) - 1 \cdot 1.2 = 0.8 > 0
\]

The load is outside of the Forbidden Region.

\[
l_1^{(1)} = 0.4102 \lambda, \quad l_2^{(1)} = 0.2211 \lambda
\]

\[
l_1^{(2)} = 0.1922 \lambda, \quad l_2^{(2)} = 0.0801 \lambda
\]

e) Yes. It is possible to achieve a match between the line and the load because,

\[
[(1 + m^2) - m^2 g_L] = (1 + 1) - 1 \cdot 0.8 = 1.2 > 0
\]

The load is outside of the Forbidden Region.

\[
l_1^{(1)} = 0.3336 \lambda, \quad l_2^{(1)} = 0.2852 \lambda
\]

\[
l_1^{(2)} = 0.0998 \lambda, \quad l_2^{(2)} = 0.0672 \lambda
\]

f) No. It is not possible to achieve a match between the line and the load because,

\[
[(1 + m^2) - m^2 g_L] = (1 + 1) - 1 \cdot 2.5 = -0.5 < 0
\]

The load is inside of the Forbidden Region.

12.8) In an experiment to determine the unknown impedance of a microwave antenna, a slotted section of transmission line is first connected to a short-circuit load. In this case, adjacent voltage minima are found 23.6 cm and 35.4 cm away from the load. After replacing the short-circuit with antenna, the location of minima shifted to points 27.8 and 39.6 cm, respectively, away from the load with the SWR equal to 2.6. If the slotted transmission line is lossless of characteristic
impedance \( Z_0 = 50(\Omega) \) and the velocity of propagation \( v = c \), determine the following:

e) Operating frequency,
f) Unknown impedance of microwave antenna,
g) It is required to match the antenna in part a section of transmission line
of characteristic impedance \( Z_0 = 50(\Omega) \). Show that the double-stub tuner available in the laboratory and shown in Figure 4 is not suitable to
achieve the required matching.
h) It is suggested that we insert a length \( d \) of a 50(\Omega) lossless transmission
line between the antenna and the side A-A of the double stub tuner. Find the \textit{minimum} length of such a transmission-line section and the lengths \( l_1 \) and \( l_2 \) of the stubs required to achieve matching.

Solution:

e) Short-circuit

\[
d_{\text{min}}^{(1)} = 23.6 \text{ cm} = 0.236 \text{ m}
\]
\[
d_{\text{min}}^{(2)} = 35.4 \text{ cm} = 0.354 \text{ m}
\]
\[
d_{\text{min}}^{(n)} = \frac{\lambda}{4\pi} \left[ \theta_\rho + (2n + 1)\pi \right]
\]
\[
d_{\text{min}}^{(2)} - d_{\text{min}}^{(1)} = \frac{\lambda}{4\pi} 2\pi = \frac{\lambda}{2}
\]
\[
\lambda = 2\left[d_{\text{min}}^{(2)} - d_{\text{min}}^{(1)} \right] = 0.236 \text{ m}
\]
\[
f = \frac{c}{\lambda} = 1.2712 \text{ (GHz)}
\]

f) Circuit with antenna

\[
d_{\text{min}}^{(1)} = 27.8 \text{ cm} = 0.278 \text{ m}
\]
\[
d_{\text{min}}^{(2)} = 39.6 \text{ cm} = 0.396 \text{ m}
\]
\[
d_{\text{min}}^{(n)} = \frac{\lambda}{4\pi} \left[ \theta_\rho + (2n + 1)\pi \right]
\]
\[
d_{\text{min}}^{(1)} = 0.278 = \frac{\lambda}{4\pi} \left[ \theta_\rho + 3\pi \right]
\]
\[\theta_\rho = 4.712\pi - 3\pi = 1.712\pi = 2\pi - 0.288\pi\]
\[
SWR = \frac{1 + \rho}{1 - \rho} \\
|\Gamma| = \frac{SWR - 1}{SWR + 1} = \frac{1.6}{3.6} = 0.444
\]

\[
\Gamma = |\Gamma| e^{j\theta}\rho = 0.444e^{j(2\pi - 0.288\pi)} = 0.444e^{-j0.288\pi}
\]

\[
\Gamma = 0.274 - j0.349
\]

\[
Z_{\text{antenna}} = Z_0 \frac{1 + \Gamma}{1 - \Gamma} = 50 \frac{1.274 - j0.349}{0.727 + j0.349} = (60 - j55)\Omega = Z_L
\]

g) With this impedance and stub location either on side A-A and B-B the matching can not be achieved because the load is in the FORBIDDEN REGION.

h) We will put between the antenna and A-A the transmission line of a 50(\Omega), with the length \( l = x\lambda \) so that the distance from A-A to the first stab is \( d_1 = 0.11\lambda + l = (0.11 + x)\lambda \), and \( a = \tan[\beta(0.11 + x)\lambda] \). Using (21) and (22) where the minimum length \( d_1 \) between the first stab and the load was calculated we obtain,

\[
a_1 = -26.4506
\]

\[
a_2 = 0.6926
\]

I) \[
\tan[\beta(0.11 + x)\lambda] = -26.4506
\]

\[0.11 + x = 0.2560142\]

\[
x = 0.1460142 \text{ or } d = 0.1460142\lambda, \quad d_{1(1)} = 0.2560142\lambda
\]

II) \[
\tan[\beta(0.11 + x)\lambda] = 0.6926
\]

\[0.11 + x = 0.096407\]

\[
x = -0.013593 \text{ or } d = -0.013593\lambda \quad \text{ (reject)}
\]
We choose,
\[ x = 0.1460142 \quad \text{or} \quad d = 0.1460142\lambda, \quad d_{1(1)} = 0.2560142\lambda \]

By [5] (Smith chart) chose:
\[ d = 0.15\lambda \]

The lengths of the stabs by [5] (Smith chart) are,
\[ l_1^{(1)} = 0.4\lambda, \quad l_2^{(1)} = 0.303\lambda \]

By this work,
\[ l_1^{(1)} = l_2^{(2)} = 0.401\lambda, \quad l_1^{(1)} = l_2^{(2)} = 0.300\lambda \]

12.9) A load of \((30 - j40)\Omega\) is connected to a 50\Omega line. The stubs are \(\lambda/8\) apart and the stub 1 is 0.3\lambda from the load.

d) Find the lengths of the stubs required for the smallest SWR between stubs.

e) Also find the value of SWR between the load and the first stub, and between stubs.

f) Verify the SWR of the total matching after the second stub which must be 1.

Solution:

d) From []
\[ l_1^{(1)} = 0.34\lambda, \quad l_2^{(1)} = 0.129\lambda \]

This work:
\[ l_1^{(1)} = 0.3400\lambda, \quad l_2^{(1)} = 0.1270\lambda \]
\[ l_1^{(2)} = 0.4333\lambda, \quad l_2^{(2)} = 0.4484\lambda \]

e) SWR near the load before the first stub (for the first couple)
\[ |\Gamma_{BS1}| = 0.5 \]
\[ SWR_{BS1} = 3 \]

SWR between the stubs

\[ | \Gamma_{S1-S2} | = 0.4383 \]

\[ SWR_{S1-S2} = 2.5605 \]

f) SWR after the second stub the stubs

\[ | \Gamma_{AS2} | = 4.7277 \cdot 10^{-4} \]

\[ SWR_{AS2} = 1.0009 \]

Thus, there is not the reflected wave, so that the calculations are confirmed.

12.10) A double stub tuner is illustrated where the positions of the stubs on the transmission line are fixed but the short stub of the lengths of the \( l_1 \) and \( l_2 \) are variable. If the load is \( Z_L = (50 + j100)\Omega \) and the characteristic impedances \( Z_0 = 100\Omega \) is equal either of the line or the stubs. If \( d_1 = \lambda/4 \) and \( d_2 = \lambda/8 \). Determine the shortest lengths \( l_1 \) and \( l_2 \) to give no reflected wave at the point \( P \) (point of the second stub). Find also SWRs between \( R \) (point of the load) and \( Q \) (point of the first stub) and between \( Q \) and \( P \).

Solution:

\[ l_1 = 0.1364\lambda, \quad l_2 = 0.1494\lambda \]

\[ l_1 = 0.3636\lambda, \quad l_2 = 0.4442\lambda \]

For the first couple,

\[ | \Gamma_R | = 0.6202 \]

\[ VSWR_{RQ} = \frac{1+| \Gamma_R |}{1-| \Gamma_R |} = 4.44266 \]

\[ | \Gamma_Q | = 0.3437 \]

\[ SWR_{QP} = \frac{1+| \Gamma_Q |}{1-| \Gamma_Q |} = 2.0474 \]

\[ Z_{QTotal} = (186.62 - j49.974)\Omega \]
\[ |\Gamma_p| = 1.8657 \cdot 10^{-4} \]

\[ SWR_p = \frac{1+|\Gamma_p|}{1-|\Gamma_p|} = 1 \]

12.11) A double stub is used to match a load of impedance \( Z_L = (100 + j100)\Omega \) to a lossless transmission line with \( Z_0 = 300\Omega \). The stubs are separated by \( 3\lambda/8 \) and one of the stubs is located at the load.

a) Obtain the lengths of the two stubs if they are terminated by an open circuit.

b) What is the value of the SWR in the main line and in the section between the two stubs?

Verify the SWR after the second stub.

Solution:

d) O/C

\[ l_1 = 0.1494\lambda, \quad l_2 = 0.4364\lambda \]

\[ l_1 = 0.4442\lambda, \quad l_2 = 0.3399\lambda \]

e) For the second solution:

\[ |\Gamma_{BS2}| = 0.6192 \]

\[ SWR_{BS2} = \frac{1+|\Gamma_{BS2}|}{1-|\Gamma_{BS2}|} = 4.2521 \]

\[ Z_{Tj2} = 78.5220 + j97.6663 \]

f) \[ |\Gamma_{AS2}| = 3.5795 \cdot 10^{-4} \]

\[ SWR_{AS2} = \frac{1+|\Gamma_{AS2}|}{1-|\Gamma_{AS2}|} = 1.0007 \]

Thus, there is not the reflected wave, so that the calculations are confirmed.

12.12) For the double-stub tuner, find the shortest values of \( l_1 \) and \( l_2 \) to match the load if \( Z_L = (100 + j50)\Omega \), \( Z_0 = Z_{S1} = Z_{S2} = 50\Omega \). The distances between
the stubs and between the first stub are equal \( d_1 = d_2 = \lambda/8 \). The stubs are short circuits.

Solution:

The Smith chart [13]

\[
\begin{align*}
l_1 &= 0.194\lambda, & l_2 &= 0.15\lambda \\
l_1 &= 0.4\lambda, & l_2 &= 0.442\lambda
\end{align*}
\]

The other possible design, where the respective lengths are found to be \(0.4\lambda\) and \(0.442\lambda\), is not recommended.

This work:

\[
\begin{align*}
l_1 &= 0.1942\lambda, & l_2 &= 0.1494\lambda \\
l_1 &= 0.3994\lambda, & l_2 &= 0.4442\lambda
\end{align*}
\]

Thus, we choose also the first couple. All results are in the excellent agreement.

12.13) A lossless 100\(\Omega\) transmission line is to be matched with a \((100 + j100)\Omega\) load using a double-stub tuner. Separation between the two stubs is \(\lambda/8\) and the characteristic impedance is 100\(\Omega\). A load is connected right at the location of the first stub. Determine the shortest possible lengths of the two stubs to obtain the matched condition and find the SWR between the two stubs.

Solution:

From this work: \( k_1 = k_2 = 1 \)

\[
\begin{align*}
l_1 &= 0.3399\lambda, & l_2 &= 0.1494\lambda \\
l_1 &= 0.4364\lambda, & l_2 &= 0.4442\lambda
\end{align*}
\]

The first couple must be chosen.

\[
|\Gamma_{BetweenS1S2}| = 0.7454 \\
SWR_{BS1S2} = \frac{1+|\Gamma_{BBetweenS1S2}|}{1-|\Gamma_{BetweenS1S2}|} = 6.8541
\]
The SWR after the second stub must be 1.

\[ Z_{TJS2} = (186.65 - j49.924)\Omega \]

\[ |\Gamma_{AS2}| = 3.2507 \cdot 10^{-4} \]

\[ SWR_{AS2} = \frac{1+|\Gamma_{AS2}|}{1-|\Gamma_{AS2}|} = 1.0007 \]

12.14) A lossless \( Z_0 = 75\Omega \) transmission line is to be matched with a \((150 + j15)\Omega\) load using a shunt-connected double-stub tuner. Separation between the two stubs is \( \lambda/8 \) and the characteristic impedance is \( Z_{S1} = Z_{S2} = 75\Omega \). The stub closest to the load (first stub) is \( \lambda/2 \) away from it.

c) Determine the shortest possible lengths of the two stubs to obtain the matched condition and find the SWR on each section of the transmission line.

d) Repeat (a) if \( Z_{S1} = Z_{S2} = 100\Omega \).

Solution:

c) \( k_1 = k_2 = 1 \)

\[ l_1 = 0.2793\lambda, \quad l_2 = 0.1482\lambda \]

\[ l_1 = 0.4233\lambda, \quad l_2 = 0.4444\lambda \]

The first couple must be chosen.

\[ |\Gamma_{ALoad}| = 0.3392 \]

\[ SWR_{BS2} = \frac{1+|\Gamma_{ALoad}|}{1-|\Gamma_{ALoad}|} = 2.0266 \]

\[ Z_{TJS1} = 150.00 + j15.00 \]

\[ |\Gamma_{S1S2}| = 0.3485 \]
\[
SWR_{BS2} = \frac{1 + |\Gamma_{S1S2}|}{1 - |\Gamma_{S1S2}|} = 2.0696
\]

\[
Z_{TJS2} = (140.77 \, - \, j38.871)\Omega
\]

\[
| \Gamma_{AS2} | = 3.0775 \cdot 10^{-4}
\]

\[
SWR_{BS2} = \frac{1 + |\Gamma_{AS2}|}{1 - |\Gamma_{AS2}|} = 1.0006
\]

d) \quad k_1 = k_2 = 1.333

\[
l_1 = 0.2888\lambda, \quad l_2 = 0.1257\lambda
\]

\[
l_1 = 0.4405\lambda, \quad l_2 = 0.4575\lambda
\]

The first couple must be chosen.

\[
| \Gamma_{A\text{Load}} | = 0.3392
\]

\[
SWR_{A\text{Load}} = \frac{1 + |\Gamma_{A\text{Load}}|}{1 - |\Gamma_{A\text{Load}}|} = 2.0266
\]

\[
Z_{TJS1} = (150.00 \, + \, j15.00)\Omega
\]

\[
| \Gamma_{S1S2} | = 0.3485
\]

\[
SWR_{BS2} = \frac{1 + |\Gamma_{S1S2}|}{1 - |\Gamma_{S1S2}|} = 2.0696
\]

\[
Z_{TJS2} = (140.72 \, - \, j38.955)\Omega
\]

\[
| \Gamma_{AS2} | = 2.3194 \cdot 10^{-4}
\]

\[
SWR_{AS2} = \frac{1 + |\Gamma_{AS2}|}{1 - |\Gamma_{AS2}|} = 1.0005
\]

12.15) A lossless \( Z_0 = 50 \Omega \) transmission line is to be matched with a \( (25 + j50)\Omega \) load using an open double-stub tuner. Separation between the two stubs
is \(3\lambda/8\) and the characteristic impedance is \(Z_{S1} = Z_{S2} = 50\Omega\). The stub closest to the load (first stub) is 0.2\(\lambda\) away from it.

c) Determine the shortest possible lengths of the two stubs to obtain the matched condition and find the SWR on each section of the transmission line.

d) Repeat (a) if \(Z_{S1} = 75\Omega, Z_{S2} = 100\Omega\).

Solution:

c) \(k_1 = k_2 = 1\)

\[
l_1 = 0.3904\lambda, \quad l_2 = 0.1476\lambda
\]

\[
l_1 = 0.3160\lambda, \quad l_2 = 0.2964\lambda
\]

The first couple must be chosen.

\[
|\Gamma_{A\text{Load}}| = 0.6202
\]

\[
SWR_{A\text{Load}} = \frac{1 + |\Gamma_{A\text{Load}}|}{1 - |\Gamma_{A\text{Load}}|} = 4.2656
\]

\[
Z_{TJS1} = (25 + j50)\Omega
\]

\[
|\Gamma_{S1S2}| = 0.5547
\]

\[
SWR_{BS2} = \frac{1 + |\Gamma_{S1S2}|}{1 - |\Gamma_{S1S2}|} = 3.4918
\]

\[
Z_{TJS2} = 90.1135 + j80.0173
\]

\[
|\Gamma_{AS2}| = 7.1621 \cdot 10^{-4}
\]

\[
SWR_{AS2} = \frac{1 + |\Gamma_{AS2}|}{1 - |\Gamma_{AS2}|} = 1.0014
\]

d) \(k_1 = 1.5, \quad k_2 = 2\)
\[ l_1 = 0.3583 \lambda, \quad l_2 = 0.1929 \lambda \]

\[ l_1 = 0.2954 \lambda, \quad l_2 = 0.2737 \lambda \]

The first couple must be chosen.

\[ | \Gamma_{ALoad} | = 0.6202 \]

\[ SWR_{BS2} = \frac{1+| \Gamma_{ALoad} |}{1-| \Gamma_{ALoad} |} = 4.2656 \]

\[ Z_{TjS1} = (25 + j50) \Omega \]

\[ | \Gamma_{S1S2} | = 0.5549 \]

\[ SWR_{BS2} = \frac{1+| \Gamma_{S1S2} |}{1-| \Gamma_{S1S2} |} = 3.4929 \]

\[ Z_{TjS2} = (89.9492 + j80.0367) \Omega \]

\[ | \Gamma_{AS2} | = 3.4674 \cdot 10^{-4} \]

\[ SWR_{AS2} = \frac{1+| \Gamma_{AS2} |}{1-| \Gamma_{AS2} |} = 1.0007 \]
13. TRIPLE TUNER STUBS EN PARALLÈLE

The load impedance $Z_L = A + jB$, $(A, B \in R)$ of the transmission line with the characteristic impedance $Z_0$ and without losses, is to be matched by a triple-stub tuner connected in parallel (Fig. 13). The first stub is distanced from the load by $d_1$. The distance between the first and the second stub is $d_2$ and the distance between the second and the third stub is $d_3$. Their characteristic impedances are respectively $Z_{S1} = k_1 Z_0$, $Z_{S2} = k_2 Z_0$ and $Z_{S3} = k_3 Z_0$. The lengths $l_1$, $l_2$ and $l_3$ are required to find. In this paper all impedances or admittances are normalized.

![Diagram](image)

Fig. 13. Triple short stubs in parallel (S/S/S; S/S/O; O/S/S; O/S/O; S/O/S; S/O/O; O/O/S; O/O/O)

The normalized load admittance is,

$$y_{LN} = \frac{1}{y_{LN}} = p_L + jq_L$$

where,

$$p_L = \frac{A}{Y_0(A^2 + B^2)}, q_L = \frac{-B}{Y_0(A^2 + B^2)}$$

Let us calculate the normalized admittance at points $A$,

$$y_A = y(d_1) = \frac{y_{LN} + j \tan(\beta d_1)}{1 + j \tan(\beta d_1)y_{LN}} = \frac{y_{LN} + ja}{1 + ja y_{LN}} = g_{LA} + j b_{LA}$$

(43)
where

\[ a = \tan(\beta d_1), \ a \in R \]
\[ g_{LA} = \frac{p_L(1 + a^2)}{(1 - a q_L)^2 + a^2 p_L^2} \]
\[ b_{LA} = \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 - a q_L)^2 + a^2 p_L^2} \]

The total admittance at points A is,

\[ y_{TotalA} = y_A + jy_{S1} = g_{LA} + j(b_{LA} + B_{S1}) = g_{LA} + jB_1 \tag{44} \]

where,

\[ y_{S1} = jB_{S1} \tag{45} \]

\[ B_1 = b_{LA} + B_{S1} \]
\[ B_{S1} = -\frac{1}{z_{S1}} = -\frac{1}{k_1 b}, \ z_{T1} = k_1 \tan(\beta l_1) = k_1 b, \left(\frac{S}{C}\right) \tag{46} \]
\[ B_{S1} = \frac{1}{z_{S1}} = \frac{b}{k_1}, \ z_{T1} = \frac{\tan(\beta l_1)}{k_1} = \frac{b}{k_1}, \left(\frac{O}{C}\right) \tag{47} \]
\[ b = \tan(\beta l_1), \]

The next step is the determination of the admittance at point B.

Now, the admittance at the point B is

\[ y_B = \frac{y_{TotalA} + j \tan(\beta d_2)}{1 + jy_{TotalA} \tan(\beta d_2)} \]
\[ = \frac{y_{TotalA} + jm}{1 + jy_{TotalA} m} = g_{LB} + j b_{LB} \tag{48} \]

where,

\[ m = \tan(\beta d_2) \]
\[ g_{LB} = \frac{g_{LA}(1 + m^2)}{(1 - m B_1)^2 + m^2 g_{LA}^2} \tag{49} \]
\[ b_{LB} = \frac{m(1 - g_{LA}^2 - B_1^2) + B_1(1 - m^2)}{(1 - m B_1)^2 + m^2 g_{LA}^2} \tag{50} \]

The total admittance at the point B is,

\[ y_{TotalB} = y_B + jy_{S2} = g_{LB} + j(b_{LB} + B_{S2}) = g_{LB} + jB_2 \tag{51} \]
with

\[ B_2 = b_{LB} + B_{S2} \]  \hspace{1cm} (52)

where,

\[ B_{S2} = - \frac{1}{Z_{S2}} = - \frac{1}{k_2 n}, \quad Z_{S2} = k_2 \tan(\beta l_2) = k_2 n, \left( \frac{S}{C} \right) \]  \hspace{1cm} (52)

\[ B_{S2} = \frac{1}{Z_{S2}} = \frac{n}{k_2}, \quad Z_{S2} = \frac{\tan(\beta l_2)}{k_2} = \frac{n}{k_2}, \left( \frac{O}{C} \right) \]  \hspace{1cm} (53)

The admittance at the point C is,

\[ y_C = \frac{y_{TotalB} + j \tan(\beta d_3)}{1 + j y_{TotalB} \tan(\beta d_3)} = \frac{y_{TotalB} + j p}{1 + j y_{TotalB} p} = g_{LC} + j b_{LC} \]  \hspace{1cm} (54)

with

\[ p = \tan(\beta d_3) \]

\[ g_{LC} = \frac{g_{LB}(1 + p^2)}{(1 - pB_2)^2 + p^2 g_{LB}^2} \]  \hspace{1cm} (55)

\[ b_{LC} = \frac{(1 - pL^2 - B_2^2) + B_2 (1 - p^2)}{(1 - pB_2)^2 + p^2 g_{LB}^2} \]  \hspace{1cm} (56)

The total admittance at the point C is,

\[ y_{TotalC} = y_C + j y_{S3} = g_{LC} + j (b_{LC} + B_{S3}) = g_{LC} + j B_3 \]  \hspace{1cm} (57)

with

\[ B_3 = b_{LC} + B_{S3} \]  \hspace{1cm} (58)

The condition of the matching at the point C is,

\[ y_{TotalC} = 1 \]  \hspace{1cm} (59)

which gives,

\[ g_{LC} = 1 \quad \text{or} \quad \frac{g_{LB}(1 + p^2)}{(1 - pB_2)^2 + p^2 g_{LB}^2} = 1 \]  \hspace{1cm} (60)

and
\[ b_{LC} + B_{S3} = 0, \text{ or} \]
\[ B_{S3} = \frac{-p(1 - g_{LB}^2 - B_{2}^2) + B_{2}(1 - p^2)}{(1 - pB_{2})^2 + p^2 g_{LB}^2} \quad (61) \]

where,
\[ B_{S3} = -\frac{1}{z_{S3}} = -\frac{1}{k_{3}q}, \quad z_{S3} = k_{3} \tan(\beta l_{3}) = k_{3}q, \quad \left( \frac{S}{C} \right) \quad (62) \]
\[ B_{S3} = \frac{1}{z_{S3}} = \frac{q}{k_{3}}, \quad z_{S2} = \frac{\tan(\beta l_{3})}{k_{3}} = \frac{q}{k_{3}}, \quad \left( \frac{O}{C} \right) \quad (63) \]

The next step is determination of all stub lengths.

From (60) we obtain,
\[ (B_{2} - \frac{1}{p})^2 + g_{LB}^2 = g_{LB}^2 \frac{g(p)}{g_{LB}} \quad (64) \]

whose solutions for \( B_{2} \) are,
\[ B_{2}^{(1,2)} = \frac{1}{p} \pm g_{LB} \sqrt{\frac{g(p)}{g_{LB}} - 1} \quad (65) \]

where,
\[ g(p) = \frac{1 + p^2}{p^2} \quad (66) \]

Equation (65) can have the real solutions if,
\[ \frac{g(p)}{g_{LB}} - 1 \geq 0 \]

or,
\[ g_{LB} \leq g(p) = \frac{1 + p^2}{p^2} \quad (67) \]

Also, equation (64) can be written in the form,
\[ g_{LB}^2 - g_{LB}g(p) + \left(B_{2} - \frac{1}{p}\right)^2 = 0 \quad (68) \]

whose solutions are,
\[ g_{LB(1,2)} = \frac{g(p) \pm \sqrt{g(p)^2 - 4\left(B_2 - \frac{1}{p}\right)^2}}{2} \]  

(69)

The following condition must be satisfied,

\[ g(p)^2 - 4\left(B_2 - \frac{1}{p}\right)^2 \geq 0 \]

which gives,

\[ -\frac{(p - 1)^2}{2p^2} \leq B_2 \leq \frac{(p + 1)^2}{2p^2} \]  

(70)

or

\[ -g_{LB(i)} - \frac{(p - 1)^2}{2p^2} \leq B_{S2}^{(i)} \leq -g_{LB(i)} + \frac{(p + 1)^2}{2p^2} \]  

(71)

This condition is important because it gives the limits for \( B_2 \) or the susceptance for the second stub.

Combining (49) and (64) we obtain,

\[ g_{LB} = \frac{g_{LA}\left(1 + m^2\right)}{(1 - MB_1)^2 + m^2g_{LA}^2} \leq g(p) \]

or

\[ \frac{g_{LA}g_{(m)}}{(B_1 - \frac{1}{m})^2 + g_{LA}^2} \leq g(p) \]  

(72)

where,

\[ g_{(m)} = \frac{1 + m^2}{m^2} \]  

(73)

Introducing the constant \( Q \) as,

\[ Q = \frac{g_{(m)}}{g_{LA}g(p)} = \frac{(1 + m^2)p^2}{m^2(1 + p^2)g_{LA}} > 0, \forall m, p, g_{LA} \]  

(74)

The expression (74) can be written in the following form,
\[
\frac{g_{LA}^2 Q}{(B_1 - \frac{1}{m})^2 + g_{LA}^2} \leq 1
\]  
(75)

This inequality presents the condition for finding the length of the first stub. Introducing the parameter \( t \geq 1 \), (75) can be written as,

\[
\frac{g_{LA}^2 Q}{(B_1 - \frac{1}{m})^2 + g_{LA}^2} = \frac{1}{t} \leq 1
\]  
(76)

Let us take the limit of (76),

\[
(B_1 - \frac{1}{m})^2 + g_{LA}^2 = g_{LA}^2 Q t
\]

or

\[
(B_1 - \frac{1}{m})^2 + g_{LA}^2 = g_{LA}^2 (Q t - 1)
\]

I) If \( 0 < Q \leq 1 \)

\[
(B_1 - \frac{1}{m})^2 = 0
\]

\[
B_1 = \frac{1}{m}
\]  
(77)

Equation (36) gives the susceptance for the first stub,

\[
B_{S1} = -b_{LA} + \frac{1}{m}
\]  
(78)

One can see that \( B_{S1} \) does not depend on \( 't' \). Moreover, the susceptance for the second and the third stubs do not depend on \( 't' \) by their definition. It is important because the unique triple stub matching is achieved.

Thus, there is the unique solution for the triple stub matching where \( 0 < Q \leq 1 \). This case is usual for \( g_{LA} > 1 \).

II) If \( Q > 1 \)

\[
(B_1 - \frac{1}{m})^2 - g_{LA}^2 (Q t - 1) = 0
\]  
(79)

Equation (79) gives the limit susceptance of the first stub for which the matching is possible.

\[
B_1 \geq \frac{1}{m} + g_{LA} \sqrt{Q t - 1} \quad \text{and} \quad B_1 \leq \frac{1}{m} - g_{LA} \sqrt{Q t - 1}
\]  
(80)
with the condition,
\[ g_{LA} < \left( \frac{1 + m^2}{m^2} \right)^{1/2} \] \hspace{1cm} (81)

The conductance \( g_{LA} \) at the point of the connection with the first stub must satisfy the condition (81).

Equations (39) gives the limits where the matching is possible for the first stub.

From the previous analysis one can conclude that there is not the unique solution for \( Q > 1 \). It could be an optimization problem. This is problem of the three variables, the lengths of the three stubs could be chosen to optimize the bandwidth of matching.

For \( Q > 1 \) the susceptance \( B_1 \) for the first stub are determined as,
\[ B_{1(1,2)} = \frac{1}{m} \pm g_{LA} \sqrt{Q - 1} \] \hspace{1cm} (82)

or
\[ B_{S1(1,2)} = -g_{LA} + \frac{1}{m} \pm g_{LA} \sqrt{Q - 1} \] \hspace{1cm} (83)

with condition (81) for \( g_{LA} \).

The corresponding lengths for the first stub can be obtained for:

a) Short circuit (stub)
\[ b_{S1(1,2,3,4)}^{(1,2,3,4)} = -\frac{1}{k_1 B_{S1(1,2,3,4)}} \] , \[ B_{S1(1)} = B_{S1(2)}, B_{S1(3)} = B_{S1(4)} \]

or
\[ b_{S1(1,2,3,4)}^{(1,2,3,4)} = -\frac{1}{k_1 B_{S1(1,2,3,4)}} \]
\[ l_{S1(1,2,3,4)}^{(1,2,3,4)} = \frac{\lambda}{2\pi} \text{atan} \left[ b_{S1(1,2,3,4)}^{(1,2,3,4)} \right] \] \hspace{1cm} (84)

b) Open circuit (stub)
\[ B_{S1(1,2,3,4)}^{(1,2,3,4)} = \frac{b_{S1(1,2,3,4)}}{k_1} \] , \[ B_{S1(1)} = B_{S1(2)}, B_{S1(3)} = B_{S1(4)} \]
or

\[ k_1 B_{S2}^{(1,2,3,4)} = b_{(C)S2}^{(1,2,3,4)} \]

\[ l_{(C)S1}^{(1,2,3,4)} = \frac{\lambda}{2\pi} \text{atan} \left[ b_{(C)S1}^{(1,2,3,4)} \right] \]  \hspace{1cm} (85)

From (79) it can be seen that

\[ B_1^{(1)} + B_1^{(2)} = \frac{2}{m} \]  \hspace{1cm} (86)

and

\[ g_{LB1} = g_{LB2} = \frac{g_{LA}(1 + m^2)}{(1 - mB_1^{(1)})^2 + m^2 g_{LA}^2} \]  \hspace{1cm} (87)

\[ b_{LB1} = \frac{m \left( 1 - g_{LA}^2 - B_1^{(1)2} \right) + B_1^{(1)}(1 - m^2)}{(1 - mB_1^{(1)})^2 + m^2 g_{LA}^2} \]  \hspace{1cm} (88)

\[ b_{LB2} = \frac{m \left( 1 - g_{LA}^2 - B_1^{(2)2} \right) + B_1^{(1)}(1 - m^2)}{(1 - mB_1^{(1)})^2 + m^2 g_{LA}^2} \]  \hspace{1cm} (89)

Now, we simply find from (52) and (65) the susceptance for the second stub,

\[ B_{S2}^{(1,2,3,4)} = -b_{LB}^{(1,2)} + \frac{1}{p} \pm \frac{g_{LA}^{(1,2)}}{\sqrt{g_{LB}^{(1,2)}}} - 1 \]  \hspace{1cm} (90)

where \( g_{LB}^{(1,2)} \), \( b_{LB}^{(1,2)} \) and \( g_{(p)} \) are given by (87), (88), (89) and (66).

Now, let us find the corresponding lengths for the second stub:

c) Short circuit (stub)

\[ B_{S2}^{(1,2,3,4)} = -\frac{1}{k_2 n_{(S)S2}^{(1,2,3,4)}} \]

or

\[ n_{(S)S2}^{(1,2,3,4)} = -\frac{1}{k_2 B_{S2}^{(1,2,3,4)}} \]  \hspace{1cm} (91)

\[ l_{(S)S2}^{(1,2,3,4)} = \frac{\lambda}{2\pi} \text{atan} \left[ n_{(S)S2}^{(1,2,3,4)} \right] \]
d) Open circuit (stub)

\[ B_{S2}^{(1,2,3,4)} = \frac{b_{(S)S2}^{(1,2,3,4)}}{k_2} \]

or

\[ k_2 B_{S2}^{(1,2,3,4)} = n_{(S)S2}^{(1,2,3,4)} \]

Finally, the susceptance of the third stub can be obtained from (60),

\[ B_{S3}^{(1,2,3,4)} = -\frac{1}{k_3 q_{(S)S3}^{(1,2,3,4)}} (1 - p^2) \]

where,

\[ g_{LB}^{(1,2)} \] is given by (87).

The corresponding lengths for the third stub are:

e) Short circuit (stub)

\[ B_{S3}^{(1,2,3,4)} = -\frac{1}{k_3 q_{(S)S3}^{(1,2,3,4)}} \]

or

\[ q_{(S)S3}^{(1,2,3,4)} = -\frac{1}{k_3 B_{S3}^{(1,2,3,4)}} \]

\[ l_{(S)S3}^{(1,2,3,4)} = \frac{\lambda}{2\pi} \text{atan} \left[ q_{(S)S3}^{(1,2,3,4)} \right] \] (94)

f) Open circuit (stub)

\[ B_{S3}^{(1,2,3,4)} = \frac{q_{(S)S3}^{(1,2,3,4)}}{k_3} \]

or

\[ k_3 B_{S3}^{(1,2,3,4)} = q_{(S)S3}^{(1,2,3,4)} \]
\[ l_{S2}^{(1,2,3,4)} = \frac{\lambda}{2\pi} \text{atan} \left[ q_{S3}^{(1,2,3,4)} \right] \]  

For stub lengths the following formula must be respected,

\[
l_s = \begin{cases} 
\frac{\lambda}{2\pi} \text{atan}[P] + \frac{\lambda}{2} & \text{for } P < 0 \\
\frac{\lambda}{2\pi} \text{atan}[P] & \text{for } P > 0 
\end{cases}
\]  

In the Table 1, the corresponding solutions are summarized as follows,

**Table 1. The stub lengths in the triple stub matching**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_{S1}^{(1)} )</td>
<td>( l_{S2}^{(1)} )</td>
<td>( l_{S3}^{(1)} )</td>
</tr>
<tr>
<td>( l_{S1}^{(2)} )</td>
<td>( l_{S2}^{(2)} )</td>
<td>( l_{S3}^{(2)} )</td>
</tr>
<tr>
<td>( l_{S1}^{(3)} )</td>
<td>( l_{S2}^{(3)} )</td>
<td>( l_{S3}^{(3)} )</td>
</tr>
<tr>
<td>( l_{S1}^{(4)} )</td>
<td>( l_{S2}^{(4)} )</td>
<td>( l_{S3}^{(4)} )</td>
</tr>
</tbody>
</table>

This schema is valuable either for the short or the open stubs. Thus, the eight possible combinations can be given by this schema \((S/S/S; S/S/O; O/S/S; O/S/O/; S/O/S; S/O/O; O/O/S; O/O/O)\).

In this work which treats the triple stub matching many authors after finding the lengths of the first stub use it for solving the rest of the problem as the double stub matching. In this paper as it is proposed after finding the length of the first stub, the lengths of the second and the third stubs are simple to find from corresponding formulas. Moreover, in this paper the characteristic impedance of each circuit is different.

### 13.1 Special cases

a1) \( d_1 = d_2 = d_3 = \lambda/4 \)

\[
a = m = p = \infty
\]

\[
g_{LA} = \frac{p_L}{p_L^2 + q_L^2}, \quad b_{LA} = -\frac{q_L}{p_L^2 + q_L^2}
\]

\[
g_{LB1} = g_{LB2} = \frac{g_{LA}}{B_{1(1)}^2 + g_{LA}^2} = g_{LB} \leq 1
\]
\[ b_{LB1} = -\frac{B_{1(1)}}{B_{1(1)}^2 + g_{LA}^2}, \quad b_{LB2} = -\frac{B_{1(2)}}{B_{1(2)}^2 + g_{LA}^2} \quad (99) \]

\[ B_{1}^{(1,2)} = b_{LA} + B_{S1}^{(1,2)} \quad (100) \]

\[ B_{S1}^{(1)} = -b_{LA} + g_{LA}\sqrt{Qt - 1} \quad (101) \]

\[ B_{S1}^{(2)} = -b_{LA} - g_{LA}\sqrt{Qt - 1} \quad (102) \]

\[ Q = \frac{1}{g_{LA}} \quad (103) \]

\[ B_{S2}^{(1,2,3,4)} = -b_{LB}^{(1,2)} \pm g_{LB} \sqrt{\frac{1}{g_{LB}} - 1} \quad (104) \]

\[ B_{S3}^{(1,2,3,4)} = -\frac{B_{2}^{(1,2,3,4)}}{\left(\frac{B_{2}^{(1,2,3,4)}}{g_{LB}}\right)^2 + g_{LB}^{(1,2,3,4)^2}} \quad (105) \]

a2) \( d_1 = d_2 = \lambda/4, \quad d_3 \neq \lambda/4 \)

\[ a = m = \infty, \quad p \neq \infty \]

We use (59)-(64)

\[ Q = \frac{g(p)}{g_{LA}} \quad (106) \]

and (49) and (52) for \( B_{S2}^{(1,2,3,4)} \) and \( B_{S3}^{(1,2,3,4)} \).

a3) \( d_1 = d_3 = \lambda/4, \quad d_2 \neq \lambda/4 \)

\[ a = p = \infty, \quad m \neq \infty \]

We use (56) for \( g_{LA}, b_{LA}, (46), (48 \text{ and } (58) \text{ for } g_{LB1}, \ g_{LB2}, \ b_{LB1}, \ b_{LB2}, \ (3) \) (40) \text{ and } (41) \text{ for } B_{1}^{(1,2)}, \ B_{S1}^{(1,2)} \). For \( B_{S2}^{(1,2,3,4)} \) and \( B_{S3}^{(1,2,3,4)} \) (63) \text{ and } (64) \text{ are used respectively with},

\[ Q = \frac{g(m)}{g_{LB}} \quad (107) \]

a4) \( d_1 = \lambda/4, \quad d_2 \neq \lambda/4, \quad d_3 \neq \lambda/4 \)

\[ a = \infty, \quad m \neq \infty, \quad p \neq \infty \]
We use (59) for \( g_{LA}, b_{LA} \). All other expressions for \( B_{S1}^{(1,2)}, B_{S2}^{(1,2,3,4)} \) and \( B_{S3}^{(1,2,3,4)} \) are calculated by the same expressions as in the general case.

a5) \( d_1 \neq \frac{\lambda}{4}, d_2 = \frac{\lambda}{4}, d_3 \neq \frac{\lambda}{4} \)

\[ a \neq \infty, \ m = \infty, \ p \neq \infty \]

We use (60) and (61) for calculating \( B_{S1}^{(1,2)} \), (57) and (58) for \( g_{LB}, b_{LB1}, b_{LB2} \) with

\[ Q = \frac{1}{g(p)g_{LB}} - 1 \quad (108) \]

All other expressions for \( B_{S2}^{(1,2,3,4)} \) and \( B_{S3}^{(1,2,3,4)} \) are calculated by the same expressions as in the general case.

a6) \( d_1 \neq \frac{\lambda}{4}, d_2 \neq \frac{\lambda}{4}, d_3 = \frac{\lambda}{4} \)

\[ a \neq \infty, \ m = \infty, \ p = \infty \]

We use (63) and (64) for calculating \( B_{S2}^{(1,2,3,4)} \) and \( B_{S3}^{(1,2,3,4)} \) with (66). All other expressions are calculated by the same expressions as in the general case.

The stub lengths either for short circuit or open circuit are calculated by the previous expressions.

\[ g_{LA} = \frac{p_L(1 + a^2)}{(1 - aq_L)^2 + a^2p_L^2} \geq 1 \quad (109) \]

or

\[ (1 - aq_L)^2 + a^2p_L^2 \leq p_L(1 + a^2) \]

from which,

\[ a^2(p_L^2 + q_L^2 - p_L) - 2q_La + 1 - p_L \leq 0 \quad (110) \]

The inequality (69) gives,

\[ a_2 \leq a \leq a_1 \quad (111) \]

where,
\[ a_{(1,2)} = \frac{q_L \pm \sqrt{\Delta}}{(p_L^2 + q_L^2 - p_L)} \]  \hspace{1cm} (112)

with,
\[ \Delta = p_L[(p_L - 1)^2 + q_L^2] > 0 \]  \hspace{1cm} (113)

If \( p_L^2 + q_L^2 - p_L = 0 \) and \( p_L \leq 1 \)
\[ a_{(1)} = \frac{1 - p_L}{q_L - \sqrt{p_L}(1 - p_L)} \]  \hspace{1cm} (114)
\[ a_{(2)} = \frac{1 - p_L}{q_L + \sqrt{p_L}(1 - p_L)} \]  \hspace{1cm} (115)
\[ a_{(1)} < a_{(2)} \]  \hspace{1cm} (116)

If \( Z_0 > A \rightarrow p_L^2 + q_L^2 - p_L > 0 \) and \( p_L > 1 \)
\[ a_{(1)} > 0 \ \forall \ a_{(1)} > a_{(2)} \]

Thus, the condition (70) is satisfied.

For previous cases,
\[ 0 < Q < 1 \]

This is the case of unique solution when the matching is achieved by the triple stub matching.

The corresponding formulas for this unique matching are,
\[ B_{S1}^{(1,2)} = B_{S1}^{(1,2)} = -g_{LA} + \frac{1}{m} \]
\[ B_{S2}^{(1)} = B_{S2}^{(3)} = -b_{LB}^{(1)} + \frac{1}{p} + g_{LB} \sqrt{\frac{g(p)}{g_{LB}} - 1} \]
\[ B_{S2}^{(2)} = B_{S2}^{(4)} = -b_{LB}^{(2)} + \frac{1}{p} + g_{LB} \sqrt{\frac{g(p)}{g_{LB}} - 1} \]
\[ B_{S3}^{(1)} = B_{S3}^{(3)} = -\frac{p \left( 1 - g_{LB}^2 - B_2^{(1,3)2} \right) + B_2^{(1,3)} \left( 1 - p^2 \right)}{\left( 1 - pB_2^{(1,3)} \right)^2 + p^2 g_{LB}^2} \]

\[ B_{S23}^{(2)} = B_{S3}^{(4)} = -\frac{p \left( 1 - g_{LB}^2 - B_2^{(2,4)2} \right) + B_2^{(2,4)} \left( 1 - p^2 \right)}{\left( 1 - pB_2^{(2,4)} \right)^2 + p^2 g_{LB}^2} \]

with

\[ B_2^{(1,3)} = b_{LB}^{(1)} + B_{S2}^{(1,3)} \]

\[ B_2^{(2,4)} = b_{LB}^{(2)} + B_{S2}^{(2,4)} \]

where \( g_{LB}, b_{LB}^{(1)}, b_{LB}^{(2)} \) are given by (46), (47) and (48) respectively.

14. Examples

To verify the validity of the analytical method, some problems solved by the Smith chart will be treated.

14.1) The normalized terminating impedance is \( z_{LN} = (1.64 + j1.97) \Omega \) and the normalized characteristic impedances of the line and the stubs are 1 \( \Omega \). The first stub is away \( d_1 = 0.154 \lambda \) from the load. The spacing between the first and second stub is stub \( d_2 = 1/8 \lambda \) and between the third and the second stub is \( d_3 = 1/8 \lambda \).

a) Determine the lengths of the short-circuited stubs as well as the open circuited stubs when the match is achieved.

b) Find the VSWR on any section of the transmission line.

Solution:

Short stubs:

In [2] the incomplete solution of this problem is given for the lengths of three stubs.

Using the Smith chart [2] gives,

First stub:

\[ l_1^{(1)} = 0.367\lambda \text{ and } l_1^{(2)} = 0.203\lambda \]

Second stub:

\[ l_2^{(1)} = 0.312\lambda, l_2^{(2)} = 0.471\lambda, l_2^{(3)} = 0.100\lambda, l_2^{(4)} = 0.457\lambda \]

Third stub:
\[ l_3^{(1)} = 0.367\lambda \text{ and } l_3^{(2)} = 0.203\lambda \]

From the presented method:

\[ a = \tan(\beta d_1) = 1.45175 \]

\[ m = \tan(\beta d_2) = 1, \quad p = \tan(\beta d_3) = 1 \]

\[ g_{LA} = 0.3540, \quad b_{LA} = 0.7132 \]

\[ Q = 2.8251 \]

Let us begin with \( t = 1 \), (38) for which the smallest limits of the first stub are given by (36) and (37),

\[ B_{S1}^{(1)} > -b_{LA} + \frac{1}{m} + g_{LA}\sqrt{Q-1} = 0.7650 \]

and

\[ B_{S1}^{(1)} < -b_{LA} + \frac{1}{m} - g_{LA}\sqrt{Q-1} = -0.1914 \]

Thus,

\[ B_{S1}^{(1)} > 0.7650 \]

\[ B_{S1}^{(2)} < -0.1914 \]

are the limits for the first stub or more accurately the domain in which the matching is possible.

Just for these limits we obtained (the case \( t = 1 \)),

\[ l_{S1}^{(1)} = l_{S1}^{(2)} = 0.3539\lambda \text{ and } l_{S1}^{(3)} = l_{S1}^{(4)} = 0.2199\lambda \]

\[ l_{S2}^{(1)} = l_{S2}^{(2)} = 0.4666\lambda \text{ and } l_{S2}^{(3)} = l_{S2}^{(4)} = 0.1526\lambda \]

\[ l_{S3}^{(1)} = l_{S3}^{(2)} = l_{S3}^{(3)} = l_{S3}^{(4)} = 0.3750\lambda \]

The same results can be obtained by [1]. In [2] using the Smith chart the susceptances for the first stub are obtained as follows,

\[ B_{S1(1)} = 0.9 \text{ and } B_{S1(1)} = -0.3 \]
from which the lengths of the first stub are previously given.

The limits for the first stub are (36),

\[ B_{S1}^{(1)} > 0.7650 \quad \text{and} \quad B_{S1}^{(1)} < -0.1914 \]

Obviously, these values [1] are in the domain of definition for the matching,

\[ 0.9 > 0.7650 \]

and

\[ -0.3 < -0.1914 \]

Possibly in [1] these values were chosen randomly to be sure that the load will be outside of the forbidden region.

By inspection and with [11] we find \( t = 1.37931 \) which could correspond to the results obtained by the earlier values for the susceptance of the first stub, [1].

By [11] the susceptances for the first stub are,

\[ B_{S1}^{(1)} = 0.8892, \quad B_{S1}^{(2)} = -0.3157 \]

These values satisfy (36) and (37).

The limits for the second stub (28) are,

\[ 3.4679 \leq B_{S2}^{(1,2)} \leq 5.4679 \]

and

\[ -1.4679 \leq B_{S2}^{(3,4)} \leq 0.5321 \]

For the second stub the calculate values are,

\[ B_{S2}^{(1)} = 5.3609, \quad B_{S2}^{(2)} = 3.5748 \]

\[ B_{S2}^{(3)} = 0.4252, \quad B_{S2}^{(4)} = -1.3609 \]

Obviously, all susceptance for the second stub belong to previously domain of the matching.
The lengths of each stub which can be summarized in the Table 2.

**Table 2. The stub lengths (S/C)**

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3657$\lambda$</td>
<td>0.4566$\lambda$</td>
<td>0.3084$\lambda$</td>
</tr>
<tr>
<td>0.3657$\lambda$</td>
<td>0.4706$\lambda$</td>
<td>0.4118$\lambda$</td>
</tr>
<tr>
<td>0.2014$\lambda$</td>
<td>0.1008$\lambda$</td>
<td>0.3084$\lambda$</td>
</tr>
<tr>
<td>0.2014$\lambda$</td>
<td>0.3141$\lambda$</td>
<td>0.4118$\lambda$</td>
</tr>
</tbody>
</table>

The shortest stub lengths are:

\[ l_{S1} = 0.2013\lambda, \quad l_{S2} = 0.1009\lambda, \quad l_{S3} = 0.1009\lambda \]

The obtained results correspond to those for the first and the second stub obtained by the Smith chart [2] but not for the third stub.

By using [1] we obtained the same results as with our analytical approach.

Let us prove the validity of the results obtained analytically for the shortest lengths of stubs.

I) VSWR before the load:

\[ d_{1zN} = 0, \quad l_{1zN} = 0 \]

\[ z_{1N} = (1.64 + j1.79)\Omega \]

\[ |\Gamma_{BLoad}| = 0.6288, \quad VSWR_{BLoad} = 4.3883 \]

II) VSWR between the first and the second stub:

\[ d_1 = 0.154\lambda, \quad l_{S1}^{(1)} = 0.2013\lambda \]

\[ z_{TotalS1} = (1.64 + j1.79)\Omega \]

\[ |\Gamma_{(S1-S2)}| = 0.5375, \quad VSWR_{(S1-S2)} = 3.3242 \]

III) VSWR after the second stub:

\[ z_{TotalS2} = (1.2501 - j1.4032)\Omega \]

\[ d_2 = 0.125\lambda, \quad l_{S2}^{(1)} = 0.1009\lambda \]

\[ |\Gamma_{(S2-S3)}| = 0.1884, \quad VSWR_{(S2-S3)} = 1.4641 \]
IV) VSWR after the third stub:

\[ z_{TS3} = (0.6863 - j0.0508)\Omega \]
\[ d_3 = 0.125\lambda, \quad l_{S3}^{(1)} = 0.3084 \lambda \]
\[ |\Gamma_{(AS3)}| = 3.9793 \cdot 10^{-4} \]
\[ VSWR_{(AS3)} = 1.0008 \]

Thus, the calculations are proved. In these calculations \( \Gamma \) and \( VSWR \) are the reflection coefficient and the standing wave ratio, respectively. The potential users can verify the reflection coefficient and the standing wave ratio for other solutions of the stub lengths.

Open stubs:

**Table 3. The stub lengths (O/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1157</td>
<td>0.2206</td>
<td>0.1618</td>
</tr>
<tr>
<td>0.1157</td>
<td>0.2066</td>
<td>0.0584</td>
</tr>
<tr>
<td>0.4513</td>
<td>0.0640</td>
<td>0.1618</td>
</tr>
<tr>
<td>0.4513</td>
<td>0.3509</td>
<td>0.0584</td>
</tr>
</tbody>
</table>

By using [1] we obtained the same results.

14.2) The terminating impedance is \( Z_L = (50 - j10)\Omega \) and the characteristic impedance \( Z_0 \) of the line and the stubs is 50 \( \Omega \). The first stub is connected to the load \( (d_1 = 0 \lambda) \). The spacing between the first and second stub is stub \( d_2 = 1/8 \lambda \) and between the third and the second stub is \( d_3 = 1/8 \lambda \). Determine the lengths of the short-circuited stubs as well as the open circuited stubs when the match is achieved.

Solution:

Short stubs:

From the presented method:

\[ a = \tan(\beta d_1) = 0 \]
\[ m = \tan(\beta d_2) = 1, \quad p = \tan(\beta d_3) = 1 \]
\[ g_{LA} = 0.9615, \quad b_{LA} = 0.1923 \]
\[ Q = 1.0400 \]
\[ g_{LB1} = g_{LB2} = 1.6 \]
\[ b_{LB1} = -1.8764, \quad b_{LB2} = -0.1236 \]
\[ B_{S1}^{(1)} = 1.3343, \quad B_{S1}^{(2)} = 0.2810 \]

The limits for the first stub are (36),
\[ B_{S1}^{(1)} \geq 1.3342 \]
\[ B_{S1}^{(2)} \leq 0.2812 \]

The limits for the second stub (28) are,
\[ B_{S2}^{(1)} = 3.6764, \quad B_{S2}^{(2)} = 2.0764 \]
\[ B_{S3}^{(3)} = 1.9236, \quad B_{S2}^{(4)} = 0.3236 \]
\[ 1.8764 \leq B_{S2}^{(1,2)} \leq 3.8764 \]

and
\[ 0.1236 \leq B_{S2}^{(3,4)} \leq 2.1236 \]

Well, all susceptances for the first and second stub are in the domain of the matching.

For \( t = 1.25 \), from this work and (11) the results are given in Table 4.

**Table 4. The stub lengths in the triple stub matching (S/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3976</td>
<td>0.4577</td>
<td>0.4064</td>
</tr>
<tr>
<td>0.3976</td>
<td>0.4286</td>
<td>0.3238</td>
</tr>
<tr>
<td>0.2936</td>
<td>0.4237</td>
<td>0.4064</td>
</tr>
<tr>
<td>0.2936</td>
<td>0.2998</td>
<td>0.3238</td>
</tr>
</tbody>
</table>

The the shortest stub lengths are:
\[ l_{S1} = 0.2936 \lambda, \quad l_{S2} = 0.2998 \lambda, \quad l_{S3} = 0.3238 \lambda \]
Using [1] we obtain the same results.

Open stubs:

Table 5. The stub lengths (O/C)

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1476</td>
<td>0.2077</td>
<td>0.1564</td>
</tr>
<tr>
<td>0.1476</td>
<td>0.1786</td>
<td>0.0738</td>
</tr>
<tr>
<td>0.0436</td>
<td>0.1737</td>
<td>0.1564</td>
</tr>
<tr>
<td>0.0436</td>
<td>0.0498</td>
<td>0.0738</td>
</tr>
</tbody>
</table>

Using [1] we obtain the same results.

14.3) The terminating impedance is $Z_L = (100 + j50)\Omega$ and the characteristic impedance $Z_0$ of the line and the stubs is 50 $\Omega$. The first stub is away $d_1 = 0.503 \lambda$ from the load. The spacing between the first and second stub is stub $d_2 = 3/8 \lambda$ and between the third and the second stub is $d_3 = 3/8 \lambda$. Determine the lengths of the short-circuited stubs as well as of the open circuited stubs when the match is achieved.

Solution:

$$a = \tan(\beta d_1) = 0.0189$$

$$m = \tan(\beta d_2) = -1, \quad p = \tan(\beta d_3) = -1$$

$$Q = 1.0400$$

$$g_{LA} = 0.3971, \quad b_{LA} = 0.1923$$

$$g_{LB1} = g_{LB2} = 1.6$$

$$b_{LB1} = -1.8764, \quad b_{LB2} = -0.1236$$

$$x = 1.75$$

Short stubs:

Table 6. The stub lengths (S/C)

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2367</td>
<td>0.3825</td>
<td>0.2288</td>
</tr>
<tr>
<td>0.2367</td>
<td>0.1351</td>
<td>0.0783</td>
</tr>
<tr>
<td>0.0912</td>
<td>0.0494</td>
<td>0.2288</td>
</tr>
<tr>
<td>0.0912</td>
<td>0.1351</td>
<td>0.0783</td>
</tr>
</tbody>
</table>
Open stubs:

**Table 7. The stub lengths (O/C)**

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4867</td>
<td>0.1325</td>
<td>0.4788</td>
</tr>
<tr>
<td>0.4867</td>
<td>0.3851</td>
<td>0.3283</td>
</tr>
<tr>
<td>0.3412</td>
<td>0.2994</td>
<td>0.4788</td>
</tr>
<tr>
<td>0.3412</td>
<td>0.2808</td>
<td>0.3283</td>
</tr>
</tbody>
</table>

Using [1] we obtain the same results.

**14.4)** Let us solve the following problem where the terminating impedance is $Z_L = (25 - j25)\Omega$ and the characteristic impedance $Z_0$ of the line and the stubs is 50 $\Omega$. The first stub is away $d_1 = 0.25 \lambda$ from the load. The spacing between the first and second stub is stub $d_2 = 1/4 \lambda$ and between the third and the second stub is $d_3 = 1/4 \lambda$. Determine the lengths of the short-circuited stubs as well as of the open circuited stubs when the match is achieved.

Solution:

This is the special case a1)

For $t = 10$

**Table 8. The stub lengths (S/C)**

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.4431</td>
<td>0.3510</td>
<td>0.4488</td>
</tr>
<tr>
<td>0.4431</td>
<td>0.2715</td>
<td>0.0512</td>
</tr>
<tr>
<td>0.0855</td>
<td>0.2285</td>
<td>0.4488</td>
</tr>
<tr>
<td>0.0855</td>
<td>0.1490</td>
<td>0.0512</td>
</tr>
</tbody>
</table>

**Table 9. The stub lengths (S/C)**

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1931</td>
<td>0.1010</td>
<td>0.1988</td>
</tr>
<tr>
<td>0.1931</td>
<td>0.0215</td>
<td>0.3012</td>
</tr>
<tr>
<td>0.3355</td>
<td>0.4785</td>
<td>0.1988</td>
</tr>
<tr>
<td>0.3355</td>
<td>0.3990</td>
<td>0.3012</td>
</tr>
</tbody>
</table>
Using [1] we obtain the same results either for the short stubs or for the open stubs.

14.5) A load with a load impedance of \( Z_L = (10 + j5)\Omega \) is to be matched to a transmission line of a microstrip line with a characteristic input impedance of 50\( \Omega \) using a triple stub. All stubs are open with the same characteristic impedances as the transmission line. The gaps between the stubs are \( d_2 = 3/8 \lambda \) and \( d_3 = 3/8 \lambda \), and the gap between the load and the first stub is fixed at \( d_1 = 0.503 \lambda \).

Solution:

In this case,

\[
\begin{align*}
a &= \tan(\beta d_1) = 0.0189 \\
m &= \tan(\beta d_2) = -1, \quad p = \tan(\beta d_3) = -1 \\
g_{LA} &= 3.6964 > 1, \quad b_{LA} = -2.1778 \\
Q &= 0.2705 < 1
\end{align*}
\]

This is the case a8) (Appendix) where \( a = 0.0189 \) satisfies the following condition (70),

\[-0.5757 \leq a \leq 0.325\]

\( g_{LB1} = g_{LB2} = 0.5411 \)

\( b_{LB1} = b_{LB2} = 1 \)

\( B_{S1}^{(1)} = B_{S1}^{(2)} = 1.1778 \)

\( B_{S2}^{(1)} = B_{S2}^{(3)} = -1.1115, \quad B_{S2}^{(2)} = B_{S2}^{(4)} = -2.8885 \)

\( B_{S3}^{(1)} = B_{S3}^{(3)} = 0.6421, \quad B_{S23}^{(2)} = B_{S3}^{(4)} = -2.6421 \)

**Table 10. The stub lengths (O/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1380</td>
<td>0.3666</td>
<td>0.0908</td>
</tr>
<tr>
<td>0.1380</td>
<td>0.3030</td>
<td>0.3076</td>
</tr>
<tr>
<td>0.1380</td>
<td>0.3666</td>
<td>0.0908</td>
</tr>
<tr>
<td>0.1380</td>
<td>0.3030</td>
<td>0.3076</td>
</tr>
</tbody>
</table>
Let us solve the same example for the short stabs.

It is important to mention again that this case gives the unique solution for the triple stub matching because the stub lengths expressions are not depending on the parameter \( t' \). All results are summarized in Tables 10 and 11.

### Table 11. The stub lengths (S/C)

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3880</td>
<td>0.1166</td>
<td>0.3408</td>
</tr>
<tr>
<td>0.3880</td>
<td>0.0530</td>
<td>0.0576</td>
</tr>
<tr>
<td>0.3880</td>
<td>0.1166</td>
<td>0.3408</td>
</tr>
<tr>
<td>0.3880</td>
<td>0.0530</td>
<td>0.0576</td>
</tr>
</tbody>
</table>

By using [1] we obtained the same results.

14.6) Finally, let us solve the following example in which all characteristic impedances are different. This example can be used to verify the triple stub matching by using the Smith chart. All characteristic impedances are different. The terminating impedance is \( Z_L = (100 + j100) \Omega \) and the characteristic impedance of the line \( Z_0 = 50 \Omega \). The characteristic impedances of stubs are respectively \( Z_{S1} = 75 \Omega \), \( Z_{S2} = 100 \Omega \) and \( Z_{S3} = 125 \Omega \). The first stub is away \( d_1 = 0.4 \lambda \) from the load. The spacing between the first and second stub is stub \( d_2 = 3/8 \lambda \) and between the third and the second stub is \( d_3 = 1/8 \lambda \). Determine the lengths of the open stubs as well as the short-circuited stubs when the match is achieved.

Solution:

**Open stubs:**

We start with the extreme case, that is, with \( x = 1 \) for which we have,

\[
a = \tan(\beta d_1) = -0.7265
\]
\[
m = \tan(\beta d_2) = -1, \quad p = \tan(\beta d_3) = 1
\]
\[
g_{LA} = 0.5436, \quad b_{LA} = -1.0726
\]
\[
Q = 1.8397
\]
\[
g_{LB1} = g_{LB2} = 2
\]
\[ b_{LB1} = -0.8327, \quad b_{LB2} = 2.8327 \]

\[ B_{S1}^{(1)} = 0.5707, \quad B_{S1}^{(2)} = -0.4255 \]

\[ B_{S1}^{(1)} \geq 0.5707 \]

\[ B_{S1}^{(2)} \leq -0.4255 \]

It was expected because \( B_{S1}^{(1)} \) and \( B_{S1}^{(2)} \) are the limits of the domain of the matching.

\[ B_{S2}^{(1)} = 1.8327, \quad B_{S2}^{(2)} = 1.8327 \]

\[ B_{S3}^{(3)} = -1.8327, B_{S3}^{(4)} = -1.8327 \]

\[ 1.8327 \leq B_{S2}^{(1,2)} \leq 3.8327 \]

and

\[ -2.8327 \leq B_{S2}^{(3,4)} \leq -1.8327 \]

The same conclusion is for \( B_{S1}^{(1,2)} \) and \( B_{S3}^{(3,4)} \) which are the limits of the domain of matching. The stub lengths are summarized in Table 12.

**Table 12. The stub lengths (O/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1127</td>
<td>0.2076</td>
<td>0.1894</td>
</tr>
<tr>
<td>0.1127</td>
<td>0.2076</td>
<td>0.1894</td>
</tr>
<tr>
<td>0.4096</td>
<td>0.2924</td>
<td>0.1894</td>
</tr>
<tr>
<td>0.4096</td>
<td>0.2924</td>
<td>0.1894</td>
</tr>
</tbody>
</table>

**Short stubs:**
In Table 13 are summarized the lengths of short stubs.

**Table 13. The stub lengths (S/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3627</td>
<td>0.4576</td>
<td>0.4394</td>
</tr>
<tr>
<td>0.3627</td>
<td>0.4576</td>
<td>0.4394</td>
</tr>
<tr>
<td>0.1596</td>
<td>0.0424</td>
<td>0.4394</td>
</tr>
<tr>
<td>0.1596</td>
<td>0.0424</td>
<td>0.4394</td>
</tr>
</tbody>
</table>
Now, let us solve the same example for \( x = 1.5 \).

In Table 14 and Table 15 are summarized the lengths for the open and short stubs, respectively.

Open stubs:

**Table 14. The stub lengths (O/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1388</td>
<td>0.2210</td>
<td>0.2134</td>
</tr>
<tr>
<td>0.1388</td>
<td>0.1634</td>
<td>0.1006</td>
</tr>
<tr>
<td>0.3772</td>
<td>0.3366</td>
<td>0.2134</td>
</tr>
<tr>
<td>0.3772</td>
<td>0.2790</td>
<td>0.1006</td>
</tr>
</tbody>
</table>

Short stubs:

**Table 15. The stub lengths (S/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3888</td>
<td>0.4710</td>
<td>0.4634</td>
</tr>
<tr>
<td>0.3888</td>
<td>0.4134</td>
<td>0.3506</td>
</tr>
<tr>
<td>0.1272</td>
<td>0.0866</td>
<td>0.4634</td>
</tr>
<tr>
<td>0.1272</td>
<td>0.0290</td>
<td>0.3506</td>
</tr>
</tbody>
</table>
15. TRIPLE TUNER STUBS IN SERIES

The load impedance $Z_L = A + jB$, $(A, B \in \mathbb{R})$ of the transmission line with the characteristic impedance $Z_0$ and without losses, is to be matched by a triple-stub tuner connected in series (Fig. 14). The first stub is distanced from the load by $d_1$. The distance between the first and the second stub is $d_2$ and the distance between the second and the third stub is $d_3$. Their characteristic impedances are respectively $Z_{S1} = k_1Z_0$, $Z_{S2} = k_2Z_0$ and $Z_{S3} = k_3Z_0$. The lengths $l_1$, $l_2$ and $l_3$ are required to find. All impedances are normalized.

The normalized load admittance is,

$$z_{LN} = p_L + jq_L$$

where,

$$p_L = \frac{A}{Z_0}, \quad q_L = -\frac{B}{Z_0}$$
Let us calculate the normalized impedance at points $A$,

$$z_A = z_{(d_1)} = \frac{z_{LN} + j \tan(\beta d_1)}{1 + j \tan(\beta d_1) y_{LN}} = \frac{z_{LN} + ja}{1 + jaz_{LN}} = \frac{z_{LN} + ja}{1 + jaz_{LN}} = r_{LA} + jx_{LA} \quad (117)$$

where,

$$a = \tan(\beta d_1), a \in R$$

$$r_{LA} = \frac{p_L (1 + a^2)}{(1 - a q_L)^2 + a^2 p_L^2}$$

$$x_{LA} = \frac{a(1 - p_L^2 - q_L^2) + q_L(1 - a^2)}{(1 - a q_L)^2 + a^2 p_L^2}$$

The total impedance at points $A$ is,

$$z_{TotalA} = z_A + jz_{S1} = r_{LA} + j(x_{LA} + x_{S1}) = r_{LA} + jX_1 \quad (118)$$

where,

$$z_{S1} = jx_{S1}$$

$$X_1 = x_{LA} + x_{S1} \quad (119)$$

$$x_{S1} = k_1 b, x_{S1} = k_1 \tan(\beta l_1) = k_1 b, \quad (S) \quad (120)$$

$$x_{S1} = -\frac{k_1}{b}, x_{S1} = -k_1 b \tan(\beta l_1) = -\frac{k_1}{b}, \quad (O) \quad (C) \quad (121)$$

$$b = \tan(\beta l_1)$$

The next step is the determination of the impedance at point $B$.

Now, the impedance at the point $B$ is

$$z_B = \frac{z_{TotalA} + j \tan(\beta d_2)}{1 + jz_{TotalA} \tan(\beta d_2)} = \frac{z_{TotalA} + jm}{1 + jz_{TotalA} m} = r_{LB} + jx_{LB} \quad (121)$$

where

$$m = \tan(\beta d_2)$$
\[ r_{LB} = \frac{r_{LA}(1 + m^2)}{(1 - mX_1)^2 + m^2r_{LA}^2} \]  \hspace{1cm} (122)

\[ x_{LB} = \frac{m(1 - r_{LA}^2 - X_1^2) + X_1(1 - m^2)}{(1 - mX_1)^2 + m^2r_{LA}^2} \]  \hspace{1cm} (123)

The total impedance at the point B is,

\[ z_{TotalB} = z_B + jz_{S2} = r_{LB} + j(x_{LB} + x_{S2}) = r_{LB} + jX_2 \]  \hspace{1cm} (124)

with

\[ X_2 = x_{LB} + x_{S2} \]  \hspace{1cm} (125)

where,

\[ x_{S2} = k_2n, \quad z_{S2} = k_2 \tan(\beta l_2) = k_2n, \quad \text{short circuit} \left(\frac{S}{C}\right) \]  \hspace{1cm} (126)

\[ x_{S2} = -\frac{k_2}{n}, \quad x = -k_2 \cotan(\beta l_2) = -\frac{k_2}{n}, \quad \text{open circuit} \left(\frac{O}{C}\right) \]  \hspace{1cm} (127)

with

\[ n = \tan(\beta l_2) \]

The impedance at the point C is,

\[ z_C = \frac{z_{TotalB} + j\tan(\beta d_3)}{1 + jz_{TotalB} \tan(\beta d_3)} = \frac{z_{TotalB} + jp}{1 + jz_{TotalB} p} = r_{LC} + jx_{LC} \]  \hspace{1cm} (128)

with

\[ p = \tan(\beta d_3) \]

\[ r_{LC} = \frac{r_{LB}(1 + p^2)}{(1 - pX_2)^2 + p^2r_{LB}^2} \]  \hspace{1cm} (129)

\[ x_{LC} = \frac{p(1 - r_{LB}^2 - X_2^2) + X_2(1 - p^2)}{(1 - pB_2)^2 + p^2g_{LB}^2} \]  \hspace{1cm} (130)

The total impedance at the point C is,

\[ z_{TotalC} = z + jz_{S3} = r_{LC} + j(x_{LC} + x_{S3}) = r_{LC} + jX_3 \]  \hspace{1cm} (131)

with

\[ X_3 = x_{LC} + x_{S3} \]  \hspace{1cm} (132)

The condition of the matching at the point C is,
\[ z_{TotalC} = 1 \]  \hspace{1cm} (133)

which gives,

\[ r_{LC} = 1 \quad \text{or} \quad \frac{r_{LB}(1 + p^2)}{(1 - pX_2)^2 + p^2r_{LB}^2} = 1 \]  \hspace{1cm} (134)

and

\[ x_{LC} + x_{S3} = 0, \quad \text{or} \quad x_{S3} = -\frac{p(1 - r_{LB}^2 - X_2^2) + X(1 - p^2)}{(1 - pX_2)^2 + p^2r_{LB}^2} \]  \hspace{1cm} (135)

where,

\[ x_{S3} = k_3 q, \quad x_{S3} = k_3 \tan(\beta l_3) = k_3 q, \]  \hspace{1cm} (136)

\[ x_{S3} = -\frac{k_3}{q}, \quad z_{S3} = -k_3 \cotan(\beta l_3) = -\frac{k_3}{q}, \]  \hspace{1cm} (137)

The next step is determination of all stub lengths.

From (19) we obtain,

\[ (X_2 - \frac{1}{p})^2 + r_{LB}^2 = r_{LB}^2 \frac{r_{(p)}}{r_{LB}} \]  \hspace{1cm} (138)

whose solutions for \( X_2 \) are,

\[ X_2^{(1,2)} = \frac{1}{p} \pm r_{LB} \sqrt{\frac{r_{(p)}}{r_{LB}} - 1} \]  \hspace{1cm} (139)

where,

\[ r_{(p)} = \frac{1 + p^2}{p^2} \]  \hspace{1cm} (140)

Equation (24) can have the real solutions if,

\[ \frac{r_{(p)}}{r_{LB}} - 1 \geq 0 \]

or,

\[ r_{LB} \leq r_{(p)} = \frac{1 + p^2}{p^2} \]  \hspace{1cm} (141)
Also, equation (23) can be written in the form,

\[ r_{LB}^2 - r_{LB} r_{(p)} + \left( X_2 - \frac{1}{p} \right)^2 = 0 \]

whose solutions are,

\[ r_{LB(1,2)} = \frac{r_{(p)} \pm \sqrt{r_{(p)}^2 - 4 \left( X_2 - \frac{1}{p} \right)^2}}{2} \tag{142} \]

The following condition must be satisfied,

\[ r_{(p)}^2 - 4 \left( X_2 - \frac{1}{p} \right)^2 \geq 0 \]

which gives,

\[ -\frac{(p - 1)^2}{2p^2} \leq X_2 \leq \frac{(p + 1)^2}{2p^2} \tag{143} \]

or

\[ -r_{LB(i)} - \frac{(p - 1)^2}{2p^2} \leq X_{S2}^{(i)} \leq -r_{LB(i)} + \frac{(p + 1)^2}{2p^2} \tag{144} \]

This condition is important because it gives the limits for \( X_2 \) or the reactance for the second stub.

Combining (7) and (26) we obtain,

\[ r_{LB} = \frac{r_{LA} \left( 1 + m^2 \right)}{(1 - mX_1)^2 + m^2r_{LA}^2} \leq r_{(p)} \]

or

\[ \frac{r_{LA}r_{(m)}}{(X_1 - \frac{1}{m})^2 + r_{LA}^2} \leq r_{(p)} \tag{145} \]

where,

\[ r_{(m)} = \frac{1 + m^2}{m^2} \tag{146} \]

Let us introduce the constant Q as,
The expression (30) can be written in the following form,

\[
\frac{r_{LA}^2 Q}{(X_1 - \frac{1}{m})^2 + r_{LA}^2} \leq 1
\]  

(147)

This inequality presents the condition for finding the length of the first stub. Introducing the parameter \( t \geq 1 \), (33) can be written as,

\[
\frac{r_{LA}^2 Q}{(X_1 - \frac{1}{m})^2 + r_{LA}^2} = \frac{1}{t} \leq 1
\]

(148)

Let us take the limit of (34),

\[
(X_1 - \frac{1}{m})^2 + r_{LA}^2 = r_{LA}^2 Q t
\]

or

\[
(X_1 - \frac{1}{m})^2 = r_{LA}^2 (Q t - 1)
\]

(149)

I) \( 0 < Q < 1, \ t \geq 1 \)

I.1) If \( 1 \leq t \leq 1/Q = r_{LA} \frac{r_{(p)}}{r_{(m)}} = t_{max} \)

\[
D = Q t - 1 = 0
\]

(150)

that gives,

\[
(X_1 - \frac{1}{m})^2 = 0
\]

(151)

Equation (37) gives the reactance for the first stub,

\[
x_{S1} = -x_{LA} + \frac{1}{m}
\]

(152)

One can see that \( x_{S1} \) does not depend on \( 't' \). Moreover, the reactances for the second and the third stubs do not depend on \( 't' \) by their definition. It is important because the unique triple stub matching is achieved for

\[
1 \leq t \leq t_{max} \ \text{and} \ 0 < Q < 1
\]
Thus, there is the unique solution (37) or (38) for the triple stub matching with (36).

I.2) If \(0 < Q < 1\), \(t > 1/Q\)

\[ D = Qt - 1 > 0 \]

that gives

\[ X_1 = \frac{1}{m} + r_{LA}\sqrt{Qt - 1} \quad \text{and} \quad X_1 = \frac{1}{m} - r_{LA}\sqrt{Qt - 1} \quad (153) \]

or

\[ x_{S_1}^{(1,2)} = -x_{LA} + \frac{1}{m} \pm r_{LA}\sqrt{Qt - 1} \quad (154) \]

with the condition,

\[ r_{LA} \leq \frac{(1 + m^2)p^2}{m^2(1 + p^2)t} \quad (155) \]

II) \(Q \geq 1, \quad t \geq 1\)

For these values, this condition \(Qt - 1 \geq 0\) is satisfied automatically and the reactance \(X_1^{(1,2)}\) or \(x_{S_1}^{(1,2)}\) for the first stub are determined by (39) and (40) with condition (41) for \(r_{LA}\).

Expressions (39) give the limits where the matching is possible for the first stub.

\[ X_1 \geq \frac{1}{m} + r_{LA}\sqrt{Qt - 1} \quad \text{and} \quad X_1 \leq \frac{1}{m} - r\sqrt{Qt - 1} \quad (156) \]

From the previous analysis one can conclude that there is not the unique solution for

\[ A) \quad \text{If} \quad 0 < Q < 1, \quad t > 1/Q \quad \text{and} \]

\[ B) \quad \text{If} \quad Q \geq 1, \quad t \geq 1 \]

It could be an optimization problem. This is problem of the three variables where the lengths of the three stubs could be chosen to optimize the bandwidth of the matching.

The corresponding lengths for the first stub can be obtained for:

\[ g) \quad \text{Short circuit (stub)} \]

\[ x_{S_1}^{(1,2,3,4)} = k_1 b_{S_1}^{(1,2,3,4)} , \quad x_{S_1}^{(1)} = x_{S_1}^{(2)} , \quad x_{S_1}^{(3)} = x_{S_1}^{(4)} \]

or
\[ b_{(\mathcal{S}C)S1}^{(1,2,3,4)} = \frac{x_{S1}^{(1,2,3,4)}}{k_1} \]

\[ l_{(\mathcal{S}C)S1}^{(1,2,3,4)} = \frac{\lambda}{2\pi} \arctan \left[ b_{(\mathcal{S}C)S1}^{(1,2,3,4)} \right] \]  

(157)

h) Open circuit (stub)

\[ x_{S1}^{(1,2,3,4)} = -\frac{k_1}{b_{(\mathcal{C})S1}^{(1,2,3,4)}} , \quad x_{S1}^{(1)} = x_{S1}^{(2)} , \quad x_{S1}^{(3)} = x_{S1}^{(4)} \]

or

\[ -\frac{k_1}{x_{S1}^{(1,2,3,4)}} = b_{(\mathcal{C})S1}^{(1,2,3,4)} \]

(158)

From (41) it can be seen that

\[ X_1^{(1)} + X_1^{(2)} = \frac{2}{m} \]

(159)

and

\[ r_{LB1} = r_{LB2} = \frac{r_{LA}(1 + m^2)}{(1 - mX_1^{(1)})^2 + m^2r_{LA}^2} \]

(160)

\[ x_{LB1} = \frac{m \left( 1 - r_{LA}^2 - X_1^{(1,2)} \right) + X_1^{(1)} \left( 1 - m^2 \right)}{(1 - mB_1^{(1)})^2 + m^2r_{LA}^2} \]

(161)

\[ x_{LB2} = \frac{m \left( 1 - r_{LA}^2 - X_1^{(1,2)} \right) + X_1^{(2)} \left( 1 - m^2 \right)}{(1 - mX_1^{(1)})^2 + m^2r_{LA}^2} \]

(162)

Now, we simply find from (10) and (23) the reactance for the second stub,

\[ x_{S2}^{(1,2,3,4)} = -x_{LB}^{(1,2)} + \frac{1}{p} \pm \frac{r_{LB}^{(1,2)}}{\sqrt{r_{LB}^{(1,2)}}} \sqrt{r_{LB}^{(1,2)}} - 1 \]

(163)

where \( r_{LB}^{(1,2)} \), \( x_{LB}^{(1,2)} \) and \( r_{p}^{(1,2)} \) are given by (46), (47), (48) and (25).

Now, let us find the corresponding lengths for the second stub:
i) Short circuit (stub)

\[ x_{S2}^{(1,2,3,4)} = k_2 n_{(S/C)S2}^{(1,2,3,4)} \]

or

\[ n_{(S/C)S2}^{(1,2,3,4)} = \frac{x_{S2}^{(1,2,3,4)}}{k_2} \]

\[ l_{(S/C)S2}^{(1,2,3,4)} = \frac{\lambda}{2\pi} \tan \left[ n_{(S/C)S2}^{(1,2,3,4)} \right] \quad (164) \]

j) Open circuit (stub)

\[ x_{S2}^{(1,2,3,4)} = -\frac{k_2}{n_{(O/C)S2}^{(1,2,3,4)}} \]

or

\[ -\frac{k_2}{x_{S2}^{(1,2,3,4)}} = n_{(O/C)S2}^{(1,2,3,4)} \]

\[ l_{(O/C)S2}^{(1,2,3,4)} = \frac{\lambda}{2\pi} \tan \left[ n_{(O/C)S2}^{(1,2,3,4)} \right] \quad (165) \]

Finally, the reactance of the third stub can be obtained from (19),

\[ x_{S3}^{(1,2,3,4)} = -\frac{p}{2} \left( 1 - r_{LB}^{(1,2)} - X_2^{(1,2,3,4)} \right)^2 + X_2^{(1,2,3,4)} \left( p - r_{LB}^{(1,2)} \right)^2 \]

where \( r_{LB}^{(1,2)} \) is given by (46).

The corresponding lengths for the third stub are:

k) Short circuit (stub)

\[ x_{S3}^{(1,2,3,4)} = k_3 q_{(S/C)S3}^{(1,2,3,4)} \]

or

\[ q_{(S/C)S3}^{(1,2,3,4)} = \frac{x_{S3}^{(1,2,3,4)}}{k_3} \]
\[
I^{(1,2,3,4)}_{S3} = \frac{\lambda}{2\pi} \tan \left[ q^{(1,2,3,4)}_{S3} \right]
\]

1) Open circuit (stub)

\[
x^{(1,2,3,4)}_{S3} = -\frac{k_3}{q^{(1,2,3,4)}_{S3}}
\]

or

\[
\frac{x^{(1,2,3,4)}_{S3}}{k_3} = q^{(1,2,3,4)}_{S3}
\]

For stub lengths the following formula must be respected,

\[
l_s = \begin{cases} 
\frac{\lambda}{2\pi} \tan[P] + \frac{\lambda}{2} & \text{for } P < 0 \\
\frac{\lambda}{2\pi} \tan[P] & \text{for } P > 0 
\end{cases}
\]

In the Table 16. the corresponding solutions are summarized as follows,

**Table 16. The stub lengths in the triple stub matching**

<table>
<thead>
<tr>
<th>(l_{S1}^{(1)})</th>
<th>(l_{S1}^{(2)})</th>
<th>(l_{S1}^{(3)})</th>
<th>(l_{S1}^{(4)})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_{S2}^{(1)})</td>
<td>(l_{S2}^{(2)})</td>
<td>(l_{S2}^{(3)})</td>
<td>(l_{S2}^{(4)})</td>
</tr>
<tr>
<td>(l_{S3}^{(1)})</td>
<td>(l_{S3}^{(2)})</td>
<td>(l_{S3}^{(3)})</td>
<td>(l_{S3}^{(4)})</td>
</tr>
</tbody>
</table>

This schema is valuable either for the short or the open stubs. Thus, the eight possible combinations can be given by this schema \((S/S/S; S/S/O; O/S/S; O/S/O; S/O/S; S/O/O; O/O/S; O/O/O)\).

In this work which treat the triple stub matching many authors after finding the lengths of the first stub use it for solving the rest of the problem as the double stub matching. In this paper as it is proposed after finding the length of the first stub, the lengths of the second and the third stubs are simple to find directly from the corresponding formulas. Moreover, in this paper the characteristic impedance of each circuit is different.
1. Special cases

a1) \( d_1 = d_2 = d_3 = \lambda/4 \)

\[ a = m = p = \infty \]

\[
\begin{align*}
  r_{LA} &= \frac{p_L}{p_L^2 + q_L^2}, & x_{LA} &= -\frac{q_L}{p_L^2 + q_L^2} \\
  r_{LB1} &= r_{LB2} = \frac{r_{LA}}{X_{1(1)}^2 + r_{LA}^2} = r_{LB} \leq 1
\end{align*}
\]

\[ (170) \]

\[
\begin{align*}
  x_{LB1} &= -\frac{X_{1(1)}}{X_{1(1)}^2 + r_{LA}^2}, & x_{LB2} &= -\frac{X_{1(2)}}{X_{1(2)}^2 + r_{LA}^2} \\
  X_1^{(1,2)} &= r_{LA} + x_{S1}^{(1,2)} & X_1^{(1)} &= -x_{LA} + r_{LA}\sqrt{Qt - 1} \\
  X_1^{(2)} &= -x_{LA} - r_{LA}\sqrt{Qt - 1}
\end{align*}
\]

\[ (171) \]

\[
\begin{align*}
  Q &= \frac{1}{r_{LA}} \quad & X_2^{(1,2,3,4)} &= -\frac{X_2^{(1,2,3,4)}}{(B_2^{(1,2,3,4)})^2 + r_{LB}^{(1,2)^2}}
\end{align*}
\]

\[ (172) \]

\[ (173) \]

\[ (174) \]

\[ (175) \]

\[ (176) \]

\[ (177) \]

\[ (178) \]

a2) \( d_1 = d_2 = \lambda/4, \ d_3 \neq \lambda/4 \)

\[ a = m = \infty, \quad p \neq \infty \]

We use (59)-(64)

\[ Q = \frac{r(p)}{r_{LA}} \]

\[ (179) \]

and (49) and (52) for \( x_{S2}^{(1,2,3,4)} \) and \( x_{S3}^{(1,2,3,4)} \).
a3) \( d_1 = d_3 = \lambda / 4, \quad d_2 \neq \lambda / 4 \)

\[ a = p = \infty, \quad m \neq \infty \]

We use (56) for \( r_{LA}, x_{LA}, (46), (48 \text{ and } (58) \text{ for } r_{LB1}, r_{LB2}, x_{LB1}, x_{LB2}, (3) (40) \text{ and } (41) \text{ for } X_{1}^{(1,2)}, X_{S1}^{(1,2)} \). For \( x_{S2}^{(1,2,3,4)} \) and \( x_{S3}^{(1,2,3,4)} \) (63) and (64) are used respectively with,

\[ Q = \frac{r(m)}{r_{LA}} \quad (180) \]

a4) \( d_1 = \frac{\lambda}{4}, \quad d_2 \neq \frac{\lambda}{4} \quad d_3 \neq \frac{\lambda}{4} \)

\[ a = \infty, \quad m \neq \infty, \quad p \neq \infty \]

We use (59) for \( r_{LA}, x_{LA} \). All other expressions for \( x_{S1}^{(1,2)}, x_{S2}^{(1,2,3,4)} \text{ and } x_{S3}^{(1,2,3,4)} \) are calculated by the same expressions as in the general case.

a5) \( d_1 \neq \frac{\lambda}{4}, \quad d_2 = \lambda / 4 \quad d_3 \neq \lambda / 4 \)

\[ a \neq \infty, \quad m = \infty, \quad p \neq \infty \]

We use (60) and (61) for calculating \( x_{S1}^{(1,2)} \), (57) and (58) for \( r_{LB}, x_{LB1}, x \) with

\[ Q = \frac{1}{r(p)r_{LB}} \quad (181) \]

All other expressions for \( x_{S2}^{(1,2,3,4)} \) and \( x_{S3}^{(1,2,3,4)} \) are calculated by the same expressions as in the general case.

a6) \( d_1 \neq \frac{\lambda}{4}, \quad d_2 \neq \frac{\lambda}{4} \quad d_3 = \lambda / 4 \)

\[ a \neq \infty, \quad m \neq \infty, \quad p = \infty \]

We use (63) and (64) for calculating \( x_{S2}^{(1,2,3,4)} \) and \( x_{S3}^{(1,2,3,4)} \) with (66). All other expressions are calculated by the same expressions as in the general case.

The stub lengths either for short circuit or open circuit are calculated by the previous expressions.

16. Examples

16.1. The terminating impedance is \( Z_L = (50 - j10)\Omega \) and the characteristic impedance \( Z_0 \) of the line and the stubs is 50 \( \Omega \). The first stub is connected to the load \( (d_1 \)
= 0 \lambda). The spacing between the first and second stub is stub \( d_2 = 1/8 \lambda \) and between the third and the second stub is \( d_3 = 1/8 \lambda \). Determine the lengths of the short-circuited stubs when the match is achieved.

Solution:

\[ r_{LA} = 1, \quad x_{LA} = -0.2, \quad Q = 1, \quad t_{\text{min}} = 1 \]

We begin by \( t = 1 \) for which,

Table 17. The stub lengths in the triple stub matching (S/C), \( t = 1 \)

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1394</td>
<td>0.1762</td>
<td>0.1250</td>
</tr>
<tr>
<td>0.1394</td>
<td>0.1762</td>
<td>0.1250</td>
</tr>
<tr>
<td>0.1394</td>
<td>0.1762</td>
<td>0.1250</td>
</tr>
<tr>
<td>0.1394</td>
<td>0.1762</td>
<td>0.1250</td>
</tr>
</tbody>
</table>

Table 18. The stub lengths in the triple stub matching (S/C), \( t = 3 \)

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1919</td>
<td>0.2099</td>
<td>0.1875</td>
</tr>
<tr>
<td>0.1919</td>
<td>0.1762</td>
<td>0.4375</td>
</tr>
<tr>
<td>0.4664</td>
<td>0.1762</td>
<td>0.4375</td>
</tr>
<tr>
<td>0.4664</td>
<td>0.0181</td>
<td>0.4375</td>
</tr>
</tbody>
</table>

Table 19. The stub lengths in the triple stub matching (S/C), \( t = 10 \)

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2128</td>
<td>0.2018</td>
<td>0.2110</td>
</tr>
<tr>
<td>0.2128</td>
<td>0.1762</td>
<td>0.3238</td>
</tr>
<tr>
<td>0.3307</td>
<td>0.1762</td>
<td>0.2110</td>
</tr>
<tr>
<td>0.3307</td>
<td>0.1074</td>
<td>0.3238</td>
</tr>
</tbody>
</table>

Table 20. The stub lengths in the triple stub matching (S/C), \( t = 100 \)

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2358</td>
<td>0.1871</td>
<td>0.2355</td>
</tr>
<tr>
<td>0.2358</td>
<td>0.1762</td>
<td>0.2677</td>
</tr>
<tr>
<td>0.2681</td>
<td>0.1762</td>
<td>0.2355</td>
</tr>
<tr>
<td>0.2681</td>
<td>0.1612</td>
<td>0.2677</td>
</tr>
</tbody>
</table>
Table 21. The stub lengths in the triple stub matching (S/C), $t=100000$

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2495</td>
<td>0.1766</td>
<td>0.2495</td>
</tr>
<tr>
<td>0.2495</td>
<td>0.1762</td>
<td>0.2505</td>
</tr>
<tr>
<td>0.2505</td>
<td>0.1762</td>
<td>0.2495</td>
</tr>
<tr>
<td>0.2505</td>
<td>0.1758</td>
<td>0.2505</td>
</tr>
</tbody>
</table>

Table 22. The stub lengths in the triple stub matching (S/C), $t=10^{10}$

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2500</td>
<td>0.1762</td>
<td>0.2500</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.1762</td>
<td>0.2500</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.1762</td>
<td>0.2500</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.1762</td>
<td>0.2500</td>
</tr>
</tbody>
</table>

From Tables 16 to 22 we can see that for different $t$ there is not the unique solution for the stub matching, but each solution satisfies the matching condition (148) that will be proved by the coefficient SWR (APPENDIX). We choose randomly the lengths of stub1,2 and 3.

Not unique solution: $t = 3$ (Second row in Table 18)

I) SWR before the load:

\[ d_{1zN} = 0, \quad l_{1zN} = 0 \]

\[ z_{1N} = (50 - j10)\Omega \]

\[ |\Gamma_{BLoad}| = 0.0995, \quad SWR_{BLoad} = 1.2210 \]

II) SWR between the first and the second stub:

\[ d_1 = 0, \quad l_{S1}^{(1)} = 0.2128\lambda \]

\[ z_{TotalS1} = (50 + j200.01)\Omega \]

\[ |\Gamma_{S1-S2}| = 0.8944, \quad SWR_{S1-S2} = 17.9456 \]

III) VSWR after the second stub:

\[ z_{TotalS2} = (9.9988 + j19.9887)\Omega \]

\[ d_2 = 0.125\lambda, \quad l_{S2}^{(1)} = 0.1762\lambda \]

\[ |\Gamma_{S2-S3}| = 0.7071, \quad SWR_{S2-S3} = 5.8282 \]

IV) VSWR after the third stub:
\[ z_{TS3} = (49.9613 - j 0.0287) \Omega \]
\[ d_3 = 0.125 \lambda, \quad l_{S3}^{(1)} = 0.3238 \lambda \]
\[ | \Gamma_{(AS3)} | = 4.8200 \cdot 10^{-4} \]
\[ SWR_{(AS3)} = 1.0010 \]

Thus, for any 't' (148) is satisfied.

16.2 The terminating impedance is \( Z_L = (300 + j100) \Omega \) and the characteristic impedance \( Z_0 \) of the line and the stubs is 50 \( \Omega \). The first stub is away \( d_1 = 0.503 \lambda \) from the load. The spacing between the first and second stub is stub \( d_2 = 3/8 \lambda \) and between the third and the second stub is \( d_3 = 3/8 \lambda \). Determine the lengths of the short-circuited stubs when the match is achieved.

Solution:
\[ r_{LA} = 6.3934, \quad x_{LA} = 1.3465, \quad Q = 0.1564 \]
\[ t_{\text{min}} = 6.3934 \]

For any 't' between 1 and 6.3934 we have the unique solution for the stub matching (Table 22). Let us take \( t = 6.4 \), (Table 23).

Short stubs:

**Table 22. The stub lengths (S/C), \( t = 1 \)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3141</td>
<td>0.3559</td>
<td>0.1470</td>
</tr>
<tr>
<td>0.3141</td>
<td>0.3059</td>
<td>0.2965</td>
</tr>
<tr>
<td>0.3141</td>
<td>0.3559</td>
<td>0.1470</td>
</tr>
<tr>
<td>0.3141</td>
<td>0.3059</td>
<td>0.2965</td>
</tr>
</tbody>
</table>

**Table 23. The stub lengths (S/C), \( t = 6.4 \)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3196</td>
<td>0.3565</td>
<td>0.1470</td>
</tr>
<tr>
<td>0.3196</td>
<td>0.3061</td>
<td>0.2965</td>
</tr>
<tr>
<td>0.3094</td>
<td>0.3553</td>
<td>0.1470</td>
</tr>
<tr>
<td>0.3094</td>
<td>0.3058</td>
<td>0.2965</td>
</tr>
</tbody>
</table>
Table 24. The stub lengths (S/C), $t = 100$

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2428</td>
<td>0.3336</td>
<td>0.2323</td>
</tr>
<tr>
<td>0.2428</td>
<td>0.3201</td>
<td>0.2645</td>
</tr>
<tr>
<td>0.2559</td>
<td>0.3279</td>
<td>0.2323</td>
</tr>
<tr>
<td>0.2559</td>
<td>0.3159</td>
<td>0.2645</td>
</tr>
</tbody>
</table>

Table 25. The stub lengths (S/C), $t = 10^{10}$

<table>
<thead>
<tr>
<th>$l_{S1}$</th>
<th>$l_{S2}$</th>
<th>$l_{S3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2500</td>
<td>0.3238</td>
<td>0.2500</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.3238</td>
<td>0.2500</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.3238</td>
<td>0.2500</td>
</tr>
<tr>
<td>0.2500</td>
<td>0.3238</td>
<td>0.2500</td>
</tr>
</tbody>
</table>

We have two types of matching solutions (see Tables 22 to 25):

a) Unique for $1 \leq t \leq t_{min} = 6.3934$

b) Not unique for $t > t_{min} = 6.3934$

Unique solution: $1 \leq t \leq t_{min} = 6.3934$ (First row in Table 22)

I) SWR before the load:

\[ d_{1zN} = 0, \quad l_{1zN} = 0 \]
\[ z_{1N} = (300 + j100)\Omega \]
\[ |\Gamma_{BLoad}| = 0.7397, \quad SWR_{BLoad} = 6.6837 \]

II) SWR between the first and the second stub:

\[ d_{1} = 0.503\lambda, \quad l_{S1}^{(1)} = 0.3141\lambda \]
\[ z_{TotalS1} = (319.67 - j50.036)\Omega \]
\[ |\Gamma_{(S1-S2)}| = 0.7352, \quad SWR_{(S1-S2)} = 6.5538 \]

III) VSWR after the second stub:

\[ z_{TotalS2} = (15.6411 - j13.7106)\Omega \]
\[ d_{2} = 0.375\lambda, \quad l_{S2}^{(1)} = 0.3559\lambda \]
\[ |\Gamma_{(S2-S3)}| = 0.5517, \quad SWR_{(S2-S3)} = 3.4610 \]
 IV) VSWR after the third stub:

\[ z_{TS3} = (50.0815 - j 0.0358) \Omega \]
\[ d_3 = 0.375\lambda, \quad l_{S3}^{(1)} = 0.1470\lambda \]
\[ | \Gamma_{(AS3)} | = 8.8948 \cdot 10^{-4} \]
\[ SWR_{(AS3)} = 1.0018 \]

No unique solution: \( t = 100 > t_{\text{min}} = 6.3934 \) (Fourth row in Table 24)

I) SWR before the load:

\[ d_{1ZN} = 0, \quad l_{1ZN} = 0 \]
\[ z_{1N} = (300 + j100) \Omega \]
\[ | \Gamma_{BLoad} | = 0.7397, \quad VSWR_{BLoad} = 6.6837 \]

II) SWR between the first and the second stub:

\[ d_1 = 0.503\lambda, \quad l_{S1}^{(1)} = 0.2559\lambda \]
\[ z_{TotalS1} = (319.67 - j1280.8) \Omega \]
\[ | \Gamma_{(S1-S2)} | = 0.9818, \quad VSWR_{(S1-S2)} = 109.1800 \]

III) VSWR after the second stub:

\[ z_{TotalS2} = (0.9884 - j59.9680) \Omega \]
\[ d_2 = 0.375\lambda, \quad l_{S2}^{(1)} = 0.3159\lambda \]
\[ | \Gamma_{(S2-S3)} | = 0.9839, \quad VSWR_{(S2-S3)} = 123.3619 \]

IV) VSWR after the third stub:

\[ z_{TS3} = (49.2535 - j 0.5696) \Omega \]
\[ d_3 = 0.375\lambda, \quad l_{S3}^{(1)} = 0.2645\lambda \]
\[ | \Gamma_{(AS3)} | = 0.0095 \]
\[ VSWR_{(AS3)} = 1.0191 \]

Thus, for any ‘t’ (148) is satisfied.

16.3. A load with a load impedance of \( Z_L = (250 + j80) \Omega \) is to be matched to a transmission line of a microstrip line with a characteristic input impedance of 50\( \Omega \) using a triple stub. All stubs are open with the same characteristic impedances as the transmission
line. The gaps between the stubs are \( d_2 = 1/8 \lambda \) and \( d_3 = 3/8 \lambda \), and the gap between the load and the first stub is fixed at \( d_1 = 0.482 \lambda \).

Solution:
In this case,
\[
\begin{align*}
 r_{LA} &= 2.9462, \quad x_{LA} = 2.9462, \quad Q = 0.3394 \\
 t_{min} &= 2.9462
\end{align*}
\]

This is the case of the unique solution where all expressions do not depend on the parameter \( 't' \) (151) and (152) for,
\[
1 \leq t \leq t_{min} = 2.9462
\]

For any \( 't' \) from the interval \( 1 \leq t \leq t_{min} = 2.9462 \) the unique solution is given for the open and the short stubs in Table 26.

**Table 26. The stub lengths (O/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0857</td>
<td>0.3707</td>
<td>0.3099</td>
</tr>
<tr>
<td>0.0857</td>
<td>0.1293</td>
<td>0.0629</td>
</tr>
<tr>
<td>0.0857</td>
<td>0.3707</td>
<td>0.3099</td>
</tr>
<tr>
<td>0.0857</td>
<td>0.1293</td>
<td>0.0629</td>
</tr>
</tbody>
</table>

For any \( 't' \) from the interval \( t = t_{min} > 2.9462 \) we obtain the different solutions as it was shown in the previous examples. The matching condition (148) is satisfied.

**16.4.** Let us solve the following problem where the terminating impedance is \( Z_L = (25 - j25) \Omega \) and the characteristic impedance \( Z_0 \) of the line and the stubs is 50 \( \Omega \). The first stub is away \( d_1 = 0.25 \lambda \) from the load. The spacing between the first and second stub is stub \( d_2 = 1/4 \lambda \) and between the third and the second stub is \( d_3 = 1/4 \lambda \). Determine the lengths of the open circuited stubs when the match is achieved.

Solution:

This is the special case a1)
\[
\begin{align*}
 r_{LA} &= 0.0917, \quad x_{LA} = -0.0200, \quad Q = 10.9039 \\
 t_{min} &= 0.0917 < 1
\end{align*}
\]
There is not the unique solution but for any \( t' \) the matching condition (34) is satisfied.

For \( t = 3 \) (Table 27)

**Table 27. The stub lengths (O/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3125</td>
<td>0.3703</td>
<td>0.4020</td>
</tr>
<tr>
<td>0.3125</td>
<td>0.2500</td>
<td>0.0980</td>
</tr>
<tr>
<td>0.0625</td>
<td>0.2500</td>
<td>0.4020</td>
</tr>
<tr>
<td>0.0625</td>
<td>0.1297</td>
<td>0.0980</td>
</tr>
</tbody>
</table>

**16.5** Finally, let us solve the following example in which all characteristic impedances are different.

The terminating impedance is \( Z_L = (60 - j80) \Omega \) and the characteristic impedance of the line \( Z_0 \) is 50 \( \Omega \). The characteristic impedances of stubs are respectively \( Z_{S1} = 75 \Omega \), \( Z_{S2} = 100 \Omega \) and \( Z_{S3} = 125 \Omega \). The first stub is away \( d_1 = 0.154 \lambda \) from the load. The spacing between the first and second stub is stub \( d_2 = 3/8 \lambda \) and between the third and the second stub is \( d_3 = 1/8 \lambda \). Determine the lengths of the open stubs as well as the short-circuited stubs when the match is achieved.

Solution:

\[ r_{LA} = 0.2649, \quad x_{LA} = -0.1835, \quad Q = 3.7746, \quad t_{\text{min}} = 0.2649 < 1 \]

Thus, there is not the unique solution.

We start with the open stubs and \( x = 5 \) for which we have in Table 28.

**Table 28. The stub lengths (O/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2818</td>
<td>0.3923</td>
<td>0.3894</td>
</tr>
<tr>
<td>0.2818</td>
<td>0.3167</td>
<td>0.1894</td>
</tr>
<tr>
<td>0.1049</td>
<td>0.1833</td>
<td>0.3894</td>
</tr>
<tr>
<td>0.1049</td>
<td>0.1077</td>
<td>0.1894</td>
</tr>
</tbody>
</table>

Let us verify these results by the coefficient SWR for the shortest lengths of stubs (Fourth row in Table 28).

I) SWR before the load:

\[ d_{1zN} = 0, \quad l_{1zN} = 0 \]
\[ z_{1N} = (60 - j80) \Omega \]
\[ | \Gamma_{BLoad} | = 0.5927, \quad SWR_{BLoad} = 3.9110 \]

II) SWR between the first and the second stub:

\[ d_1 = 0.154\lambda, \quad l_{S1}^{(1)} = 0.1049\lambda \]
\[ z_{TotalS1} = (10.8822 - j9.7414)\Omega \]
\[ | \Gamma_{(S1-S2)} | = 0.9089, \quad SWR_{(S1-S2)} = 20.9535 \]

III) VSWR after the second stub:

\[ z_{TotalS2} = (20.0070 + j10.0690)\Omega \]
\[ d_2 = 0.375\lambda, \quad l_{S2}^{(1)} = 0.1077\lambda \]
\[ | \Gamma_{(S2-S3)} | = 0.4473, \quad SWR_{(S2-S3)} = 2.6187 \]

IV) VSWR after the third stub

\[ z_{TS3} = (50.1488 + j0.0526i)\Omega \]
\[ d_3 = 0.125\lambda, \quad l_{S3}^{(1)} = 0.1894\lambda \]
\[ | \Gamma_{(AS3)} | = 0.0016 \]
\[ SWR_{(AS3)} = 1.0032 \]

Short stubs:

Now, for the short stubs and \( t = 5 \) the results are given in Table 29.

**Table 29. The stub lengths (S/C)**

<table>
<thead>
<tr>
<th>( l_{S1} )</th>
<th>( l_{S2} )</th>
<th>( l_{S3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0318</td>
<td>0.1423</td>
<td>0.1394</td>
</tr>
<tr>
<td>0.0318</td>
<td>0.0667</td>
<td>0.4394</td>
</tr>
<tr>
<td>0.3549</td>
<td>0.4333</td>
<td>0.1394</td>
</tr>
<tr>
<td>0.3549</td>
<td>0.3577</td>
<td>0.4394</td>
</tr>
</tbody>
</table>

\[ r_{LA} = 0.2649, \quad x_{LA} = -0.1835, \quad Q = 3.7746 \]
\[ \varepsilon_{max} = 0.2649 \]

Let us verify these results by the coefficient SWR (Second row in Table 29).
I) SWR before the load:
\[ d_{1zN} = 0, \quad l_{1zN} = 0 \]
\[ Z_1 = (60 - j80)\Omega \]
\[ |\Gamma_{BLoad}| = 0.5927, \quad SWR_{BLoad} = 3.9110 \]

II) SWR between the first and the second stub:
\[ d_1 = 0.154\lambda, \quad l_{S1}^{(1)} = 0.0318\lambda \]
\[ z_{TotalS1} = (13.2464 + j6.0124)\Omega \]
\[ |\Gamma_{(S1-S2)}| = 0.5862, \quad SWR_{(S1-S2)} = 3.8333 \]

III) VSWR after the second stub:
\[ z_{TotalS2} = (19.9924 + j10.0100)\Omega \]
\[ d_2 = 0.375\lambda, \quad l_{S2}^{(1)} = 0.0667\lambda \]
\[ |\Gamma_{(S2-S3)}| = 0.4474, \quad SWR_{(S2-S3)} = 2.6192 \]

IV) VSWR after the third stub:
\[ z_{TS3} = (50.0086 - j0.0067)\Omega \]
\[ d_3 = 0.125\lambda, \quad l_{S3}^{(1)} = 0.4394\lambda \]
\[ |\Gamma_{(AS3)}| = 1.0914 \cdot 10^{-4} \]
\[ SWR_{(AS3)} = 1.0002 \]

Thus, in all examples is confirmed by SWR the validity of obtained results.

17. Conclusion
In this approach we presented the useful analytical method for the single, double, and triple tuning stub impedance matching on transmission line. The stubs can be short circuits, open circuits, and all their combinations. The characteristic impedances of the stubs are different between them and, they are different from the characteristic impedance of the transmission line. The SWR is calculated at any section of the line. The special cases have been treated with detailed analysis. Many representative examples are given, and their solutions are in the excellent agreement whit those obtained by the Smith chart and those found in the literature. Also, proposed method can be useful for the students as the educational tools because the solvers either in MATLAB or MATHEMATICA programming give the fast and accurate results. The presented method is also useful for engineers and
physicists especially the triple tuner stub matching. In this paper all calculations have been made by the solver [16].

18. References
1) S. J. Orfanidis,'Electromagnetic Waves and Antennas', www.ece.rutgers.edu/~orfanidi/ewa.
2) F. A. Benson & T. M. Benson, 'Fields, Waves and Transmission lines', Springer-Science+Business Media, BV.
11) Electromagnetic Field Theory – A Problem-Solving Approach – Chapter 8, MIT Open Course Ware.