Clean self-supervised MRI reconstruction from noisy, sub-sampled training data with Robust SSDU

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Abstract

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Index Terms—Deep Learning, Image Reconstruction, Magnetic Resonance Imaging

I. INTRODUCTION

Magnetic Resonance Imaging (MRI) has excellent soft tissue contrast and is the gold standard modality for a number of clinical applications. A hindrance of MRI, however, is its lengthy acquisition time, which is especially challenging when high spatio-temporal resolution is required, such as for dynamic imaging [1]. To address this, there has been substantial research attention on methods that reduce the acquisition time without significantly sacrificing the diagnostic quality [2]–[4]. In MRI, measurements are acquired in the Fourier representation of the image, referred to in the MRI literature as “k-space”. Since the acquisition time is roughly proportional to the number of samples, acquisitions can be accelerated by sub-sampling. A reconstruction algorithm is then employed to estimate the image from the sub-sampled data.

In recent years, reconstructing sub-sampled MRI data with neural networks has emerged as the state-of-the-art [5]–[7]. The majority of existing methods assume that a fully sampled dataset is available for fully-supervised training. However, for many applications, no such dataset is available, and may be difficult or even infeasible to acquire in practice [8]–[10]. In response, there has been a number of self-supervised methods proposed, which train on sub-sampled data only [11]–[14]. Most existing training methods assume that the measurement noise is small and do not denoise sampled data. Section III shows that this implies that the performance substantially degrades when the measurement noise increases. This is a particular concern for low SNR applications, such as for the increasing interest in low-cost, low-field scanners [15]–[17].

In response, this paper presents a modification of Self-Supervised Learning via Data Undersampling (SSDU) [13] that also removes measurement noise. Our method, which we term “Robust SSDU”, combines SSDU with additive Noisier2Noise [18]. In brief, Robust SSDU trains a network to map from a further sub-sampled and further noisy version of the training data to the original sub-sampled, noisy data. Then, at inference, a correction is applied to the network that ensures that the clean image is recovered in expectation. We find that Robust SSDU performs competitively with a best-case benchmark where the network is trained on clean, fully sampled data, despite training on noisy, sub-sampled data only. We also propose a related method that recovers clean images for the simpler task of when fully sampled, noisy data is available for training, which we term “Noisier2Full”. Both Noisier2Full and Robust SSDU are fully mathematically justified and have minimal additional computational expense compared to standard training.

The existing method most similar to Robust SSDU is Nois2Recon-Self-Supervised (Noise2Recon-SS) [19]. We find that Robust SSDU offers substantial image quality im-
provements over Noise2Recon-SS in practice and a two-fold reduction in computational cost at training: see Section V.

II. THEORY: BACKGROUND

This section reviews key works from the literature that form the bases of the methods proposed in this paper.

A. Self-supervised denoising with Noisier2Noise

Denoising with deep learning concerns the task of recovering a clean $N_y$-dimensional vector from noisy measurements

$$y_t = y_{0,t} + n_t,$$

where $n_t$ is noise and $t$ indexes the training set. In MRI, noise in k-space is modeled as complex Gaussian with zero mean, $n_t \sim \mathcal{CN}(0, \Sigma_n^2)$, where $\Sigma_n^2$ is a covariance matrix that can be estimated, for instance, with an empty pre-scan [20]. In this paper, the noise is modeled as white, $\Sigma_n^2 = \sigma_n^2 1$. Although multi-coil MR data has non-trivial $\Sigma_n^2$ in general, we note that the noise can be whitened by left-multiplying $y_t$ with $\Sigma_n^{-1}$.

Self-supervised denoising concerns the task of training a network to remove noise when the training data is itself noisy [21]–[24]. This paper focuses on additive Noisier2Noise [18] because we find that it offers a natural way to extend image reconstruction to low SNR data: see Section III.

Noisier2Noise’s training procedure consists of corrupting the noisy training data with further noise, and training a network to recover the singly noisy image from the noiser image. Concretely, further noise is introduced to $y_t$,

$$\tilde{y}_t = y_t + \tilde{n}_t = y_{0,t} + n_t + \tilde{n}_t,$$

where $\tilde{n}_t \sim \mathcal{CN}(0, \alpha^2 \sigma_n^2 1)$ for a constant $\alpha$. Then, a network $f_\theta$ with parameters $\theta$ is trained to minimize

$$\hat{\theta} = \arg \min_\theta \sum_t \|f_\theta(\tilde{y}_t) - y_t\|^2_2.$$  

The following result states that a simple transform of the trained network yields the clean $y_{0,t}$ in expectation.

Result 1: Let $y_t = y_{0,t} + n_t$ and $\tilde{y}_t = y_t + \tilde{n}_t$ be instances of the random variables $Y = Y_{0} + N$ and $\tilde{Y} = Y + \tilde{N}$ respectively. Minimizing

$$\theta^* = \arg \min_\theta \mathbb{E}[\|f_\theta(\tilde{Y}) - Y\|^2_2|\tilde{Y}]$$  

yields a network that satisfies

$$\mathbb{E}[Y_0|\tilde{Y}] = \frac{(1 + \alpha^2) f_{\theta^*}(\tilde{Y}) - \tilde{Y}}{\alpha^2}.$$  

Proof: See Section 3.3 of [18].

B. Self-supervised reconstruction with SSDU

This section focuses on the case where the data consists of noise-free, sub-sampled data

$$y_t = M_{\Omega_t} y_{0,t}.$$  

Here, $M_{\Omega_t}$ is a sampling mask, a diagonal matrix with $j$th diagonal $1$ when $j \in \Omega_t$ and $0$ otherwise for sampling set $\Omega_t \subseteq \{1, 2, \ldots, N_y\}$.

Self-supervised reconstruction consists of training a network to recover images when only sub-sampled data is available for training [25]. This work focuses on the popular method SSDU [13], which was theoretically justified in [26] via the multiplicative noise version of Noiser2Noise [18]. In this framework, analogous to the further noise used in (1), the data is further sub-sampled by applying a second mask with sampling set $\Lambda_t \subseteq \{1, 2, \ldots, N_y\}$ to $y_t$,

$$\tilde{y}_t = M_{\Lambda_t} y_t = M_{\Lambda_t \cap \Omega_t} y_{0,t},$$  

where $M_{\Lambda_t \cap \Omega_t} = M_{\Lambda_t} M_{\Omega_t}$. Training consists of minimizing a loss function on indices in $\Omega_t \setminus \Lambda_t$, such as

$$\hat{\theta} = \arg \min_\theta \sum_t \|M_{\Omega_t \setminus \Lambda_t} (f_\theta(\tilde{y}_t) - y_t)\|^2_2,$$

where $M_{\Omega_t \setminus \Lambda_t} = \mathbb{I} - M_{\Lambda_t} M_{\Omega_t}$. Although for theoretical ease we state SSDU with an $\ell_2$ loss here, it is known that other losses are possible [13].

Let $p_j = \mathbb{P}[j \in \Omega]$ and $\tilde{p}_j = \mathbb{P}[j \in \Lambda]$. Assuming that

$$p_j > 0 \quad \forall j \quad \text{and} \quad \tilde{p}_j < 1 \quad \forall \{j : p_j < 1\},$$  

the following result from [26] proves that SSDU recovers the clean image in expectation.

Result 2: Let $y_t = M_{\Omega_t} y_{0,t}$ and $\tilde{y}_t = M_{\Lambda_t} y_t$ be instances of the random variables $Y = M_{\Omega_t} Y_0$ and $\tilde{Y} = M_{\Lambda_t} Y$ respectively. When (7) and (8) hold, minimizing

$$\theta^* = \arg \min_\theta \sum_t \|M_{\Omega_t \setminus \Lambda_t} (f_\theta(\tilde{Y}) - Y)\|^2_2$$  

yields a network with parameters that satisfies

$$M_{(\Lambda \cap \Omega)^c} \mathbb{E}[Y_0|\tilde{Y}] = M_{(\Lambda \cap \Omega)^c} f_{\theta^*}(\tilde{Y}).$$  

Proof: See Appendix B of [26].

Result 2 states that the network correctly estimates $Y_0$ in conditional expectation for indices not in $\Lambda \cap \Omega$. To estimate everywhere in k-space one can overwrite sampled indices or use a data consistent architecture; see [26] for details.

III. THEORY: PROPOSED METHODS

The reminder of this paper considers the task of training a network to recover images from noisy, sub-sampled data

$$y_s = M_{\Omega_s} (y_{0,s} + n_s).$$  

It has been stated that when a network reconstructs noisy MRI data with a standard training method, there is a denoising effect

1Where [26] uses $\mathbb{I} - M_{\Lambda_t} M_{\Omega_t}$, this uses paper the more compact notation $M_{\Lambda_t \cap \Omega_t}^c$, where superscript $c$ denotes the complement of a set.
[16]. In the following, we motivate the need for methods that explicitly remove noise by showing that the apparent noise removal is in fact a “pseudo-denoising” effect due to the correct estimation in expectation of indices in \( \Omega \).

Consider the standard approach of training a network to map from noisy, sub-sampled \( y_t \) to noisy, fully sampled \( y_{0,t} + n_t \). In terms of random variables, training consists of minimizing

\[
\theta^* = \arg \min_{\theta} E[\|f_\theta(Y) - (Y_0 + N)\|_2^2 | Y],
\]

which gives a network that satisfies

\[
f_{\theta^*}(Y) = E[Y_0 + N | Y].
\]  \hspace{1cm} (11)

It is instructive to examine how \( E[Y_0 + N | Y] \) depends on the sampling mask \( \Omega \). Firstly, for \( j \notin \Omega \),

\[
E[Y_{0,j} + N_j | Y, j \notin \Omega] = E[Y_{0,j} | Y] + E[N_j]
\]

where we have used the independence of \( N_j \) from \( Y \) when \( j \notin \Omega \) and \( E[N_j] = 0 \) by assumption. For the alternative \( j \in \Omega \),

\[
E[Y_{0,j} + N_j | Y, j \in \Omega] = E[Y_j | Y] = Y_j
\]

where \( Y_{0,j} + N_j = Y_j \) for \( j \in \Omega \) has been used. The trained network therefore satisfies

\[
f_{\theta^*}(Y) = E[Y_0 + N | Y] = M_{\Omega^c} E[Y_0 | Y] + M_{\Omega} Y.
\]

Therefore the network targets the noise-free \( Y_{0,j} \) in regions in \( \Omega^c \) but recovers the noisy \( Y \) otherwise. As there is less measurement noise present than \( Y_0 + N \), this gives the impression of noise removal, however, we emphasize that the network does not remove the noise in \( Y \); hence the use of the term “pseudo-denoising”.

We refer to this method described in this section as “Noisier2Full” throughout this paper. In the following we propose methods that explicitly recover \( Y_0 \) in conditional expectation from noisy, sub-sampled inputs in two cases: A) the training data is noisy and fully sampled; B) the training data is noisy and sub-sampled. For tasks A and B we propose “Noisier2Full” and “Robust SSDU” respectively.

### A. Noisier2Full for fully sampled, noisy training data

This section proposes Noisier2Full, which extends additive Noisier2Noise to non-supervised tasks for noisy, fully sampled training data. Based on (11), we propose corrupting the measurements \( y_t \) with further noise on the sampled indices,

\[
\tilde{y}_t = y_t + M_{\Omega_t} n_t.
\]

Then we minimize the loss between \( \tilde{y}_t \) and the noisy, fully sampled training data \( y_{0,t} + n_t \). In terms of random variables,

\[
\theta^* = \arg \min_{\theta} E[\|f_\theta(\tilde{Y}) - (Y_0 + N)\|_2^2 | \tilde{Y}].
\]  \hspace{1cm} (13)

Minimizing the \( \ell_2 \) norm gives a network that satisfies

\[
f_{\theta^*}(\tilde{Y}) = E[Y_0 + N | \tilde{Y}],
\]

which is recognizable as (11) with \( Y \) replaced by \( \tilde{Y} \). Similarly to (12), \( N_j \) is independent of \( \tilde{Y} \) when \( j \notin \Omega \), so the ground truth is estimated in such regions:

\[
E[Y_{0,j} | \tilde{Y}, j \notin \Omega] = E[Y_{0,j} | \tilde{Y}].
\]

However, crucially, the expectation is conditional on \( \tilde{Y} \), not \( Y \), so the additive Noisier2Noise correction stated in Result 1 is applicable when \( j \in \Omega \):

\[
E[Y_{0,j} | \tilde{Y}, j \in \Omega] = \frac{(1 + \alpha^2)f_{\theta^*}(\tilde{Y}) - \tilde{Y}_j}{\alpha^2}
\]

Therefore \( Y_0 \) can be estimated with

\[
E[Y_0 | \tilde{Y}] = M_{\Omega^c} \left( \frac{(1 + \alpha^2)f_{\theta^*}(\tilde{Y}) - \tilde{Y}}{\alpha^2} \right) + M_{\Omega} f_{\theta^*}(\tilde{Y}).
\]  \hspace{1cm} (14)

In summary, Noisier2Full recovers \( Y_0 \) in conditional expectation by introducing further noise to the sampled indices during training, and correcting those indices at inference via additive Noisier2Noise. In the subsequent section, we show how this approach can extended to the more challenging case where the training data is also sub-sampled.

### B. Robust SSDU for sub-sampled, noisy training data

This section proposes Robust SSDU, which recovers clean images in conditional expectation when the training data is both noisy and sub-sampled. Robust SSDU combines the approaches from Sections [II-A] and [II-B] to simultaneously
reconstruct and denoise the data; see Fig. [1] for a schematic. We propose combining [1] and [5] to form a vector that is further sub-sampled and additionally noisy,
\[ \tilde{y}_t = M_{\Lambda'\cap\Omega}(y_t + \tilde{n}_t). \]
Recall that SSDU employs \( M_{\Omega'\cap\Omega} \) in the loss, which yields a network that estimates indices in \((\Lambda \cap \Omega)^e\); see Result 2 For Robust SSDU we replace \( M_{\Omega'\cap\Omega} \) with \( M_{\Omega} \), so that the loss is
\[ \hat{\theta} = \arg \min_{\theta} \sum_{t} \| M_{\Omega}(f_\theta(\tilde{y}_t) - y_t) \|_2^2. \]
In the following we show that this change leads to estimation everywhere in k-space, not just indices in \((\Lambda \cap \Omega)^e\).

Claim 1: Let \( y_t = M_{\Omega}(y_{0,t} + n_t) \) and \( \tilde{y}_t = M_{\Lambda'\cap\Omega}(y_t + \tilde{n}_t) \) be instances of the random variables \( Y = M_{\Omega}(Y_0 + N) \) and \( \tilde{Y} = M_{\Lambda'\cap\Omega}(Y + \tilde{N}) \) respectively. When (7) and (8) hold, minimizing
\[ \theta^* = \arg \min_{\theta} E[\| M_{\Omega}(f_\theta(\tilde{Y}) - Y) \|_2^2 \tilde{Y}]. \]
yields a network with parameters that satisfies
\[ f_\theta(\tilde{Y}) = E[Y_0 + N|\tilde{Y}]. \]
Proof: See the appendix.

At inference, we can use a similar approach to Section III-A applying the additive Noisier2Noise correction on indices sampled in \( \tilde{Y} \). Since the indices sampled in \( \tilde{Y} \) are \( \Lambda \cap \Omega \), the clean image \( Y_0 \) is estimable with
\[ E[Y_0|\tilde{Y}] = M_{\Lambda'\cap\Omega} \left( \frac{(1 + \alpha^2) f_{\theta^*}(\tilde{Y}) - \tilde{Y}}{\alpha^2} \right) + M_{(\Lambda'\cap\Omega)^e} f_{\theta^*}(\tilde{Y}). \]
Roughly speaking, Robust SSDU can be thought of as a generalization of Noisier2Full to sub-sampled training data. Specifically, Robust SSDU is mathematically equivalent to Noisier2Full when \( \Omega = \{1, 2, \ldots, N_c\} \) and there is the change of notation \( \Lambda \rightarrow \Omega \). More broadly, Robust SSDU can be interpreted as the simultaneous application of additive and multiplicative Noisier2Noise [18], [26].

C. Weighted variants of Noisier2Full and Robust SSDU

For Noisier2Full and Robust SSDU, the task at training and inference is not identical: at training the network maps from \( \tilde{Y} \) to \( Y_0 + N \) or \( Y \), while at inference it maps from \( \tilde{Y} \) to \( Y_0 \) via the \( \alpha \)-based correction term. This section describes how this can be compensated for by modifying the loss function.

For Noisier2Full, we propose replacing (13) with
\[ \theta^* = \arg \min_{\theta} E[\| W_A(f_\theta(\tilde{Y}) - (Y_0 + N)) \|_2^2 \tilde{Y}], \]
where \( W_A \) is a diagonal loss weighting defined as
\[ W_A = \frac{(1 + \alpha^2)^2}{\alpha^2} M_{\Omega} + M_{(\Lambda'\cap\Omega)^e}. \]
The matrix \( W_A \) increases the weight of the indices in \( \Omega \). Its role is to compensate for the difference between the noise removed at training, which has variance \( \text{Var}(\tilde{N}) = \alpha^2 \sigma_n^2 \), and the noise removed at inference, which has variance \( \text{Var}(N + \tilde{N}) = (1 + \alpha^2) \sigma_n^2 \).

The correction term is the ratio of these variances, where a square root is included due to the square in the norm. The weighting is large when \( \alpha \) is small and tends to 1 for large \( \alpha \).

For Robust SSDU, we propose replacing \( \theta_0 \) with
\[ \theta^* = \arg \min_{\theta} E[\| W_B M_{\Omega}(f_\theta(\tilde{Y}) - Y) \|_2^2 \tilde{Y}], \]
where \( W_B \) is the diagonal matrix
\[ W_B = \frac{(1 + \alpha^2)^2}{\alpha^2} M_{\Omega} + (1 - K)^{-\frac{1}{2}} M_{(\Lambda'\cap\Omega)^e}. \]
Here, \( K \) is defined as
\[ K = (1 - \tilde{P} P)^{-1}(1 - P) \]
for \( P = E[M_A] \) and \( \tilde{P} = E[M_A] \). The \( M_{(\Lambda'\cap\Omega)^e} \) coefficient has a similar role to the \( M_{\Omega} \) coefficient in (17). The \( M_{(\Lambda'\cap\Omega)^e} \) coefficient compensates for the variable density of \( \Omega \) and \( \Lambda \), and was shown in [26] to improve the reconstruction quality and robustness to the distribution of \( \Lambda \).

The weightings can be thought of as entry-wise modifications of the learning rate [26]. Neither \( W_A \) nor \( W_B \) change \( \theta^* \), so the proofs of Noisier2Full and Robust SSDU from Sections III-A and III-B hold. Rather, the role of the weights is to improve the finite-sample case in practice, where \( \theta^* \) is estimated with \( \hat{\theta} \); see Section V for an empirical evaluation.

IV. MATERIALS AND METHODS

A. Description of data

We used the multi-coil brain data from the publicly available fastMRI dataset [28]. We only used data that had 16 coils and normalized so that the cropped RSS estimate had maximum 1. Here, the cropped RSS is defined as \( Z((\sum_{j=1}^{N_c} |F^H y_{ij}|^2)^{\frac{1}{2}}, \) where the subscript \( c \) refers to all entries on the cth coil, \( F \) is the discrete Fourier transform, \( N_c \) is the number of coils and \( Z \) is a cropping operator. We retrospectively sub-sampled column-wise with the central 10 lines fully sampled and randomly drawn with polynomial density otherwise, with the probability density scaled to achieve a desired acceleration factor \( R_{\Omega} = N_C/\sum_j p_j \). The data was treated as noise-free, and we generated white, complex Gaussian measurement noise with standard deviation \( \sigma_n \) to simulate noisy conditions. An implementation of our method in PyTorch will be made available on GitHub.

B. Comment on the proposed methods in practice

The theoretical guarantees for Noisier2Full and Robust SSDU use the further noisy, possibly further sub-sampled \( y_t \) as the input to the network at inference. In practice, as suggested in the original Noisier2Noise [18] and SSDU [13] papers, we used \( y_s \) as the input to the network at inference, so that the estimate
\[ \hat{y}_s = M_{\Omega}(y_s) \]
for_recon

available from https://fastmri.med.nyu.edu
https://github.com/charlesmillard/Noisier2Noise_for_recon
is used in place of (14) and (16). Although this deviates from strict theory, and is not guaranteed to be correct in conditional expectation, we have found that it achieves better reconstruction performance in practice: see [18] and [26] for a detailed evaluation. All subsequent results for the proposed methods use this estimate at inference.

C. Comparative training methods

For noise-free, fully sampled training data, fully supervised training can be employed. Although it is possible in principle to have higher SNR data at training than at inference by acquiring multiple averages [29], such datasets would require an extended acquisition time and are rare in practice. Nonetheless, training a network on this type of data via simulation is instructive as a best-case target. This method is referred to as the “best-case benchmark” throughout this paper.

For noisy, fully sampled training data, we employed three training methods: the proposed Noisier2Full and Weighted Noisier2Full and the standard approach Noise2Full, as described in Section III.

For the more challenging scenario where noisy, sub-sampled training data is available, we compared Robust SSDU and Weighted Robust SSDU with the original version of SSDU, which reconstructs sub-sampled data but does not denoise. We refer to this as “Standard SSDU”. We also compared with Noise2Recon-SS [19], which, like Robust SSDU, includes adding further noise to the sub-sampled data. However, Noise2Recon-SS has a number of key differences to the method proposed in this paper. With an \( \ell_2 \) k-space loss, training with Noise2Recon-SS consists of minimizing

\[
\hat{\theta} = \arg \min_{\theta} \sum_t \left\| M_{\Omega_t} \Lambda_t (f_\theta(M_{\Lambda_t} y_t) - y_t) \right\|^2_2 + \lambda \left\| f_\theta(y_t + M_{\Omega_t} \bar{n}_t) - f_\theta(M_{\Lambda_t} y_t) \right\|^2_2,
\]

where \( \lambda \) is a hand-selected weighting. We used \( \lambda = 1 \) throughout. The \( \ell_2 \) loss in k-space was used so that it could be fairly compared to the other methods, but we note that [19] used image-domain losses. The first term is based on SSDU, and the second ensures that \( f_\theta(y_t + M_{\Omega_t} \bar{n}_t) \) and \( f_\theta(M_{\Lambda_t} y_t) \) yield similar outputs, so that the method is in a sense robust to \( \bar{n}_t \). At inference, Noise2Recon-SS uses \( \hat{y}_s = f_\theta(y_s) \); there is no correction term. We emphasize that, unlike the proposed Robust SSDU, there is no theoretical evidence that Noise2Recon-SS recovers the clean image in expectation.

D. Network architecture and training details

All training methods considered in this paper are agnostic to the network architecture. We employed a network architecture based on the Variational Network (VarNet) [7], [30], which is available as part of the fastMRI package [28]. VarNet consists of a coil sensitivity map estimation module followed by a series of “cascades”. The k-space estimate at the \( k \)th cascade takes the form

\[
\hat{y}_{k+1} = \hat{y}_k - \eta_k M_{in}(\hat{y}_k - y_{in}) + G_{\theta_k}(\hat{y}_k)
\]

where \( y_{in} \) and \( M_{in} \) are the input k-space and sampling mask respectively and the \( t \) or \( s \) index has been dropped for legibility. We use the generic subscript \( in \) here because the input is not the same for every method: for instance, fully-supervised and Noisier2Full have \( M_{in} = M_{\Omega_t} \) and \( M_{in} = M_{\Lambda_t \cap \Omega_t} \) respectively. Here, \( \eta_k \) is a trainable parameter and \( G_{\theta_k}(\hat{y}_k) \) is a neural network with cascade-dependent parameters \( \theta_k \), referred to as a “refinement module”, which was an image-domain U-net [31] in [7], [30].

VarNet was originally constructed for reconstruction only, without explicit denoising. For joint reconstruction and denoising, we propose partitioning \( G_{\theta_k}(\hat{y}_k) \) into two functions,

\[
G_{\theta_k}(\hat{y}_k) = M_{in} G_{\theta_k^D}(\hat{y}_k) + (1 - M_{in}) G_{\theta_k^R}(\hat{y}_k).
\]

This refinement module is illustrated in Fig. 2. We refer to the architecture with the proposed refinement module as “Denoising VarNet” throughout this paper. We used a U-net [31] for both \( G_{\theta_k^D}(\hat{y}_k) \) and \( G_{\theta_k^R}(\hat{y}_k) \), although we note that in general these functions need not be the same. We used 5 cascades, giving a network with 2.5 \( \times \) 10^7 parameters.

We used the Adam optimizer [32] and trained for 100 epochs with learning rate \( 10^{-3} \). The \( \Omega_t \) and \( n_t \) were fixed but the \( \Lambda_t \) and \( \bar{n}_t \) were re-generated once per epoch [33]. As in [26], we used the same distribution of \( \Lambda_t \) as \( \Omega_t \) but with parameters selected to give a sub-sampling factor of \( R_\Lambda = N_y / \sum_j \bar{p}_j = 2 \). The choice of \( \alpha \) is discussed in Section V-B. Unless otherwise stated, the training methods were evaluated on data generated with \( \sigma_n \in \{0.02, 0.04, 0.06, 0.08\} \) and \( R_\Omega \in \{4, 8\} \). For each training method, \( \sigma_n \) and \( R_\Omega \), we trained a separate network from scratch.
C. Task A: Fully sampled, noisy training data

Fig. 6 shows the how the test set loss of networks trained on fully sampled, noisy data compares with the best-case benchmark. Noise2Full’s performance significantly degrades as $\sigma_n$ increases: for $R_{\Omega} = 8$ and $\sigma_n = 0.08$, Noise2Full’s performance is approximately double that of the best-case benchmark. In contrast, Weighted Noisier2Full performs similarly to the benchmark: for all $\sigma_n$ and $R_{\Omega}$, Weighted Noisier2Full was within 0.05dB of the best-case benchmark. The performance of unweighted Noisier2Full was very similar to the weighted version, and is not shown in Fig. 6 for clarity.

Two reconstruction examples are shown in Fig. 5. Here, and throughout this paper, the example reconstructions show the image domain RSS cropped to a central $320 \times 320$ region.

The k-space Normalized Mean-Squared Error (NMSE) and Structural Similarity (SSIM) [35] are also shown, where the SSIM is computed on the magnitude image with the background excluded via the mask from the ESPIRiT algorithm [36] which we implemented with the BART toolbox [37].

D. Task B: Sub-sampled, noisy training data

For Robust SSDU, Weighted Robust SSDU and Noise2Recon-SS we trained with both $\alpha = 0.5, 1$ and show the results for the better test set NMSE. Fig. 8 shows the test set loss. The weighted and unweighted variants of the proposed Robust SSDU performed within 0.17dB of the best-case benchmark, despite only having access to noisy, sub-sampled training data. Noise2Recon-SS performs well in some cases, particularly at $R_{\Omega} = 4$, but is consistently outperformed by both variants of Robust SSDU. Fig. 7 shows example reconstructions, demonstrating similar performance to the best-case benchmark qualitatively.

Fig. 9 compares Standard SSDU and Weighted Robust SSDU using clinical expert bounding boxes from fastMRI+ [34], which shows that the proposed method has substantially enhanced pathology visualization.

VI. Discussion and conclusions

Fig. 3 shows that the proposed Denoising VarNet consistently outperforms the Standard VarNet architecture. We understand this to be a consequence of the difference between the distributions of errors due to sub-sampling or measurement noise: the Standard VarNet removes both contributions to the error in a single U-net per cascade, while the Denoising VarNet simplifies the task by decomposing the contributions to the error, so that each of the two U-nets per cascade are specialized for the two distinct error distributions.

The improvement in robustness for the weighted version of Noisier2Full shown in Fig. 4 is especially prominent for small $\alpha$, such as the roughly 4 times improvement in proximity to the benchmark at $\alpha = 0.25$. For large $\alpha$ the $\alpha$-based weighting is closer to 1, so the weighted method tends to the unweighted method and the difference in performance is small. For instance, when $\alpha = 1.75$, the $\alpha$-based weighting is 1.15, so has a relatively marginal effect. Although the performance of the methods are quite similar for tuned $\alpha$, we recommend

![Fig. 3: The difference in decibels between the test set loss of Standard VarNet, $L_{\text{Standard}}$, and the proposed Denoising VarNet, $L_{\text{Proposed}}$, is shown for the benchmark training method. All differences are positive, showing that Denoising VarNet outperforms Standard VarNet, especially for large $\sigma_n$.](image)

![Fig. 4: The robustness of Noisier2Full and Weighted Noisier2Full to $\alpha$ at $R_{\Omega} = 8$ and $\sigma_n = 0.06$. The difference in decibels between the test set loss $L$ and the best-case benchmark $L_{\text{Benchmark}}$ is shown. The weighted version is substantially more robust, especially for small $\alpha$.](image)
Fig. 5: Reconstructions at $\sigma_n = 0.06$ when fully sampled, noisy data is available for training. “Noisy” and “Noisy & sub-sampled” refer to the RSS reconstruction of $y_0 + n_s$ and $M_{\Omega}(y_0 + n_s)$ respectively. While there is clear noise in Noise2Full’s reconstruction, the proposed methods, which are indicated with an asterisk, perform very similarly to the best-case benchmark.

Fig. 6: The test set loss in decibels for methods that use noisy, fully-sampled training data. Weighted Noisier2Full performs comparably with the best-case benchmark for all $\sigma_n$. We emphasize that the flatness does not imply that the test set loss is unchanged for changing $\sigma_n$, as the best-case benchmark test set loss is different for each $\sigma_n$.

using the weighted version in practice due to its improved robustness to $\alpha$.

The examples in figures 5, 7 and 9 show that proposed methods are qualitatively very similar to the best-case benchmark, and substantially improve over methods without denoising, whose reconstructions are visibly corrupted with measurement noise. The examples exhibit some loss of detail and blurring at tissue boundaries, especially at $R_{\Omega} = 8$. However, the extent of detail loss is similar in the benchmark, indicating that the loss of detail is not a limitation of the proposed methods. Rather, the qualitative performance is limited by the other factors such as the architecture, dataset and choice of loss function. This can also be explained in part by noting that the high-frequency regions of k-space, which provide fine details, typically have smaller signal so are particularly challenging to recover in the presence of significant measurement noise.

The pseudo-denoising effect described in Section III is visible in Fig. 5, which shows less noise in Noise2Full than Noisy. We note that the NMSE of Noisy is roughly 4 and 8 times larger than Noise2Full for $R_{\Omega} = 4, 8$ respectively, consistent with the theoretical finding that Noise2Full does not remove the measurement noise in the input. A comparison of figures 6 and 8 shows that Standard SSDU performs very similarly to Noise2Full quantitatively, exhibiting a similar pseudo-denoising effect.

Although Noise2Recon-SS improves over Standard SSDU, there is a substantial difference between its performance and the proposed Robust SSDU both qualitatively and quantitatively. In [19], Noise2Recon-SS was not compared with a best-case benchmark; it was only shown to have improved performance compared to Standard SSDU, consistent with
Table:

<table>
<thead>
<tr>
<th>Method</th>
<th>RMSE</th>
<th>SSIM</th>
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<tbody>
<tr>
<td>Reference Standard SSDU</td>
<td>0.699</td>
<td>0.74</td>
</tr>
<tr>
<td>Noisy &amp; sub-sampled benchmark</td>
<td>0.393</td>
<td>0.92</td>
</tr>
<tr>
<td>Standard SSDU</td>
<td>0.546</td>
<td>0.77</td>
</tr>
<tr>
<td>Noise2Recon-SS</td>
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<td>0.78</td>
</tr>
<tr>
<td>Unweighted Robust SSDU*</td>
<td>0.415</td>
<td>0.81</td>
</tr>
<tr>
<td>Weighted Robust SSDU*</td>
<td>0.415</td>
<td>0.81</td>
</tr>
</tbody>
</table>

Fig. 7: Example reconstructions for networks trained on noisy, sub-sampled data at $\sigma_n = 0.06$. The proposed methods, highlighted with an asterisk, perform very similarly to the best-case benchmark, even at $R_\Omega = 8$.

Fig. 8: The test set loss in decibels for methods that use noisy, sub-sampled training data. The difference between the loss and the best-case benchmark is shown.

Robust SSDU requires only a few additional cheap computational steps compared to standard training: the addition or multiplication of the further noise and sub-sampling mask respectively, and the $\alpha$-based correction at inference. Accordingly, the compute time and memory requirements of the proposed methods was found to be very similar to Noise2Full or Standard SSDU. In contrast, Noise2Recon-SS uses both $M_Ny_t$ and $y_t + M_{\Omega_t}\tilde{n}_t$ as the network inputs at training, so requires twice as many forward passes to train the network compared to Robust SSDU. Accordingly, we found that Noise2Recon-SS required approximately twice as much memory and took around two times longer per epoch as the proposed methods.

Another existing method designed for noisy, sub-sampled training data is Robust Equivariant Imaging (REI) method [38], [39]. We did not compare with REI as it was designed for reconstruction tasks with a fixed sampling pattern: the $\Omega_t$ was the same for all $t$. This sampling set assumption is central to their use of equivariance, and contrasts with the methods proposed here, which assume that the sampling mask is an instance of a random variable that satisfies $p_j > 0$ everywhere. However, REI’s suggestion to use Stein’s Unbiased Risk Estimate (SURE) [40] to remove measurement noise would be feasible in combination with SSDU and warrants further investigation in future work.

In [16], an untrained denoising algorithm or pre-trained denoising network was appended to a reconstruction network, which was found to perform well in practice despite not modeling the task-specific noise characteristics [41]. To test the former, we denoised the magnitude image output of
Standard SSDU with BM3D [42]. We found that this approach was not competitive with the proposed methods: for instance, at $R_Ω = 8$ and $σ_n = 0.06$, we found that the NMSE improvement compared to Standard SSDU was only around 0.5dB. Although appending a pre-trained denoising network to a reconstruction network was beyond the scope of this work, we note that employing transfer learning in conjunction with Robust SSDU to fine-tune a pre-trained denoisier to a specific reconstruction network was beyond the scope of this work, we note that employing transfer learning in conjunction with Robust SSDU to fine-tune a pre-trained denoisier to a specific data and noise model is a potential direction for future work.

The theoretical work presented in this paper only applies to the case of $ℓ_2$ minimization, which can lead to blurry reconstructions. However, it has been established that Standard SSDU can be applied with other losses such as an entry-wise mixed $ℓ_1$-$ℓ_2$ loss in k-space [13]. We have found that Robust SSDU with an $ℓ_2$ loss on $Λ \cap Ω$ and mixed $ℓ_1$-$ℓ_2$ loss on $Ω \setminus Λ$ also performs competitively with a suitable benchmark in practice (results not shown for brevity). Future work includes establishing whether Robust SSDU can be modified to be applicable to alternative loss functions, including potentially losses on the RSS image.

It was assumed in this paper that the measurement noise is white with known variance. Future work includes evaluating the performance of the proposed methods for the more realistic case where the measurement noise has non-trivial covariance matrix and is estimated retrospectively. It would also be desirable to develop an approach that automatically tunes $α$ and the distribution of $M_Ω$, whose optimal values are specific to the noise model, $M_Ω$ distribution and dataset.

**APPENDIX**

This appendix proves Claim 4 using a similar approach to Appendix B of [26]. Minimization according to [15] yields a network that satisfies

$$E[M_Ω(f_{θ^∗}; (Y) - Y)|Y] = 0$$  \hspace{1cm} (19)$$

We split the conditional expectation into two cases: $\tilde{Y}_j \neq 0$ and $\tilde{Y}_j = 0$.  

*Case 1* ($E[m_j(f_{θ^∗};(Y))_j - Y_j)|\tilde{Y}_j \neq 0]$): When $\tilde{Y}_j \neq 0$, the measurement model implies that $m_j = 1$ and $Y_j = Y_{0,j} + N_j$. Therefore

$$E[m_j(f_{θ^∗};(Y)|Y)_j - Y_j)|\tilde{Y}_j \neq 0] = E[f_{θ^∗};(Y)|Y)_j - Y_{0,j} - N_j]\tilde{Y}_j | \tilde{Y}_j \neq 0]$$  \hspace{1cm} (20)$$

*Case 2* ($E[m_j(f_{θ^∗};(Y))_j - Y_j)|\tilde{Y}_j = 0]$): We can use the result derived from equation (27) to (29) in [26], with $Y_{0,j}$ replaced by $Y_{0,j} + N_j$:

$$E[m_j(f_{θ^∗};(Y)|Y)_j - Y_j)|\tilde{Y}_j = 0] = E[f_{θ^∗};(Y)|Y)_j - Y_{0,j} - N_j]\tilde{Y}_j = 0] \cdot (1 - k_j)$$  \hspace{1cm} (21)$$

where

$$k_j = P[Y_j = 0]|\tilde{Y}_j = 0] = \frac{1 - p_j}{1 - p_j p_j}$$  \hspace{1cm} (22)$$

is the $j$th diagonal of the matrix $K$ in [18].

**Combining Cases 1 and 2:** Consider the candidate

$$E[m_j(f_{θ^∗};(Y)|Y)_j = \{1 - k_j(1 - m_j m_j]\} E[f_{θ^∗};(Y)|Y)_j - Y_{0,j} - N_j]\tilde{Y}_j].$$  \hspace{1cm} (23)$$

To verify that this expression is correct, we can check that it is consistent with Cases 1 and 2. For Case 1: if $Y_j \neq 0$, $\bar{m}_j m_j = 1$ and the term in curly brackets is 1, so (23) is consistent with (20). For Case 2: if $Y_j = 0$, $\bar{m}_j m_j = 0$ and the term in curly brackets is $1 - k_j$, so (23) is consistent with (21) as required. By (19),

$$\{1 - k_j(1 - \bar{m}_j m_j]\} E[f_{θ^∗};(Y)|Y)_j - Y_{0,j} - N_j]\tilde{Y}_j] = 0$$

The term in the curly brackets is non-zero for all $j$ if $1 - k_j$ is non-zero for $j \notin Ω \cap Λ$, which true when (7) and (8) hold, where we note that the special case $p_j = p_j = 1$ is also allowed since $\bar{m}_j m_j = 1$ always. Given this assumption, dividing by the term in the curly brackets:

$$E[f_{θ^∗};(Y)|Y)_j - Y_{0,j} - N_j]\tilde{Y}_j] = 0.$$  \hspace{1cm} (24)$$

Vectorizing gives the required result.
REFERENCES


