A Novel One-bit Dynamic Quantizer for Event-Triggered Control Systems

Dhafer Almakhles ¹ and Mahmoud Abdelrahim ²

¹Prince Sultan University
²Affiliation not available

October 30, 2023

Abstract
We investigate the design of quantized event-triggered controllers for LTI systems with unknown initial states. The proposed approach is novel and incorporates interesting features. First, the quantizer that we synthesize is dynamic with exponential behaviour, which enables us to capture the state in a finite time. Second, once the state is captured, the quantization error is guaranteed to remain bounded thanks to the sliding mode feature. Third, the quantizer dynamics only depend on the sign of the quantization error, which allows for one-bit transmission rather than packet-based transmission. Moreover, the event-triggering mechanism that we consider is also dynamic and depends only on locally available information. The overall system is modeled as a hybrid dynamical system and the closed-loop stability is investigated using Lyapunov functions. The approach ensures global asymptotic stability property for the closed-loop system. Moreover, Zeno behavior is prevented by means of time regularization techniques. The effectiveness of the proposed approach is illustrated by numerical simulation.
A Novel One-bit Dynamic Quantizer for Event-Triggered Control Systems

Dhafer Almakhles\textsuperscript{a} and Mahmoud Abdelrahim\textsuperscript{a,b}

\textsuperscript{a}Department of Communications and Networks Engineering, Prince Sultan University, Riyadh, Saudi Arabia
\textsuperscript{b}Department of Mechatronics Engineering, Faculty of Engineering, Assiut University, Assiut, Egypt
dalmakhles@psu.edu.sa, \{m.abdelrahim@psu.edu.sa or m.abdelrahim@aun.edu.eg\}

Abstract—We investigate the design of quantized event-triggered controllers for LTI systems with unknown initial states. The proposed approach is novel and incorporates interesting features. First, the quantizer that we synthesize is dynamic with exponential behavior, which enables us to capture the state in a finite time. Second, once the state is captured, the quantization error is guaranteed to remain bounded thanks to the sliding mode feature. Third, the quantizer dynamics only depend on the sign of the quantization error, which allows for one-bit transmission rather than packet-based transmission. Moreover, the event-triggering mechanism that we consider is also dynamic and depends only on locally available information. The overall system is modeled as a hybrid dynamical system and the closed-loop stability is investigated using Lyapunov functions. The approach ensures global asymptotic stability property for the closed-loop system. Moreover, Zeno behavior is prevented by means of time regularization techniques. The effectiveness of the proposed approach is illustrated by numerical simulation.

I. INTRODUCTION

Networked control systems (NCS) have received great development and research attention in the last few decades due to their advantages compared to traditional control systems, see [1] and the references therein. The insertion of a communication channel in the feedback loop enabled more convenience of implementation, flexibility, reconfiguration, and ease of maintenance. This architecture has been developed recently for other applications such as the Internet of Things (IoT) and cyber-physical systems (CPS). Although NCS brings appealing control features, the introduced network induces new challenges in the analysis and the design of the control systems such as quantization, variable inter-transmissions, delays, packet dropout, scheduling, and security issues [2]–[5]. In this paper, we focus on the first two aspects.

Due to the digital nature of the network, any transmitted information will be discretized both in time and magnitude. Hence, sampling and quantization strategies need to be designed such that the use of the network is reduced and the closed-loop stability is maintained. To cope with quantization, several techniques of the literature have been proposed including static and dynamic quantizers, see [6] and the references therein. In contrast to the former type, dynamic quantizers are designed such that the quantizer range and the number of quantization levels are dynamically adapted according to the magnitude of the input signal and are not fixed as in static quantizers, see e.g., [7], [8]. This has a particular advantage when the initial magnitude of the quantized signal is unknown and/or when the control system is affected by exogenous input. To cope with sampling, event-triggered control (ETC) has proven its advantage to enlarge the inter-transmission times compared to periodic sampling setup, see [9], [10] and the references therein. The idea of ETC is to close the loop and transmit new information only when a state-dependent criterion is violated. Indeed, the triggering condition is synthesized such that closed-loop stability is guaranteed.

In this paper, we consider state feedback linear time-invariant (LTI) systems, where the plant state is transmitted over the network to the controller and subject to both quantization and sampling. We assume that the magnitude of the initial state measurement is unknown and we design a dynamic quantizer and an ETC to ensure the closed-loop stability. The quantizer that we propose is novel and involves an exponential function to allow capturing the state in a finite time. The design of the quantizer is based on the sliding mode technique to ensure that the quantization error remains bounded once the state is captured. Moreover, the proposed quantizer depends only on the sign of the quantization error, i.e., has two levels, which enable it to transmit the quantized information using only a single bit rather than multiple digits (packets). After the quantization policy is established, we synthesize an ETC to generate the inter-transmission times for the quantized state measurements. The ETC that we consider is also dynamic in the sense of [11] such that the triggering condition relies on a dynamic variable rather than a static threshold [12], [13]. Note that the design of event-triggered controllers based on quantized information induces non-trivial technical challenges since the sampling-induced error is not reset to zero at each transmission instant.

Few results of the literature have considered both ETC and dynamic quantization [13]–[18]. All the aforementioned techniques are based on the dynamic quantizer of [7] using a finite number of quantization regions and the transmitted information is sent in a packet-based fashion. Compared to those techniques, we propose a novel dynamic quantization strategy, the encoding-decoding mechanism allows for 1-bit transmissions, and the combined quantized ETC approach is different. Note that the idea of binary quantizers has been reported in other techniques such as delta-sigma ($\Delta\Sigma$), adaptive
differential modulators [19], [20] and sliding mode quantizers [21]. However, none of these results consider the situation where the magnitude of the initial state is unknown, in contrast to our strategy. To the best of our knowledge, this is the first work to consider all the aspects of dynamic quantization, event-triggered implementation, and 1-bit data transmission.

The overall system is modeled as a hybrid dynamical system which allows us to consider both continuous-time and discrete-time dynamics as naturally appear in NCS. Sufficient conditions are provided in terms of linear matrix inequality (LMI) to ensure closed-loop stability. The approach is illustrated by simulation on a dynamical system. The results show that the quantizer captures the initial state in a finite time as expected. Moreover, the amount of transmissions is less than those generated by periodic sampling.

The main contributions of this paper are summarized below:

- A novel dynamic quantizer is proposed, which acts exponentially to capture the state in a finite time;
- The proposed quantizer has only two levels, allowing for 1-bit data transmission;
- The approach is presented in a systematic way by solving an LMI condition to determine the design parameters for the quantizer and the ETC;
- The coupling between the quantizer and the ETC is investigated and the tradeoff between sampling and quantization is provided.

II. Preliminaries

Let $\mathbb{R} := (-\infty, \infty)$, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{N}_{>0} := \{1, 2, \ldots\}$. We denote the minimum and maximum eigenvalues of the real symmetric matrix $A$ as $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. We write $A^T$ to denote the transpose of $A$, and $I_n$ stands for the identity matrix of dimension $n$. The symbol $\star$ stands for symmetric blocks. We write $(x, y) \in \mathbb{R}^{n_\alpha+n_\beta}$ to represent the vector $[x^T, y^T]^T$ for $x \in \mathbb{R}^{n_\alpha}$ and $y \in \mathbb{R}^{n_\beta}$. For a vector $x \in \mathbb{R}^{n_\alpha}$, we denote by $|x| := \sqrt{x^T x}$ its Euclidean norm and, for a matrix $A \in \mathbb{R}^{n_\alpha \times n_\beta}$, $|A| := \sqrt{\lambda_{\max}(A^T A)}$. The sign function is defined as $\text{sgn}(s) = \{ -1, 0, 1 \} \forall \{s > 0, s = 0, s < 0\}$. We denote by $\text{Re}(a)$ the real part of the complex number $a$.

We consider hybrid systems of the following form [22]

$$
\dot{x} = F(x), \quad x \in C, \quad x^+ \in G(x) \quad x \in D, \tag{1}
$$

where $x \in \mathbb{R}^{n_\alpha}$ is the state, $C$ is the flow set, $F$ is the flow map, $D$ is the jump set and $G$ is the jump map. Solutions to the system (1) are defined on hybrid time domains.

III. Problem Formulation

Consider the following LTI system

$$
\dot{x} = Ax + Bu, \quad x(0) = x_0 \tag{2}
$$

where $x \in \mathbb{R}^{n_\alpha}$ is the plant state, $u \in \mathbb{R}^{n_\beta}$ is the control input and $K \in \mathbb{R}^{n_\beta \times n_\alpha}$ is the controller gain matrix. We assume that the pair $(A, B)$ is stabilizable. We consider the case where the full state measurement $x$ is available for feedback and transmitted over the digital channel while the control input is directly fed to the plant. Such an implementation scheme has many practical applications and has been considered in previous research works, see [13], [23] for instance. Before transmissions, the state value is subject to quantization by means of the proposed adaptive sliding mode quantizer. Consequently, the quantized state value $\bar{x}$ is only available for feedback. Then, the quantized state $\bar{x}$ is sent to the controller at discrete time instants $t_k$, $k \in \mathbb{N}$, see Figure 1.

![Figure 1. Setup description.](image)

We assume that the magnitude of the initial condition of the state $x_0$ is unknown. Then, the quantizer that we design will be capable of capturing the initial state in finite time, as will be explained later. The controller is designed using the emulation approach in the sense we first synthesize the feedback law to stabilize the closed-loop system in the absence of sampling and quantization. Then, we take those aspects into account and provide design conditions for the proposed quantizer and the event-triggering mechanism such that the closed-loop stability is maintained. We assume that the communication channel is delay-free and noiseless. Moreover, in this implementation scenario, we assume that the event-triggering mechanism has access to both the current (actual) value of the state $x(t)$ and the last transmitted (quantized) value $\bar{x}(t_k)$, $k \in \mathbb{N}$ as shown in Figure 1.

IV. The Proposed Exponential Sliding Mode Quantizer

The proposed exponential sliding mode quantizer (ESMQ) must be designed to convert the input signals, including the exponentially increasing input signals with arbitrarily initial values into binary signals at the encoder side. The transmitted binary signals are to be processed by the decoder to generate the quantized signal. The quantizer signals should be accurate enough to replace the original (true) signals, so-called “certainty equivalence”, see [24].

The ESMQ has the following dynamics

$$
s(t) = x(t) - \bar{x}(t), \tag{3a}
$$

$$
\bar{x}(t) = z(t) e^{\beta(t)}, \tag{3b}
$$

$$
\beta(t) = \lambda_q |z(t)| \int_{\tau_0}^{t} z(\tau) d\tau \quad \lambda_q > 0 \tag{3c}
$$

The ESMQ has the following dynamics
in which \( x(t) \in \mathbb{R} \) denotes the input signal and \( \dot{x}(t) \) denotes ESMQ output, and
\[
\ell \xi(t) = \text{sgn}(s(t)) - z(t) \tag{4}
\]
where \( \ell \) is the time constant (relativity small) for which \( \lim_{\xi \to 0} |z(t) - \text{sgn}(s(t))| \leq O(\ell) \).

The first equation \((3a)\) represents the quantization error and will be used later to define the sliding surface \((s = 0)\). Equations \((3b)-(3c)\) demonstrate how the quantized value \( \bar{x}(t) \) is computed based on an exponential function. The dynamics of \((4)\) is a linear first-order filter, which generates the average of the switching signal \( \text{sgn}(s(t)) \), see [25]. The exponential function in \((3b)\) enables the quantized state \( \bar{x}(t) \) to evolve rapidly in order to capture the state \( x(t) \) in a finite time. Note that the quantizer \((3b)-(4)\) only depends on the sign of the quantization error \( s(t) \). Hence, the quantizer needs to have only two levels, which allows transmitting the quantized information using only one bit. Then, the transmitted signal will be a sequence of binary data defined as \( \delta_t = \frac{1}{2} \left[ 1 + \text{sgn}(s(t)) \right] \in \{1, 0\} \), which provides a great advantage in practice for the encoder-decoder implementation. The realization of the quantizer.

We follow a similar quantization strategy as in [7] in the sense that we set the control input \( u = 0 \) in \((2)\) until the state is captured by the quantizer, we call this stage the transient stage. Then, we run the system in a closed-loop by setting \( u = -K\bar{x} \) and we call this stage the steady state stage. This procedure is formulated as follows
\[
u(t) = \begin{cases} 0 & t \leq t_c \\ -K\bar{x} & t > t_c, \end{cases} \tag{5}
\]
where \( t_c \in \mathbb{R}_{>0} \) denotes the capturing time of the state.

**Definition 1** (Practical sliding mode): The system \((3)\) is said to be in practical sliding mode if given any \( \varepsilon > 0 \) there exists a finite time \( t_f \) such that the system trajectory remains bounded in the \( \varepsilon \)-neighborhood of sliding manifold \( S \) for all \( t > t_f \), i.e., for all \( \|s(t)\| \leq \varepsilon \) for all \( t > t_f \). The system is said to be in ideal sliding mode if the trajectory remains on \( S \) (i.e., \( s(t) = \varepsilon = 0 \)).

**A. Stability analysis of the quantizer**

The following proposition establishes the sliding mode property of the ESMQ, where the quantization error converges to zero in a finite time.

**Proposition 1.** Consider system \((2)\) with unknown initial condition \( x_0 \). Take \( \lambda_q \) in \((3)\) such that \( \lambda_q > \text{Re}(\lambda_{\max}(A)) \) \( \) and let \( z(t_0) = \text{sgn}(x_0) \). Then, there exists a finite capturing time \( t_c \geq 0 \) such that
\[
\begin{align*}
1) & \quad |z| = 1 \quad \forall t \leq t_c \\
2) & \quad |x(t) - \bar{x}(t)| \leq \varepsilon \quad \forall t \geq t_c \\
3) & \quad \text{the capturing time } t_c \text{ is upper bounded by } t_c \leq \frac{\text{Re}(\lambda_q)}{\text{Re}(\lambda_{\max}(A)) - \lambda_q} + 5\ell.
\end{align*}
\]

**Proof:** Assume there exists a small parameter \( \varepsilon \in \mathbb{R}^+ \) such that the operating modes of the switching function \((3b)\) can be divided into two main modes: steady-state \( \Omega_s = \{ (x, \bar{x}) : |s(t)| < \varepsilon \Rightarrow |z(t)| < 1 \} \) and transient state \( \Omega_t = \{ (x, \bar{x}) : |s(t)| \geq \varepsilon \Rightarrow |z(t)| = 1 \} \). The stability and the dynamical behavior of ESMQ in these two modes will be investigated.

The proof of item \((1)\) follows from the fact that equation \((4)\) is a first-order filter, which produces the average value of the switching signal \( \text{sgn}(s) \), see [25]. Hence, when \( s(0) \in \Omega_t \), we have two situations. If \( s(0) \in \Omega_t \) with \( x_0 > \bar{x}(t) \), then the rate of change of the quantized signal \( \bar{x}(t) \) will increase and the output of \( \text{sgn}(s(t)) \) becomes a continuous of \(+1/\)'s until the state \( x(t) \) is successfully captured at \( t_c \). Consequently, in view of \((4)\) and since \( z(t_0) = \text{sgn}(x_0) \), we have \( |z(t)| = 1 \) after \( 5\ell \), the relativity small time constant of the low-pass filter (LPF) in \((4)\), until the state \( x(t) \) is successfully captured. Similarly, if \( s(0) \in \Omega_t \) with \( x_0 < \bar{x}(t) \), then the rate of change of \( \bar{x}(t) \) will decrease and the output of \( \text{sgn}(s(t)) \) becomes a continuous of \(-1/\)'s, which again leads to \( |z(t)| = 1 \) after \( 5\ell \) and until the state \( x(t) \) is successfully captured.

To prove the item \((2)\) of Proposition 1, it is sufficient to show that the sliding surface \( s(t) = x(t) - \bar{x}(t) \) achieves the fundamental properties of sliding mode dynamics, namely: finite-time reachability and sustaining property of the desired condition \( s(t) = 0 \). In other words, for any arbitrary initial conditions \( x_0 \in \mathbb{R} \) and \( \bar{x} = 0 \), the desired property \( s(t) = 0 \) is satisfied in a finite time and it remains sustained once the sliding surface is achieved.

To show the first property, let \( s(t) \in \Omega_s \), which implies that \(|z| = 1 \). Then, in view of \((3)\), we have \( \beta(\ell) = \lambda_q t \) and \( \bar{x} = z(t)e^{\lambda_q t} \). Consequently,
\[
\dot{\bar{x}} = \lambda_q e^{\lambda_q t} + \frac{z(t)\bar{x}}{|z(t)|} \Rightarrow z(t) = \lambda_q e^{\lambda_q t}.
\]

Recall that in view of \((5)\), we have \( u(t) = 0 \) for all \( s(t) \in \Omega_t \). Consequently, in view of \((2)\), it holds that \( \dot{x} = x_0e^{At} \). Now, consider \( V(t) = \frac{1}{2}s(t)^2 \), it follows that
\[
\dot{V}(t) = s(t)\dot{s}(t) \leq s(t)(x_0e^{At} - \text{sgn}(z(t))\lambda_q e^{\lambda_q t}).
\]

Assume \( x_0 \gg 0 \) and \( \bar{x} = 0 \), implies that \( s(t) > 0 \) and \( z(t) \rightarrow 1 \) after \( t > 5\ell \), which makes \( \beta(\ell) = \lambda_q t \) in \((3c)\). It is evident to see that \( s(t) = x_0e^{At} - \lambda_q e^{\lambda_q t} \). Since \( \lambda_q > \text{Re}(\lambda_{\max}(A)) \), the second term will exponentially evolve faster than the first term leading to \( s(t) < 0 \) in a finite time, i.e., \( \dot{V}(t) = s(t)\dot{s}(t) < 0 \).

Similarly, for \( x_0 \ll 0 \) and \( \bar{x} = 0 \), implying that \( s(t) < 0 \) and \( z(t) \rightarrow -1 \) after \( t > 5\ell \). Consequently, we have \( s(t) = x_0e^{At} + \lambda_q e^{\lambda_q t} > 0 \) and the second term will evolve faster than the first term leading to \( s(t) > 0 \), i.e., \( \dot{V}(t) = s(t)\dot{s}(t) < 0 \). This ensures the existence condition of sliding mode and \((3b)\) will ultimately force the operating region to shift from \( \Omega_t \) to \( \Omega_s \) within finite time.

Now it remains to show that this property is sustained. Once the trajectory enters \( \Omega_s \) region, \(|z(t)| < 1 \) and \((3b)\) will act to ensure the existence of sliding mode, i.e., \( \text{sgn}(s(t)) \neq \text{sgn}(\dot{s}(t)) \) in \( \Omega_s \). In other words, from the stability conditions in \( \Omega_s \), the system will be pushed to \( \Omega_s \), the region in either case and sliding mode existence will be preserved within finite time if \( \lambda_q > \text{Re}(\lambda_{\max}(A)) \) holds.
Finally, we prove the item (3) of Proposition 1. Given that \( \bar{x}_0 = 0 \Rightarrow \sqrt{2V(s(0))} = x_0 \), solving the differential equation when \(|x_0| > 0\), yields
\[
0 \leq \sqrt{2V(s(t))} \leq |x_0|e^{At} - e^{\lambda_s t} + 5\ell
\]
and since \( V(s(t)) \to 0 \) at finite time \( t_f \), then one can show that the estimated finite time is bounded for any \(|x_0| > 1\) by
\[
t_f \approx \frac{\ln(|x_0|)}{|Re(\lambda_{\max}(A))| - \lambda_q|} + 5\ell
\]
which completes the proof.

It shows that \( s(t) \) converges to zero exponentially. For known \( x_0 \) and by making \( \bar{x}_0 = x_0 \Rightarrow s(0) = 0 \), which gives \( t_f = 0 \), i.e., sliding motion starts in \( \Omega_s \) and stays there forever as long as \( Re(\lambda_{\max}(A)) < \lambda_q \) holds.

It is evident that if the proposed quantizer ESMQ initiates in transient state, i.e., cases (II), it will move to cases (I) in a finite time as per Proposition 1.

V. HYBRID DYNAMICAL MODEL

In this section, we develop the dynamic model of the overall system. Recall that, in view of (3) and (4), during the open-loop stage, i.e., when \( u = 0 \), the quantizer variable \( |z(t)| = 1 \) for all \( 0 \leq t \leq t_c \) until the state is successfully captured and during the closed-loop stage, i.e., when \( u = -K\bar{x} \) we have \(|z(t)| < 1\) for all \( t > t_c \). Consequently, the closed-loop system (2) becomes
\[
\dot{x}(t) = \begin{cases} 
Ax & \quad |z(t)| = 1 \\
Ax - BK\bar{x}(t) & \quad |z(t)| < 1,
\end{cases}
\]
\[ \tag{7} \]

We now take into account the sampling error. Since the quantized state \( \bar{x}(t) \) is sent over a digital network, the feedback signal can be only transmitted at discrete time instants \( t_k, k \in \mathbb{N} \). In other words, at any transmission instant \( t_k \) the value of \( \bar{x}(t_k) \) is sent to the controller to update the control input \( u = -K\bar{x}(t_k) \). Between two consecutive transmission instants \( t_{k+1} - t_k \) the value of \( \bar{x}(t_k) \) is kept constant by means of Zero-Order-Hold (ZOH) implementation. Define the sampling-induced error as follows
\[
\omega(t) = \bar{x}(t) - \bar{x}(t_k) \quad \forall t \in [t_k, t_{k+1}). \tag{8}
\]

Note that at each discrete instant \( t_k, k \in \mathbb{N} \), the current quantized value \( \bar{x}(t_k) \) is sent to the controller, and the sampling error is reset to zero.

In this work, we adopt the scenario of event-triggered control to generate the transmission instants \( t_k, k \in \mathbb{N} \) and we opt to design a state-dependent sampling rule such that transmissions only occur when the closed-loop stability is jeopardized. In this regard, we assume that the triggering mechanism has continuous access to both the actual state \( x(t) \) and the quantized value \( \bar{x}(t) \) at any time instant \( t \in \mathbb{R} \). Hence, in view of (3) and (8), we define the total error \( e(t) \) due to both quantization and sampling as
\[
e(t) = s(t) + \omega(t) = x(t) - \bar{x}(t_k) \quad \forall t \in [t_k, t_{k+1}). \tag{9}
\]

As a consequence, in view of (5), the closed-loop system (7) becomes
\[
\dot{x}(t) = \begin{cases} 
Ax & \quad |z(t)| = 1 \\
A_1x(t) + B_1e(t) & \quad |z(t)| < 1,
\end{cases}
\]
\[ \tag{10} \]
where \( A_1 := A - BK \) and \( B_1 := BK \). Moreover, since \( \bar{x}(t_k) \) is held constant due to ZOH, the inter-transmission dynamics of \( e(t) \) is
\[
\dot{e}(t) = \dot{x}(t) \quad \forall t \in [t_k, t_{k+1}). \tag{11}
\]

Then, according to (10), we have
\[
\dot{e}(t) = \begin{cases} 
Ax & \quad |z(t)| = 1 \\
A_1x(t) + B_1e(t) & \quad |z(t)| < 1,
\end{cases}
\]
\[ \tag{12} \]

Note that, in view of (3) and (8), at transmission instants \( t_k, k \in \mathbb{N} \) the most recent value of \( \bar{x}(t) \) is quantized and transmitted to the controller and consequently, the total error \( e(t) \) is rest to the quantization error. This only occurs during the closed-loop stage, i.e., after the initial state is successfully captured by the quantizer, while during the open-loop stage, the total error \( e(t) \) is not subject to reset. Hence, the discrete-time dynamics of \( e(t) \) for all \( t_k, k \in \mathbb{N}_{>0} \) is
\[
e(t_k^+|t_k^-) = \begin{cases} 
e(t_k) & \quad |z(t)| = 1 \\
s(t_k) & \quad |z(t)| < 1.
\end{cases}
\]
\[ \tag{13} \]

It is worth mentioning here that since the total error \( e(t) \) is not reset to zero at each transmission due to the effect of quantization, the analysis of the closed-loop system is far from trivial.

Since the closed-loop system exhibits both continuous-time and discrete-time dynamics due to the effect of sampling and quantization, it is intuitive to describe the behavior as a hybrid dynamical system. We need first to define an auxiliary variable \( \tau \in \mathbb{R}_{>0} \), which represents the time elapsed since the last transmission instant and has the following dynamics
\[
\dot{\tau} = 1 \quad t \in [t_k, t_{k+1}), \quad \tau((-k)^+) = 0 \quad \text{for } k \in \mathbb{N}.
\]
\[ \tag{14} \]
Let \( \xi := (x, e, \eta, \tau) \in \mathbb{X} \) with \( \mathbb{X} = \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \). Then, in view of (10), (12), (13), (25) and (22), the overall hybrid system is given by, we drop the arguments for ease of notation,
\[
\xi \in F(\xi) \quad \text{and} \quad \xi^+ \in G(\xi) \quad \xi \in D, \tag{15}
\]
where \( C := C_1 \cup C_2 \) is the flow set and \( D \) is the jump set with
\[
C_1 := \{ \xi \in \mathbb{X} : |z| = 1 \}\]
\[
C_2 := \{ \xi \in \mathbb{X} : |z| < 1 \quad \text{and} \quad \tau \leq T \}
\]
\[
D := \{ \xi \in \mathbb{X} : |z| < 1 \quad \text{and} \quad \eta = 0 \quad \text{and} \quad \tau > T \}
\]
Moreover, \( F(\xi) \) is the flow map which is given by
\[
F(\xi) := \begin{cases} 
\{F_1(\xi)\} & \quad \text{for } \xi \in C_1 \\
\{F_2(\xi)\} & \quad \text{for } \xi \in C_2
\end{cases}
\]
\[ \tag{17} \]
be carefully designed to maintain closed-loop stability. An efficient approach to synthesize such constant time $T$ is by adopting the time-triggering technique of [27], [28] which derive an upper bound $T > 0$ on the \textit{maximally allowable transmission interval} (MATI). Then, under time-triggering implementation, the sampling interval $T \in (0, T)$ maintains the closed-loop stability. However, in our case, we just implant this constant time $T$ in the event-triggering mechanism to exclude Zeno. By following the approach of [27], [28], the MATI bound is given by

$$T(\lambda, \bar{\lambda}, \lambda_q, \bar{L}) := \begin{cases} \frac{1}{L} \arctan \left( \frac{r(1-\bar{\lambda})}{2 \bar{\lambda} \bar{\lambda}(\bar{\lambda}+1+\bar{\lambda})} \right) & \lambda_q > \bar{L} \\ \frac{1}{L} \left( \frac{r-\lambda}{\bar{\lambda}(\bar{\lambda}+1+\bar{\lambda})} \right) & \lambda_q = \bar{L} \\ \frac{1}{L} \arctanh \left( \frac{r(1-\bar{\lambda})}{2 \bar{\lambda} \bar{\lambda}(\bar{\lambda}+1+\bar{\lambda})} \right) & \lambda_q < \bar{L} \end{cases}$$

(23)

where $r := \sqrt{(\frac{\lambda}{L})^2 - 1}$ with $\lambda_q$ comes from Assumption 1, $\bar{L} := L + \delta$ with $L := |B_2|$ and $\delta > 0$ is arbitrarily constant typically small, $\lambda \in (0, 1)$ and $\bar{\lambda} \in [\lambda, \lambda^{-1}]$. Note that the derived constant time $T$ corresponds to the time it takes for a function $\phi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ to decrease from $\phi(0) = \frac{1}{\bar{L}}$ to $\phi(T) = \lambda$ under the following dynamics

$$\frac{d\phi}{dT} = \begin{cases} -2\bar{L}\phi(\tau) - \lambda_q(\phi^2(\tau) + 1) & \tau \in [0, T] \\ 0 & \tau > T \end{cases}$$

(24)

Finally, we introduce the following dynamic function $\eta : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, which we use to generate the transmission instants. The variable $\eta$ has the following dynamics

$$\dot{\eta} = \Psi(x, e, \tau) \quad \tau \in (t_k, t_{k+1}), \quad \eta((t_k)^+) = \eta_0,$$

(25)

where

$$\Psi(x, e, \tau) := \varepsilon_\eta |x|^2 - (1 - \kappa(\tau))\lambda_q|e|^2 - \alpha \eta,$$

$$\eta_0 := \max \left\{ eR(x, e, \eta), \lambda_q|\eta|^2 - \frac{\alpha}{2} |s|^2 \right\}$$

(26)

with

$$R(x, e, \eta, \tau) := x^TPx + \lambda_q\phi(\tau)|e|^2 + \eta,$$

$$\kappa(\tau) := \begin{cases} \{1\}, & \text{for } \tau \in [0, T) \\ \{0, 1\}, & \text{for } \tau = T \\ \{0\}, & \text{for } \tau > T \end{cases}$$

(27)

where $\varepsilon_\eta \in (0, \varepsilon_x)$, $\alpha > 0$ is arbitrary small, $P, \lambda_q, \varepsilon_x$ come from Assumption 1, $\phi(\tau)$ as defined in (24) and $\lambda \in (0, 1)$ as given in (23). Note that $R$ is positive definite and the function $\eta$ is a decreasing function and hence it is updated to a positive value $\eta_0$ at each transmission instant $t_k$, $k \in \mathbb{N}$. Then, the event-triggering mechanism is defined as follows

$$t_{k+1} = \inf_{\tau \in \mathbb{R}_{\geq 0}} \{ t_k + \tau : \eta = 0 \text{ and } \tau \geq T \}.$$  

(28)

In other words, the next transmission instant is defined by the time it takes for $\eta(\tau)$ to decrease from $\eta(0) = \eta_0$ to $\eta(\tau) = 0$ provided that $\tau \geq T$. In this way, we ensure that $t_{k+1} - t_k \geq T$.

Remark 1. Note that the parameter $\lambda_q$ in Assumption 1 generates a tradeoff between the capturing time $t_c$ of the quantizer (3c) and the enforced minimum time $T$ of the ETC
(23), (28). When $\lambda_\eta$ is decreased, the capturing time $t_c$ will increase while the enforced lower bound $T$ on the inter-transmission times will increase and vice versa. Moreover, the decrease of $\lambda_\eta$ might increase the time it takes for triggering variable $\eta$ to reach 0, i.e., larger inter-transmission times. Hence, the parameter $\lambda_\eta$ allows the user to tune this tradeoff according to the implementation setup.

VII. Stability Result

We now ready to state the main result.

**Theorem 1.** Consider the hybrid system (15)-(19). Let the dynamic quantizer (3b)-(4) be designed such that $\lambda_\eta > |\lambda_{max}(A)| \lor |\lambda_{max}(A - BK)|$ and $l > 0$. Suppose that Assumption 1 hold and let the constant time $T > 0$ be designed as in (23). Then

(i) there exist a finite time $t_f > 0$ and a small value $\epsilon > 0$ such that $|x - \bar{x}| < \epsilon$ holds for $t \geq t_f$;

(ii) the set $\mathcal{A} := \{\xi \in \mathbb{X} : |x - \bar{x}| \leq \epsilon\}$ is globally pre-asymptotically stable.

The proof is provided in the appendix. It is important to mention that the design parameter $\epsilon$ provides a tradeoff between the closed-loop stability and the number of transmissions. When the value of $\epsilon$ is increased, the dynamic variable $\eta$ will be updated to a larger value of $\eta_0$ at transmissions, see (25)-(26), which could enlarge the inter-transmission intervals. However, increasing the value of $\epsilon$ might lead to a larger increase of the Lyapunov function at jumps due to quantization, as shown in the proof, which affects the asymptotic convergence of the state towards the set $\mathcal{A}$, and vice versa.

VIII. 1-bit Transmission Over the Network

In this section, we explain how the feedback signal is transmitted over the network at each transmission instant. More precisely, we present the encoding-decoding technique for the state value $x$ to be sent to the controller via the communication channel at any triggering instant $t_k, k \in \mathbb{N}$. Traditional techniques require 8-bits or 16-bits for example in order to encode the feedback information in packet-based transmission. In our case, using the proposed ESMQ, we only need to encode the sign of the quantization error, i.e., $\text{sgn}(s(t))$, and send it to the decoder. This consequently requires only one-bit signal transmission, which is one of the main advantages of the proposed quantizer.

The encoding-decoding mechanism in this case works as shown in Algorithm 1.

**Algorithm 1: Encoding-Decoding Mechanism**

forall $0 \leq t \leq t_c$ do

Encoder:
1: $\text{sgn}(s(t)) = \text{sgn}(x(t) - \bar{x}(t))$
2: Compute $\bar{x}(t)$ from (3)

Decoder:
1: $\text{sgn}(s(t)) = \text{sgn}(s(t_0))$
2: Compute $\bar{x}(t)$ from (3)

Controller:
1: $u(t) = 0$

forall $t > t_c$ do

Encoder:
1: $\text{sgn}(s(t)) = \text{sgn}(x(t) - \bar{x}(t))$
2: Compute $\bar{x}(t)$ from (3)
3: at $t_k, k \in \mathbb{N}$ submit $\text{sgn}(s(t_k))$ to decoder

Decoder:
1: at $t_k, k \in \mathbb{N}$ update $\text{sgn}(s(t_k))$
2: $\text{sgn}(s(t)) = \text{sgn}(s(t_k)), \forall t \in [t_k, t_{k+1})$
3: Compute $\bar{x}(t)$ from (3)

Controller:
1: $u(t) = -K\bar{x}(t), \forall t \in [t_k, t_{k+1})$

We then find that the LMI condition in Assumption 1 is feasible with $\varepsilon_x = 4.2419$, $\lambda_\eta = 5.6392$ and $P = 5.1674$. Then, we pick $\lambda = 0.7$, $\hat{\lambda} = 0.84$ and $\nu = 1$, which leads to $\bar{L} = L + \nu = 4$ with $L = |B_2| = 3$. Hence, in view of (23), we obtain $T = 0.0427$. Then, for the dynamics of the variable $\eta$ in (25) and (26), we take $\varepsilon_\eta = 5 \times 10^{-7}$, $\alpha = 0.01$ and $\epsilon = 5 \times 10^{-4}$. Hence, all the required parameters have been defined. We set the initial conditions as $x_0 = 20$ and we run the simulation for 10 seconds. We found that the minimum inter-transmission time is $\tau_{\min} = 0.0430$ and the average inter-transmission time is $\tau_{\avg} = 0.0602$. The fact that $\tau_{\min} > T$ implies that the event-triggering mechanism produces fewer transmissions compared to time-triggering with sampling period $T$. The observed capturing time is $t_c = 1.2238$ sec which is less than the estimated bound $t_f = 1.6457$, which supports the analysis of the ESMQ.

Figure 2. Actual and quantized state

Figure 3 shows that the norm of the quantizer state $|z| = 1$ during transient and then it decreases to zero as expected.

IX. Illustrative Example

Consider the following scalar system

$$\dot{x} = x + u, \quad u = -Kx.$$  \hspace{1cm} (29)

We first ignore the effect of the network and we design $K = 3$, which stabilizes the closed-loop system and places the eigenvalue of $A - BK$ at $\lambda_{\text{cl}} = -2$. Hence, for the exponential sliding mode quantizer (3)-(4), we take $\ell = 1$ in (4). We found that the LMI condition in Assumption 1 is feasible with $\varepsilon_x = 4.2419$, $\lambda_\eta = 5.6392$ and $P = 5.1674$. Then, we pick $\lambda = 0.7$, $\hat{\lambda} = 0.84$ and $\nu = 1$, which leads to $\bar{L} = L + \nu = 4$ with $L = |B_2| = 3$. Hence, in view of (23), we obtain $T = 0.0427$. Then, for the dynamics of the variable $\eta$ in (25) and (26), we take $\varepsilon_\eta = 5 \times 10^{-7}$, $\alpha = 0.01$ and $\epsilon = 5 \times 10^{-4}$. Hence, all the required parameters have been defined. We set the initial conditions as $x_0 = 20$ and we run the simulation for 10 seconds. We found that the minimum inter-transmission time is $\tau_{\min} = 0.0430$ and the average inter-transmission time is $\tau_{\avg} = 0.0602$. The fact that $\tau_{\min} > T$ implies that the event-triggering mechanism produces fewer transmissions compared to time-triggering with sampling period $T$. The observed capturing time is $t_c = 1.2238$ sec which is less than the estimated bound $t_f = 1.6457$, which supports the analysis of the ESMQ.
The behavior of the total error $e(t)$ is given in Figure 4, where we note that both the total error increase during the transient stage (open-loop) and then once the state is captured, the feedback loop is closed and the error $e$ is reset to $e_q$ at each transmission instant.

Figure 5 shows the inter-transmission times generated by the event-triggering mechanism. The horizontal red line in the figures corresponds to the value of the enforced minimum time $T = 0.0273$. The figure clearly shows that the inter-transmission times are larger than $T$, which justifies the advantage of the event-triggering technique over periodic sampling.

Figure 6 presents the tradeoff between the capturing time $t_c$ and the enforced lower bound $T$ on transmissions.

**X. CONCLUSIONS**

In this work, we have developed a novel quantized event-triggered controller for LTI systems. The proposed quantizer is a 2-level quantizer used in various telecommunication applications, including NCSs and cyber-physical systems. The dynamic quantizer has enabled exponential capture of the state in a finite time. Once the state is captured, the quantization error is guaranteed to remain bounded thanks to the sliding mode feature. The event-triggering mechanism that we consider is also dynamic and depends only on locally available information. For the sake of completeness, we have investigated the attraction region and stability analysis of the proposed quantizer where the system is modeled as a hybrid dynamical system. The approach ensures global asymptotic stability and prevents the occurrence of Zeno behavior.

**APPENDIX**

**Proof of Theorem 1.** We have three cases.

**Case I:** $\xi \in C_1$. In this stage, we run the system open-loop to allow for capturing the initial state in a fast manner. Hence, no transmission of the state and no control input is executed during this stage. We only need to ensure that the state will be captured within finite time $t_f > 0$, which has been established in Proposition 1.

**Case II:** $\xi \in C_2$. After the initial state is captured in $\Omega_s$ we have that $|z_i(t)| < 1$ and we run the system in closed-loop with $u(t) = -K \tilde{x}(t)$. To investigate the closed-loop stability, we define the following Lyapunov function as in (27)

$$R(\xi) = x^T P x + \lambda q \phi(\tau)|e|^2 + \eta.$$  

(30)

We have two cases based on whether $\tau > T$ or not.

**Case II-A:** $\tau \in [0, T]$  
In this case we have $\kappa(\tau) = 1$ and $\frac{d \phi}{dt} = -2 \bar{L} \phi(\tau) - \lambda q (\phi^2(\tau) + 1)$. In view of (17), it holds for all $\xi \in C_0$

$$\langle \nabla R, F(\xi) \rangle = \langle \nabla V(x), A_1 x + B_1 e \rangle + 2 \lambda q \phi(\tau)|e| \langle \frac{d \phi}{dt}, A_1 x + B_1 e \rangle + \lambda q \phi(\tau)|e|^2 + \eta,$$  

(31)

where $\langle \nabla R, F(\xi) \rangle$ denotes the Clarke generalized gradient of $R$ at $F(\xi)$. In view of (15), it holds that $\langle \frac{d \phi}{dt}, A_1 x + B_1 e \rangle \leq$
$|A_1 x| + L |e|$, where $L = |B_2|$. Recall that $\hat{L} = L + \delta$ as defined in (23). Consequently, in view of (21), we obtain

$$\langle \nabla R, F(x) \rangle = -\varepsilon_x |x|^2 - |A_1 x|^2 + \lambda_2^2 |e|^2 + 2 \lambda_\hat{R} x |e| + \lambda_\eta |e|^2 (2 \hat{L} \phi(\tau) - \lambda_\eta \hat{\phi}^2(\tau) + 1) + \Psi(x, e, \eta)$$

By using the fact that $2 \lambda_\eta \phi(\tau)|e| |A_1 x + B_1 e| \leq \lambda_2^2 \phi^2(\tau) |e|^2 + |A_1 x|^2$, we obtain (recall that $\omega(\tau) = 1$ when $\tau \in [0, T]$)

$$\langle \nabla R, F(x) \rangle = -\varepsilon_x |x|^2 - 2 \delta \lambda_\eta \phi(\tau)|e|^2 + \varepsilon_\eta |x|^2

$$= -\varepsilon_x |x|^2 - 2 \lambda_\eta \phi(\tau)|e|^2 - \alpha \eta

$$= -(e_x - \eta) |x|^2 - 2 \lambda_\eta \phi(\tau)|e|^2 - \alpha \eta \leq -\rho R(x),$$

where $\rho := \min \{ \varepsilon_x - \varepsilon_\eta, 2 \delta, \alpha \}$.

**Case II-B: $\tau > T$**

In this case we have $\kappa(\tau) = 0, \frac{d \phi}{d \tau} = 0$, $\phi(\hat{T}) = \hat{\lambda}$ and $\eta \geq 0$. As result, (33) becomes $R(\xi) = x^T P x + \lambda_\hat{R} \hat{\lambda} |e|^2 + \eta$. By following similar lines as in the previous case, we obtain (33). Hence, we conclude that (33) holds for all $\xi \in \mathcal{C}_2$.

**Case III: $\xi \in \mathcal{D}$**

We now study the dynamics of the Lyapunov function (30) at jumps. In this case we have $\phi(\tau) = \lambda$ and $\eta(\tau) = 0$. In view of (24), (25) and since the state $x$ does not change at jumps, we have

$$R(G(\xi)) - R(\xi) = (x^T P x + \lambda_\eta \phi(\tau)|e|^2 + \eta) - (x^T P x + \lambda_\eta \phi(\tau)|e|^2 + \eta)

= \frac{\lambda_\eta}{\lambda} |x|^2 + \eta - \lambda_\eta \hat{\lambda} |e|^2.$$

By substituting of $\eta_0$ from (26), we get

$$R(G(\xi)) - R(\xi) = \frac{\lambda_\eta}{\lambda} |x|^2 - \lambda_\eta \hat{\lambda} |e|^2 + \max \{ \epsilon R(\xi), \lambda_\eta \hat{\lambda} |e|^2 - \frac{\lambda_\eta}{\lambda} |x|^2 \}.$$ (34)

If $\lambda_\eta \hat{\lambda} |e|^2 - \frac{\lambda_\eta}{\lambda} |x|^2 > \epsilon R(\xi)$, then

$$R(G(\xi)) - R(\xi) = \frac{\lambda_\eta}{\lambda} |x|^2 - \lambda_\eta \hat{\lambda} |e|^2 + \lambda |e|^2 - \frac{\lambda_\eta}{\lambda} |x|^2 = 0.$$ (35)

If $\epsilon R(\xi) > \lambda_\eta \hat{\lambda} |e|^2 - \frac{\lambda_\eta}{\lambda} |x|^2$, then

$$R(G(\xi)) - R(\xi) = \frac{\lambda_\eta}{\lambda} |x|^2 - \lambda_\eta \hat{\lambda} |e|^2 + \epsilon R(\xi) \leq \epsilon R(\xi).$$ (36)

As a result, we conclude for all $\xi \in \mathcal{D}$

$$R(G(\xi)) \leq (1 + \epsilon) R(\xi).$$ (37)

In view of (33) and (38), we notice that the Lyapunov function $R(\xi)$ strictly decreasing during flow, however, it might increase at jumps. To conclude about the closed-loop stability, according to Proposition 3.29 in [22], we need to show that the possible increase at jumps is compensated by decreases during flow. We note that property (3.10) in Proposition 3.29 is satisfied with $\lambda_c = -\rho$ and $e^{\lambda_c t} = (1 + \epsilon)$. Let $\beta > 0$ and $(t, j) \in \Omega$. The last condition of Proposition 3.29 is equivalent to show that $\ln(1 + \epsilon) - \rho t \leq -\beta (t + j)$. Since the inter-transmission times are lower bounded by the constant time $T$, it holds that $j \leq \frac{T}{\beta}$, i.e., the number of inter-transmission instants is less than or equal periodic transmissions with the sampling period $T$. Consequently, to verify the previous condition, it suffices to show that $\ln(1 + \epsilon) - \rho T \leq -\beta \frac{T}{\beta}$, which is equivalent to $\ln(1 + \epsilon) + \beta \frac{T}{\beta} \leq (\rho - \beta) T$, i.e.,

$$\beta \left(1 + \frac{1}{T}\right) \leq \rho - \ln(1 + \epsilon) \frac{T}{T}.$$ (39)

If we take $\epsilon$ sufficiently small such that $\rho - \ln(1 + \epsilon) / T > 0$, then condition (39) can be verified with $\beta \in (0, \frac{\rho - \ln(1 + \epsilon) / T}{\frac{T}{T + 1}})$. As a result, all the required conditions of Proposition 3.29 are satisfied, which implies that the closed-loop system is globally pre-asymptotically stable.

**REFERENCES**


