On the correct mathematical description of polarization dependent impairments in Optical Communications

Carlos L. Janer

1Escuela Superior de Ingenieros

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Abstract

In this paper it is shown that the mathematical framework that correctly describes the combined effects of polarization mode dispersion and polarization dependent losses is the irreducible spinor representation of the extended Lorentz Group.

There are two kinds of states of polarizations that are relevant for the description of PMD-PDL effects. The optical process that allows to convert one kind into the other is identified as optical phase conjugation. The utility of the proposed mathematical framework is proven proposing a simple technique (based on optical phase conjugation) that cancels the PDL part of the impairments at the receiving end of the optical system.
Abstract—In this paper it is shown that the mathematical framework that correctly describes the combined effects of polarization mode dispersion and polarization dependent losses (combined PMD-PDL effects or impairments) in optical fibers is the irreducible spinor representation of the extended Lorentz Group. Combined PMD-PDL effects are shown to be formally identical to restricted Lorentz Transformations acting on spin $\frac{1}{2}$ zero mass particles. Since in this theory there are two different irreducible spinor representations of the restricted Lorentz group, there must also exist two kinds of states of polarization (SOPs) that are relevant for the description of PMD-PDL effects. The optical process that allows to convert one kind into the other is identified as optical phase conjugation. Optical phase conjugation is the optical process that implements the time inversion operator of the extended Lorentz Group. A practical and important example of utility of the proposed mathematical framework is presented. It is a perfectly feasible technique that cancels the PDL part of the impairments.

Index Terms—Optical fiber communication, polarization, optical fibers, mathematics, optical phase conjugation, modeling.

I. INTRODUCTION

Polarization plays a very important role in optical communications. Single mode fibers support two orthogonally polarized modes that are fully degenerate. In Polarization Multiplexed (PM) optical communication systems [1], these two modes are used to transmit information coming from two independent sources. The optical carrier is usually modulated both in amplitude and phase in order to increase the capacity of the communication channel. As light propagates along the fiber, random birefringence breaks the symmetry of the two degenerate modes, induces an energy exchange between these two modes and produces, in sufficiently long fibers, polarization mode dispersion (PMD). PMD introduces crosstalk and a differential delay between the two modes that carry the two multiplexed signals. In coherent detection schemes, PMD induced impairments can be compensated with digital processing (DSP) techniques [2].

Polarization dependent loss (PDL) typically arises in optical components such as optical isolators and couplers and introduces in optical systems (where attenuation is usually compensated with Erbium Doped Fiber Amplifiers, EDFAs) polarization dependent loss and gain. PDL cannot be compensated in coherent receivers by DSP processing and is the most important source of system degradation [3]. PDL induces polarization dependent power fluctuations resulting in unequal optical signal-to-noise ratio (OSNR) on each received polarization.

When the fiber PMD is intertwined with PDL elements, the resulting polarization effects are more complicated than PDL or PMD alone and the communication impairments can be, and usually are, more severe than either isolated effect. An accurate mathematical description of these effects is crucial in PM coherent optical networks.

The main purpose of this paper is to show that the mathematical formulation that correctly describes polarization issues in optical communication is the spinor irreducible representation theory of the extended (as opposed to restricted) Lorentz Group. Although the idea of the Lorentz Group describing polarization in Optics is well known, [4]-[6], it has never, to the best of my knowledge, been considered relevant to the field of optical communications. In this paper it will be analyzed why this happened, it will be proven that not taking into account these ideas is a serious mistake and it will be shown that polarization dependent (combined PMD-PDL) impairments can be easily reduced once the full implications of this mathematical formulation are understood. The main issue that has been ignored in this field is that there are two different kinds of states of polarization (SOPs) (that are mutually related by optical phase conjugation) that correspond to the two irreducible spinor representations of the extended Lorentz Group. These two kinds of SOPs carry all the relevant information about how combined PMD-PDL affects the polarization states of the propagated waves. Optical phase conjugation is identified as the optical process implementing the time inversion symmetry of the extended Lorentz Group and connecting the two different kinds of SOPs that have to be considered in optical communications. The PDL part of the impairments cancels when both kinds of SOPs are taken into account because, as it will be proven in this paper, the conjugated SOPs remain orthogonal at both ends of the fiber and, therefore, must be related by a pure rotation.

Carlos L. Janer is with the Escuela Superior de Ingenieros de Sevilla, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain (e-mail: janer@us.es).
II. PMD AND PDL IN OPTICAL COMMUNICATIONS

In this section, for the sake of clarity and completeness, a very sketchy introduction of polarization dependent penalties in Optical Communications will be made. This section introduces the mathematical notation that will be used in subsequent sections. For a more detailed introduction to polarization mode dispersion (PMD) and polarization dependent losses (PDL) effects in optical fibers the reader is referred to [7]-[8].

The reader should bear in mind that the state of polarization (SOP) of the transmitted wave depends on the complex valued information of the two channels that are being multiplexed. These two channels correspond to the two entries of the spinor (that is to say, the two components of the SOP vector). In PM coherent systems the transmitted information codifies the instantaneous SOP of the transmitted wave. This state of polarization will be in the rest of the paper indistinctly designated as “spinor” or SOP.

There always exists an orthogonal pair of polarization states at the output of a lossless concatenation of birefringent elements which are stationary to first order in frequency. These two states are called the Principal States of Polarization (PSPs). A differential delay exists between signals launched along one PSP and its orthogonal complement. The pointing direction of the PMD vector is aligned to the slow PSP, the PSP that imparts more delay than the other. The length of the PMD vector is the differential-group delay between the slow and fast PSP’s. The traceless Hermitian operator (or matrix) whose eigenvectors define the PSPs is [7]:

\[ H_r = jU_\omega U^\dagger = \frac{1}{2} (\vec{r}, \vec{\sigma}) \]  

(1)

where \( U \) is the 2x2 unitary matrix that defines the birefringence of the fiber. The equation that describes how the output SOPs change with frequency is [7]:

\[ |t\rangle_\omega = \exp[-\frac{1}{2} (\vec{r}, \vec{\sigma})\omega] |t\rangle_0 \]  

(2)

The eigenvalue-eigenvector equation of (1) states that the output polarization rotates about the principal-state of the fiber even if very slight changes in the optical frequency are considered. Only if the output state is aligned along one of the PSPs its state will not change with frequency. The rotation angle depends on the value of excursion frequency as: \( \vec{q} = \omega \vec{r} \) and the Unitary SU(2) rotation matrix can be obtained from the traceless 2x2 Hermitian matrix (1) by the exponential map [9]-[10] as expressed in (2). From this point of view, the PMD vector generates rotations of the output polarization states and, if we choose to use Jones vectors to describe polarization states, we can regard these as a proper (restricted) Lorentz transformations of spin \( \frac{1}{2} \) particles [9]-[12].

The reader should bear in mind that the previous observation is almost trivial. The polarization properties of an optical fiber are described by a very general group of matrices, those 2x2 complex valued matrices whose determinants are equal to 1 that are known in Group Theory [9]-[10] as the rank 2 complex special linear group of matrices, SL(2,C). Since there is a close relationship between the proper Lorentz group, \( SO^+(1,3) \), and SL(2,C) (SL(2,C) is the double cover of \( SO^+(1,3) \)) this seems to be just a mathematical curiosity devoid of any practical interest. Possibly this is the reason why, although this relationship is known in Optics [4]-[6], it has never attracted any attention in the field of optical communications. In the following sections it will be show that this was a mistake because combined PMD-PDL induced impairments can be easily alleviated (the PDL part can be cancelled) when this fact and its precise mathematical formulation are taken into account.

The vectors \( \vec{\Omega}_r \) and \( \vec{\Omega}_s \) are real Stokes vectors (whose components are the six free parameters that characterize the polarization properties of the fiber) that relate to the system birefringence and differential attenuation, but not in a straightforward way [8]. In this analysis it is assumed that there are no isotropic effects, that is to say, no common gain or loss or, at least, the optical amplifiers exactly compensate for the SOP independent loss.

The complex eigenvalues of this operator are of equal module and opposite signs as a direct consequence of it having a zero trace. If we denote the eigenvalues as: \( \lambda = \pm (\tau + j\eta) \), then the real part \( \tau \) is the familiar differential-group delay magnitude; the imaginary part \( \eta \) is the differential-attenuation slope (DAS), which is the frequency derivative of the differential attenuation along the two eigenvectors [8]. In the presence of PDL, the PSP’s are not orthogonal due to the complex value of the eigenvectors [8]. The equation that describes how the output SOPs change with frequency is:

\[ |t\rangle_\omega = \exp[-\frac{1}{2} (\vec{\Omega}_r, \vec{\sigma} + j\vec{\Omega}_s, \vec{\sigma})] |t\rangle_0 \]  

(4)

This equation describes the combined rotation and Lorentz boost (a proper Lorentz transformation) of a zero mass spin \( \frac{1}{2} \) particle [9]-[12]. So, the evolution of the SOPs at the fiber output is formally the Lorentz transformation of a two component spinor. Only if the output state is aligned along one of the PSPs its state will not change with frequency. The Lorentz angle depends on the value of excursion frequency as:

\[ \vec{q} = \frac{1}{2} \left[ (\omega \vec{\Omega}_r + j \omega \vec{\Omega}_s), \vec{\sigma} \right] \] 

and the Lorentz transformation can be obtained from the traceless 2x2 matrix (3) by the exponential map as shown in (4). From this point of view, the combined PDM and PDL vector generates a restricted Lorentz transformation of the output polarization states and, if we choose to use Jones vectors to describe polarization states, we can regard these as a proper (restricted) Lorentz transformations of spin \( \frac{1}{2} \) particles [9]-[12].
III. IRREDUCIBLE REPRESENTATIONS OF THE LORENTZ GROUP

Equations (3) and (4) show that the six free parameters that define the combined PMD and PDL properties of a fiber are formally equivalent to the six parameters that define a restricted Lorentz transformation of a spin \( \frac{1}{2} \) particle \([11]-[12]\). The Representation Theory of the Lorentz Group is of common knowledge in the field of Particle Physics \([9]-[12]\). This theory shows that the restricted Lorentz Group has two representations of dimension 2, IRs, which are not equivalent that are denoted as \( \mathbf{D}^{(1/2, 0)} \) and \( \mathbf{D}^{(0,1/2)} \). This can be traced back to the fact that a matrix belonging to the, rank two complex group of matrices with determinant equal to 1, SL(2,C), and its complex conjugate, A and A* are not equivalent in the full group (they are only in the special unitary, SU(2), subgroup) because one cannot find a similarity transformation relating them \([9]-[12]\). These matrices constitute two nonequivalent irreducible representations of the restricted Lorentz Group and from these two any other higher order representation (either spinor or tensor) can be built. The spinor IR’s are more fundamental than the tensor IR’s. From the mathematical point of view, it makes little sense to use Stokes space, \( \mathbf{D}^{(1/2, 1/2)} = \mathbf{D}^{(1/2, 0)} \otimes \mathbf{D}^{(0,1/2)} \) in representation theory, to describe combined PMD and PDL issues in optical communications without any explicit mention to the two IRs, \( \mathbf{D}^{(1/2, 0)} \) and \( \mathbf{D}^{(0,1/2)} \), upon which it is built. We have then two nonequivalent bases, and, in general, two kind of spinors (spin \( \frac{1}{2} \) particles) denoted by \( \xi \) and \( \xi^* \) that transform according to following transformation matrices:

\[
A = \exp \left[ -\frac{i}{2} \left( \Omega, \vec{\sigma} + i \Omega, \vec{\sigma} \right) \right]
\]

and the representation provided by \( A^* \) is equivalent to:

\[
B = \exp \left[ -\frac{i}{2} \left( \Omega, \vec{\sigma} - j \Omega, \vec{\sigma} \right) \right]
\]

The irreducible representations of the extended Lorentz Group can be obtained from the restricted Lorentz Group by inclusion of the discrete symmetries \( I_s \) (parity inversion), \( I_t \) (temporal inversion) and \( I_{st} \) (inversion symmetry) \([12]\). These discrete symmetries are very important because they transform one kind of spinor into the other and the optical processes implementing them should be identified. This is a novel contribution of this paper since, to the best of my knowledge, the discrete symmetries of the extended Lorentz Group had never been considered in Optics before and neither had the optical processes implementing them been identified. The fact that these discrete symmetries correspond to optical processes make this mathematical formulation useful and the fact that they had not been identified before as such helps to explain why this mathematical formulation, although known in Optics, had not attracted any interest in the field of optical communications.

A. Time Inversion

The time inversion symmetry affects the spinors in the following way \([12]\): \( I_T: \xi \rightarrow \eta^i \) and \( I_T: \eta^i \rightarrow -\xi \). It transforms the regular contravariant spinor \( \xi \) into the conjugate covariant spinor \( \eta^i \). If a contravariant spinor \( \xi \) transforms as \( A \xi \) then its covariant spinor \( \eta \) transforms as \( \eta (A^{-1})^\dagger \) (this follows the standard notation in which covariant spinors are represented by row vectors acting on the left of matrices) so time inversion conjugates the spinor and changes “variance” (from either contravariant to covariant or from covariant to contravariant). In optical links that use EDFAs to compensate for the fiber attenuation, covariant spinors would not be allowed to propagate along the whole fiber due to the presence of the isolators placed at both ends of the optical amplifiers (these isolators are placed to prevent these amplifiers from lasing). However, since the time inversion symmetry relates covariant to contravariant spinors and vice-versa, and time inversion is a symmetry that can be optically implemented, this difficulty will be ignored in this paper.

The covariant spinor \( \eta \) associated to \( \xi \) is given by the expression \([12]\): \( \eta^i = C \xi \) (or \( \eta = \xi^* C \)) where \( C \) is the antisymmetric matrix:

\[
C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

The optical process that implements this discrete symmetry is optical phase conjugation \([13]\). Optical phase conjugation would reverse, in counter-propagation, the polarization distortion introduced by the combined effect of PMD and PDL if it were not for the presence of isolators. An optical conjugator, therefore, implements the polarization transformation:

\[
\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \rightarrow \eta^t = \begin{pmatrix} \xi_2 \\ -\xi_1 \end{pmatrix}
\]

Please, notice that the original and transformed SOPs are mutually orthogonal, that is, \( \langle \eta | \xi \rangle = 0 \) all along the fiber. In the representation theory of the Lorentz Group it is well known (Ref. \([10]\)) that \( \langle \eta^* | \xi^* \rangle + \langle \eta | \xi \rangle \) and \( \langle \eta^* | \xi^* \rangle - \langle \eta | \xi \rangle \) behave as scalars (or pseudo-scalar) invariants of the group, depending on the considered discrete symmetry.

B. Frame Inversion

The frame inversion transforms spinors in the following way \([12]\): \( I_{st}: \xi \rightarrow j\xi^i \) and \( I_{st}: \eta^i \rightarrow j^i \eta \) would not change the SOP and, therefore, would be very difficult to detect. Although it is tempting to speculate and try to assign physical relevance to the “j” factor, it would probably be unjustified. The frame inversion operator will be considered as the identity operator in this paper.

C. Parity Inversion

The parity inversion transforms the spinors as follows: \( I_S: \xi \rightarrow j^i \eta \) and \( I_S: \eta^i \rightarrow j \xi \) \([12]\). Since the frame inversion operator is the identity, the optical process that implements parity inversion is optical phase conjugation.
IV. APPLICATION: PDL CANCELLATION

The purpose of this paper is to show that the natural mathematical framework that describes polarization issues in optical communications is the irreducible spinor representation of the Lorentz Group. Since it is a mathematically complex matter, the only convincing way to prove its need is to either solve some complicated problem or propose something new and very useful based on it. It is now straightforward to do so.

At the receiver end of the optical link the received spinor, $\xi_R$, is used to estimate what the transmitted spinor was, $\xi_T$. The combined PMD and PDL effects and noise make this estimation process very difficult. The previous sections make it clear that we can perform a time inversion on the received spinor by means of optical phase conjugation to obtain $\eta_R$. If this spinor could counter-propagate along the fiber, it would end at the fiber input as $\xi_T$. However, this spinor can also be considered as the SOP of a propagated wave. Our mathematical framework allows us to identify it with the unique input SOP perpendicular to $\xi_T$. Please, notice that perpendicular spinors belong to conjugated representations.

Both spinors, $\xi_R$ and $\eta_R$, can and should be used to estimate the transmitted information and that is exactly what this paper proposes. In this restricted sense, it is as if we had received two different spinors that can be used to estimate the information of a single transmitted symbol, significantly improving the estimation process. Notice also that $\xi$ and $\eta$ remain orthogonal at both ends of the fiber optic, so that the joint motion of both spinor is considered, the PDL penalty part disappears. This stems from the fact that the Lorentz boosts have opposite signs in the two conjugated representations, as shown in Eq. (5) and (6).

$\eta_R$ is the spinor that would be measured at the receiver end, if the orthogonal to $\xi_T$ spinor had been transmitted instead of $\xi_T$.

Two mutually orthogonal SOPs (the transmitted SOP and its conjugated SOP) at the fiber input can only be related to two mutually orthogonal SOPs at the fiber output (the received SOP and its conjugated SOP) by a rotation which proves that the PDL part of the impairments cancels when the correct mathematical framework is used to describe the problem.

Notice that it has been assumed that the conjugated SOP can be exactly obtained by an ideal phase conjugate mirror. No attempt has been made to describe the consequences of a less than perfect reconstruction of the conjugated SOP because of the theoretical nature of this paper.

V. CONCLUSIONS

The idea of interpreting polarization changes as restricted Lorentz Transformations was well known in Optics but had been completely ignored in the field of optical communications. However, this idea provides the key to a correct mathematical description of polarization issues in this field. The polarization transformations that take place in optical communications can be regarded as restricted Lorentz transformations. The restricted Lorentz Group can be extended to the complete Lorentz by including the three discrete symmetries when optical phase conjugation is identified as the optical process that allows to have access to the conjugate representation of polarization. These two representations provide all the relevant information about combined PMD-PDL impairments. When both spinors are taken into account, PDL impairments can be cancelled. This mathematical framework proves to be essential to understand the polarization transformation that take place in fiber optic communication systems and to realize what can be and cannot be done in these systems. The PDL can be cancelled even though each input spinor is related to each output spinor by (5) and (6) that are not rotational relationships. The joint motion of both input spinors is a rotation.

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