Directive Properties of Radiating Source Systems in Massive Electromagnetism

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Abstract

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Index Terms—Massive electromagnetism, Proca theory, massive photons, antennas, perfectly isotropic radiation, directivity, Proca metamaterial.

I. INTRODUCTION

Massive electromagnetism studies the physics and applications of electromagnetic processes where the carrier of the electromagnetic force, the photon, is massive (has small but nonzero mass), whether manifested classically [1] or quantum mechanically [2]. As such, this is a very broad field with numerous applications in particle physics, condensed-matter physics, plasma physics, astrophysics, cosmology, and general relativity, and others [3]–[6]. The most successful and widely studied theory in massive electromagnetism is Proca theory in which the field components $A_\nu$ satisfy a relativistic field equation (Proca equation) of the form $(\Box + m^2)A_\nu = j_\nu$, where $m$ is a mass parameter and $j_\nu$ is the field source [7]. A particularly interesting scenario for applications is the existence of special nonlocal material models that provide exact realization of Proca theory of massive photons. In such theory, the electromagnetic fields possess a non-Abelian invariant structure where photons have a non-vanishing mass [1], [2]. However, it was recently pointed out that a special type of nonlocal metamaterials (MTMs), hereafter Proca MTMs, can lead to exact equivalence between Maxwell’s and Proca’s field theories, where the electromagnetic fields exactly solves Proca’s equation (massive electromagnetism) in vacuum [8]. In other words, massive Proca fields in vacuum are formally equivalent to Maxwell’s theory in a nonlocal (spatially dispersive) medium, the Proca MTM. In this case, one can study Proca theory using conventional electromagnetism in continuous media, where the special medium is often called Proca metamaterial (MTM). The dispersion relations of photons in Proca MTMs is the same as the relativistic massive particle in Proca theory. However, Proca MTMs’ individual field components as functions of space and time all follow the exact same laws of Proca theory of massive photons in vacuum. Therefore, the physics of Proca MTMs provides a rich theoretical and experimental potential for exploring consequences and applications of massive electromagnetism.

The present paper’s main focus is on investigating the behaviour of classical external source systems embedded into Proca MTMs. These sources can be thought of as classical (non-quantum) “Proca antennas,” i.e., externally-controlled source systems radiating massive photons instead of the traditional massless photons of classical electromagnetism.2 As will be demonstrated below, a Proca antenna can lead to exact perfectly isotropic radiator (PIR), a situation that is impossible in antennas radiating in free space. The PIR condition applies to the total radiated energy/power, and hence lends itself naturally to generalization to the quantum Proca antenna case (not treated in the present paper). From the physical viewpoint, a PIR condition is interesting because it suggests the existence of what might be called “exact scalar electromagnetism,” i.e., an area of electromagnetism where the radiation field properties behave like scalar sources. Obviously, this is not possible in vacuum since there the field possesses a polarization structure so any scalar behaviour in free space is an approximation, e.g., as in modeling uniformly distributed random sources in propagation and scattering in fluctuating media.

Fundamentally speaking, all radiating systems operating in free space or in a temporally-dispersive media (both must be spatially 3-dimensional), whether receiver or transmitters, can never realize perfectly isotropic radiation patterns. The reason is that in such media the far-field is known to be transverse, i.e., the asymptotic limit of the longitudinal component is exactly zero [9]–[11], while the hairy ball theorem asserts that a continuous vector field on the 2-sphere cannot have a non-vanishing 3-dimensional pattern [12].3 However, such no-go theorem does not preclude attempts to construct practical approximations of isotropic radiators. Indeed, due to the recent rise of attractive and promising technologies such as Radio-frequency identification (RFID), Internet of Things (IoT) and Ubiquitous Computing [13], coupled with the already old trend in electromagnetic applications to push toward antenna miniaturization [14], [15], the general inclination of applied research now favors implanting or embedding small antennas into many (if not all) key objects found in our surrounding

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2In classical massive electromagnetism in material domains, a “massive photon” is recognized by the existence of a dispersion relation of an electromagnetic plane-wave mode that is identical in form to the quantized relativistic massive particle-wave [1], [7].

3Cf. Sec. II for complete details.
environments. However, since everyday objects are randomly oriented in 3-dimensional space, while the transmit/receive properties of antennas can be very sensitive to polarization data [16], it is essential that these antennas possess a perfectly isotropic radiation (PIR) pattern in order to ensure reliability and consistency in the overall system performance. Indeed, signals received or transmitted along nulls in the radiation pattern readily lead to jammed reception or fading in the corresponding wireless communication link [17].

In this article, we tackle the fundamental physics behind the possibility of realizing PIR conditions in arbitrary radiating structures in classical electromagnetism. It is shown that massive electromagnetism and nonlocal radiation theory jointly enact a major shift in the scope and possibilities of the theory of radiating structures compared with the classical antenna theory in vacuum. Furthermore, we examine the directive features of non-isotropic radiators (Proca sources) in massive electromagnetism by focusing on arrays of point sources with variable complex excitations, which breaks the radiation pattern symmetry and leads to directive transfer of energy into specific directions.

The paper is structured as follows. In Sec. II, we start by rigorously formulating the problem of the existence of PIR pattern for local antennas in terms of a no-go theorem, the hairy ball theorem. (Nonlocal antenna systems are defined as sources radiating into free space or temporally dispersive media [18], [19].) The main conditions blocking the existence of PIR patterns in this class of radiators are formally identified and then exploited in Sec. III by introducing the concept of the Proca antenna, a special example of a nonlocal antenna discovered recently [8]. It is carefully shown that for this class of problems a theorem asserting the existence of PIR patterns can be proved by using the concept of complementary radiation patterns introduced in Sec. II. The physics behind point-like PIR systems is then introduced formally in Sec. IV-A under the rubric of the Proca source system. Next, a brief proposal for a possible future realization of PIR antennas through an approximation of the ideal electrically-small (point source) Proca receiver is described in Sec. IV-B. The problem of constructing the general directive profile of more general (large) source systems is then considered in details in Sec. V. For simplicity, we focus there on discrete arrays where the interelement spacing between the radiating elements can be controlled. In Sec. V-A, the fundamental theorem on the existence of a PIR Proca source developed in Sec. III is exploited to derive an exact radiation formula for the Proca array directivity profile. The theoretical formulas for the special but important case of a linear uniform array are derived in Sec. V-B, with several examples illustrating the basic behaviour of Proca radiators, together with its difference from free-space arrays, given in Sec. V-C. Finally, we end up with conclusions.

II. ON THE NONEXISTENCE OF PERFECTLY ISOTROPIC ANTENNAS IN LOCAL ELECTROMAGNETIC DOMAINS

In what follows, massive electromagnetism will be developed mainly from the perspective of the Fourier spacetime domain where

\[
F(k, \omega) := \int_{\mathbb{R}^4} d^3x \, dt \, F(x, t) e^{i k \cdot x - i \omega t},
\]

is the Fourier transform of the spatiotemporal field \( F : \mathbb{R}^4 \to \mathbb{R}^3 \). Here, the position vector \( x \in \mathbb{R}^3 \) is transformed to the wavevector (spatial frequency) \( k \in \mathbb{R}^3 \), while the time variable \( t \in \mathbb{R} \) is mapped into the dual spectral variable, the radian frequency \( \omega \in \mathbb{R} \). To simplify the mathematical treatment, we work in the time-harmonic frequency domain regime, i.e., it is assumed that all electromagnetic fields are excited by a sinusoidal source \( \exp(-i \omega t) \) with frequency \( \omega \). The fields below are then complex harmonic vector fields, but with the time-dependence suppressed to simplify the notation, and are denoted by \( \mathbf{E}(x), \mathbf{H}(x) \in \mathbb{R}^3 \), for the electric and magnetic fields, respectively. That is, while we suppress the explicit dependence on \( \omega \) whenever possible, all fields remain frequency dependent.

The far-zone fields, or far-fields for short, are defined by the asymptotic limits [11], [23],

\[
\mathbf{E}_{\text{far}}(\Omega) := ||x|| \lim_{x \to \infty} \mathbf{E}(x), \quad \mathbf{H}_{\text{far}}(\Omega) := ||x|| \lim_{x \to \infty} \mathbf{H}(x).
\]

All other fields are often treated as near-fields [24], [25]. Mathematically speaking, the fundamental problem of how to describe a perfectly isotropic radiator (PIR condition) is then equivalent to asking whether the far-zone fields have no variations in the angular directions \( \Omega := (\theta, \varphi) \), where \( \theta \) and \( \varphi \) are the spherical coordinates angles. More precisely, we define the perfectly isotropic radiator (PIR) system as follows:

**Definition 1. (Perfectly isotropic radiators)** For a given set of far fields \( \mathbf{E}_{\text{far}}(\Omega) \) and \( \mathbf{H}_{\text{far}}(\Omega) \), we say that the radiating system possesses a perfectly isotropic radiation (PIR) pattern when the two conditions

\[
\nabla_{\Omega} \mathbf{E}_{\text{far}}(\Omega; \omega) = \nabla_{\Omega} \mathbf{H}_{\text{far}}(\Omega; \omega) = 0
\]

are satisfied simultaneously for every \( \omega > 0 \) at which a nonzero total electromagnetic power is radiated into the far zone.\(^4\) If only the electric (magnetic) field condition is satisfied, we say we have an electric (magnetic) type perfectly isotropic radiating system.

Next, we present the main result from the theory of continuous manifolds prohibiting the existence of PIR systems in local domains. In both algebraic and differential topology, the following theorem is fundamental [12]:

**Theorem 1. (The hairy ball theorem)** Consider a continuous vector field

\[
g : \mathbb{R}^n \to \mathbb{R}^{n+1},
\]

where \( n \) is a positive integer. Let \( N(x) \in \mathbb{R}^{n+1} \) be the vector normal to the \( n \)-sphere \( S_n \), at the point \( x \in S_n \). Assume that the following two conditions are satisfied:

1) \( n \) is even.
2) \( g(x) \cdot N(x) = 0 \) for all \( x \in S_n \), where \( a \cdot b \) is the Euclidean inner product for vectors in the ambient space \( \mathbb{R}^{n+1} \).

\(^4\)This restriction is important since not every nonzero field system necessarily implies effective radiation into the far zone.
Then \( g(x) \) must vanish at least at one point \( x \in \mathcal{S}_n \).

For the problem of electromagnetic radiation, the configuration space is the position space \( \mathbf{x} \in \mathbb{R}^3 \) where the frequency \( \omega \) is fixed. The relevant far-field zone is the 2-sphere \( \mathcal{S}_2 \) parameterized by \( \Omega \) (a 2-dimensional manifold [12].) Strictly speaking, a sphere cannot be fully parameterized by the spherical angles \( \theta \) and \( \varphi \) but one must use a system of overlapping coordinates [12]. However, this subtlety has no bearing on what follows so whenever needed a coordination of the sphere by \( \Omega \) is to be understood as a system of complete differential atlas. Consequently, \( n \) in Theorem 1 is 2, i.e., it is even (first condition is met).

Regarding the second condition in Theorem 1, the nature of the medium surrounding the antenna plays a fundamental role. First we need to review some basic definitions. Let the source region be defined as

\[
D_s := \text{supp}\{\mathbf{J} (\mathbf{x})\},
\]

where \( \text{supp}\{\mathbf{f}(\mathbf{x})\} \) is the support of the spatial function \( \mathbf{f}(\mathbf{x}) \).

Next, recall that a temporally dispersive medium is defined as a material domain whose frequency-domain constitutive parameters (scalar or tensors) depend only on \( \omega \) [26], [27]. If the medium’s response parameters also depend on \( \mathbf{k} \), the wavevector, then the medium becomes nonlocal (or spatially dispersive if the material is also homogeneous) [27]–[30]. Spatial dispersion often leads to unusual physical behaviour like negative group velocity [28], [31], nonreciprocity [32], [33], topological materials [34], and energy localization [35].

Now, if the antenna’s exterior region \( D_{\text{ext}} := \mathbb{R}^3/D_s \) is filled with either free space or a temporally dispersive medium, then it can be shown based on general principles that the far-fields are tangential to the infinite sphere \( \mathcal{S}_2 \), i.e., we have in this case:

\[
\mathbf{E}^{\text{far}} (\Omega) \cdot \hat{r} = \mathbf{H}^{\text{far}} (\Omega) \cdot \hat{r} = 0.
\]

Here, \( \hat{r} \) is the spherical coordinates radial vector plays the role of \( \mathcal{N}(\mathbf{r}) \) in Theorem 1. The relation (6) can be easily proved from Wilcox expansion, e.g., see [11], [23], [24]. Therefore, the two conditions of Theorem 1 are met and we may conclude the following:

**Theorem 2. (Nonexistence of perfectly isotropic local antennas)** Consider an arbitrary local antenna, i.e., a current source distribution \( \mathbf{J}(\mathbf{x}) \) supported in \( D_s \subset \mathbb{R}^3 \) such that it radiates continuous fields into an exterior domain \( D_{\text{ext}} \) where the latter is either free space or a material domain filled with a local medium (i.e., a temporally dispersive material).

Then there exists no current source \( \mathbf{J}(\mathbf{x}) \) such that the antenna can give rise to a perfectly isotropic radiator in the sense of Definition 1.

Then Theorem 2 provides a rigorous modern mathematical statement of the often encountered engineering difficulty of constructing high-quality fully isotropic radiators in free space [14]. It also extends this fundamental impossibility to generic (isotropic or anisotropic) media with arbitrary dependence of their material tensors on \( \omega \). Indeed, in all such radiating/receiving systems the far field cannot possess a radial component and hence the hairy ball theorem requires the existence of at least one null in the radiation pattern. Since the far-field pattern is continuous, it follows that the radiation characteristics of the system, whether in the transmission mode or (using reciprocity) receive mode, must experience rapid degradation in at least a small angular sector around the null’s location.

**III. PROCA ANTENNAS, NONLOCALITY, AND THE EXISTENCE OF PERFECTLY ISOTROPIC RADIATORS**

A way to evade the No-Go Theorem 2 is provided by the fact that the classical asymptotic Wilcox expansion [23] is no longer valid in generic nonlocal or spatially dispersive media [20], [26], [28]. New or “additional waves” must be incorporated into the solution of Maxwell’s equations in such spatially-dispersive domains to maintain consistency with the nonlocal nature of the propagation medium [36], [37]. To the best of the author’s knowledge, a complete mathematical treatment of the general structure of radiation fields in generic nonlocal media currently does not exist. However, this problem was reconsidered recently in homogeneous nonlocal media, i.e., the important case of spatial dispersion. The general structure of the radiation field was derived in [18] using a momentum (Fourier) space method [19]. In such an approach, all fields are expanded in temporal and spatial Fourier frequencies \( \omega \) and \( \mathbf{k} \), respectively. When applications and examples of various nonlocal antennas were developed using this general theory, it was initially noted that if optimization methods are applied to nonlocal antenna systems, then it might lead to perfectly isotropic radiators [38].

The logic behind this conjecture is briefly the following. In a generic nonlocal metamaterial (MTM), both transverse (T) and longitudinal (L) waves can be excited [27]. However, similar to what happens in plasma waves [26], longitudinal modes in a generic nonlocal domain may carry energy to the far zone [18], [20]. Now, following the construction of the radiation pattern using the energy method in [18], [19], one may look for special combinations of L and T modes such that the total radiation pattern is perfectly symmetric. The hope is that an optimization of a series of properly excited L and T modes may lead to the formation of exact or nearly-exact complementary radiation patterns. Motivated by the above train of thought, we propose the following formal definition:

**Definition 2. (Complementary transverse-longitudinal field system)** Let \( \mathbf{E}^T (\Omega) \) and \( \mathbf{E}^L (\Omega) \) denote the total radiation patterns due to all available transverse (T) and longitudinal (L) modes, respectively. We say that the pair \( (\mathbf{E}^T (\Omega), \mathbf{E}^L (\Omega)) \) constitutes a T-L complementary field system if the condition

\[
\nabla_{\Omega} \left[ \mathbf{E}^T (\Omega; \omega) + \mathbf{E}^L (\Omega; \omega) \right] = 0
\]

is satisfied at all angles \( \Omega \in 4\pi \) and every frequency \( \omega > 0 \). In other words, while each of \( \mathbf{E}^T (\Omega) \) and \( \mathbf{E}^L (\Omega) \) can be a strong function of \( \Omega \), the two fields’ respective angular dependencies exactly cancel when added to each other.
Note that the mere definition of the commentary field system in Definition 2 does not guarantee the existence of interesting physical examples. After all, the fields radiated by a physically realizable antenna system must satisfy Maxwell’s equations and the material constitutive relations. However, it was discovered recently that a complementary systems may exist in Proca theory, i.e., the theory of massive electromagnetism [8]. Most noteworthy is the fact that the total radiated field energy emitted by an infinitesimal (point) source are perfectly isotropic radiators (PIRs) regardless to the orientation of the infinitesimal dipole:

**Theorem 3. (Existence of perfectly isotropic nonlocal antennas)** Let the vector function

\[ \mathbf{J}(x, t) = \mathbf{\alpha}_s J_0 \delta(x-x_s) \exp(-i \omega_s t) \]  

(8)

be the current of a time harmonic infinitesimal dipole oriented along \( \mathbf{\alpha}_s \) and located at \( x_s \), while resonating at frequency \( \omega_s \). Let \( P(\omega_s, \mathbf{\alpha}_s; \Omega_s) \) be the corresponding momentum space radiation power density.\(^6\) Then if the source is embedded into a Proca MTM, it follows that

\[ \nabla_{\mathbf{\alpha}_s} \nabla_\Omega P(\omega, \mathbf{\alpha}_s; \Omega_s) = 0 \]  

(9)

for all frequencies \( \omega_s > 0 \) and dipole orientations \( \Omega, \mathbf{\alpha}_s \in 4\pi \).

**Proof sketch.** The proof is a direct consequence of applying Definition 2 and Theorem 3, and Appendix A.4 in [8]. The result is obtained after utilizing the radiation formulas (A.16) and (A.17) in [8] to compute the Proca source radiation fields. Note that in (9) we treat \( \mathbf{\alpha}_s \) and \( \Omega \) as independent direction variables.

Note that the expression (9) indicates that the perfectly isotropic nature of the Proca source is not an accident of orienting the antenna along a special direction but is valid for any dipole orientation (as it should be in a truly perfectly symmetric radiation configuration).

**IV. Perfectly Isotropic Radiators Through Proca Source Systems**

**A. Proca Source Systems**

Let us first review the basic theory of Proca MTMs (for extensive discussion, see [8]). The generalized (effective) relative dielectric tensor of the Proca MTM in such formulation is expressed by the following surpriselnigly simple analytical dyadic form [8]:

\[ \varepsilon_{\text{proca}}(\mathbf{k}, \omega) = \left( 1 - \frac{m^2 c^2}{\omega^2} \right) \mathbf{I} - \frac{k^2 c^2}{\omega^2} \mathbf{k} \mathbf{k}. \]  

(10)

Here, \( \mathbf{I} \) is the unit dyad, \( c \) speed of light in vacuum, \( k := |\mathbf{k}| \) is the wavenumber, and \( k := k/k \) is the unit vector pointing along the direction of propagation of a plane-wave mode of the form \( \exp\{-i(\omega t - \mathbf{k} \cdot \mathbf{x})\} \). The normalized mass \( m \) is the fundamental parameter of Proca theory [39], which is given in terms of the photon mass \( m_{\text{ph}} \) through the relation

\[ m = \frac{m_{\text{ph}} c}{\hbar}, \]  

(11)

where \( \hbar \) is the reduced Plank constant [1], [2], [40]. The quantity \( m \) possesses inverse length units (in SI we use \( m^{-1} \)), and hence \( 1/m \) is a characteristic length scale. From (10), it is clear that because of the explicit dependence on \( \mathbf{k} \), the Proca MTM is nonlocal. In fact, since the medium possess continuous spatiotemporal translation symmetry (homogeneous and time-invariant), the domain is spatially dispersive [28], [30]. It can be shown that the normalized mass \( m \) is related to the nonlocality radius of the Proca of MTM [8]. Extensive in-depth discussion of the physical meaning of the quantities \( m \) and \( 1/m \) can be found in [8]. For our immediate purposes here, we mention that the Proca MTM’s material tensor (10) may in principle be realized provided a method to synthesize both the \( T \) and \( L \) response tensors (the components proportional to \( \mathbf{I} \) and \( \mathbf{k} \mathbf{k} \), respectively) is introduced. Several Proca MTM design algorithms were proposed in [8]; inherent related design tradeoffs were discussed there too. However, the experimental realization of the Proca system is outside the scope of the present article.

Only a special type of electromagnetic waves, *Proca waves*, can be excited in a Proca MTM. These are a combination of \( T \) and \( L \) waves with identical dispersion relations given by

\[ \omega^2 = c^2 k^2 + c^2 m^2. \]  

(12)

It should be noted that for a Proca source system to effectively radiate into the Proca MTM, i.e., into the exterior domain \( D_s \), the operating frequency must exceed the cutoff frequency of the Proca system, see the (24) and also [8] for additional details.

Fig. 1(left) illustrates a basic example of an elementary Proca source system comprised of an infinitesimal dipole embedded into an infinite and homogeneous nonlocal Proca MTM with a dielectric tensor as in (10). We used the transmitting mode characteristic of the ideal Proca source, computed by (61) and (62) in [8], after which the Lorentz reciprocity theorem [41] was invoked in order to relate the receive power pattern signal to the transmitting source radiation pattern.
The reciprocity theorem [41], [42] is applicable because the Proca MTM is both passive and linear. The Proca antenna system is placed in the receive mode in order to test its perfectly isotropic pattern, where an incident plane wave system \((\mathbf{E}^{tr}, \mathbf{H}^{tr})\) is applied. Using the momentum space formalism, the receive power pattern \(P_{rx}(\Omega)\) can be computed directly in terms of the T and L mode patterns \(P_{tr}^{T}(\Omega)\) and \(P_{tr}^{L}(\Omega)\), respectively, as shown in the top right side of Fig. 1. Here, the plane wave is applied along the direction \(-\hat{k}\), while \(\hat{k} := k / k\) is the momentum-space spectral variable [18]. This allows applying the same expressions derived previously using the momentum space formula of the radiation pattern [19] to the problem of the receive mode characterization of isotropic dipoles. For an extensive discussion of the algorithm of how to compute radiation characteristics of nonlocal source systems illustrated with examples, see [18], [19], [38].

B. On Future Experimental Realization of Perfectly Isotropic Proca Sources

While physically well established, the technological realization of a perfectly isotropic antennas, e.g., as proposed in Fig. 1, faces several challenges. First, one needs to actually construct the Proca MTM itself. While design tradeoffs and algorithms were presented in [8], building a physical layout requires solving several technological problems. Some of them include the ability to control both the L and T mode constitutive relations simultaneously; overcoming the trade-off between the size of nonlocality (spatial dispersion strength) in the L mode dielectric tensor and bandwidth already noted in [8]; and, most importantly, ensuring that the MTM works over a large bandwidth with manageable losses. These issues are outside the scope of the present article but we hope that some or all of these problems will be overcome by future experimental research. Additional aspects of the realization of a PIR Proca system in the lab will be discussed in the remaining parts of this section, where it is assumed that a good model of the Proca MTM itself is available at hand.

Fig. 2(a) shows a practical realization of the receive system of Fig. 1 effectuated via the use of a finite perfectly symmetrical spherical Proca MTM. Note that in theory, a Proca MTM, as defined by the constitutive formula (10), is infinite and homogeneous. On the other hand, for lab-based scenarios the Proca system must be truncated. The finite spherical realization, where an approximation of (10) is placed within a sphere of radius \(a\) (Fig. 2a), destroys the perfect symmetry of the problem, and that is for the following reason. When an incident plane wave interacts with the enclosing surface \(S_{proca}\) of the Proca sphere shown in Fig. 2(a), new waves will emerge into the nonlocal domain, i.e., \((\mathbf{E}^{tr}, \mathbf{H}^{tr})\), while other waves \((\mathbf{E}^{ref}, \mathbf{H}^{ref})\) will be reflected. This is a general feature of interaction of external waves with interfaces involving nonlocal or spatially dispersive domains [36], [37]. In what follows, we are more interested now in the transmitted field system \((\mathbf{E}^{tr}, \mathbf{H}^{tr})\).

It is well known that one can establish from general considerations alone that reflection and transmission of waves at material interfaces where one medium is nonlocal (spatially dispersive) leads to the production of new “additional waves” that cannot be predicted by the usual paradigm of electromagnetic boundary-value problems in mathematical physics. Indeed, while the latter paradigm of boundary-value problem in mathematical physics has been successfully applied so far mainly to local electromagnetics [9], [43], [44], it cannot be extended in a straightforward manner to deal with nonlocal electromagnetic problems [20], [26], [28], [30], [45]. The problem of additional waves, whose discovery is essentially due to Pekar [36], [37], makes the computational or even the theoretical analysis of the problem in Fig. 2(a) extremely difficult in general. This is because all of the so-called Additional Boundary Conditions (ABCs) proposed so far to provide a complete description of the electromagnetism of the system rely on specific details of the concrete medium used, whether meta, plasma, crystal, superconductor, and so on. To the best knowledge of the author’s knowledge, current full-wave numerical electromagnetic simulators, e.g., those based on Finite Element Method (FEM) [46], Method of Moment [47], or Finite-Difference-Time-Domain (FDTD) method [48], cannot deal with a generic nonlocal interface scenario. It is conceivable then that the most economical way for dealing with this problem might eventually prove to be actually performing measurements on some possible future realization of a Proca MTMs.

Meanwhile, the exact Theorem 3 can not be applied when strong transmitted (spurious) modes such as \((\mathbf{E}^{tr}, \mathbf{H}^{tr})\) reach the dipole at the center as shown in Fig. 2(a). To solve this problem, we propose using a modified inhomogeneous locally-spatially-dispersive Proca MTM, whose dielectric tensor is given by

\[
\varepsilon_{eff}(\mathbf{k}, \omega; \mathbf{x}) = \varepsilon_{proca}(\mathbf{k}, \omega) \cdot \tilde{f}(\mathbf{x}; a). \tag{13}
\]

Here, the tensor \(\tilde{f}(\mathbf{r}, a)\) satisfies the following conditions:

(i) \(\lim_{|\mathbf{x}| \to \infty} \tilde{f}(\mathbf{x}; a) := \varepsilon_{proca}^{-1}(\mathbf{k}, \omega)\),

(ii) \(\forall |\mathbf{x}| < a, \tilde{f}(\mathbf{x}; a) = 1\),

(iii) \(\tilde{f}(\mathbf{x}; a)\) is continuous.

Collectively, these conditions guarantee that for \(r < a\), the medium behaves like an ideal Proca MTM (condition (ii)).

\(^{3}\)For a general theoretical discussion of these types of advanced nonlocal metamaterials, see [30].
On the other hand, far away from the dipole, the medium gradually decays or changes into free space (condition (i)). Condition (iii) ensures that the transition does not introduce discontinuities that may lead to the generation of new spurious modes. Collectively, and after a proper optimization, the non-local homogeneous model proposed in (13) and (14) should introduce a smooth transition region (Fig. 2(b)) acting like a “matching layer” minimizing the generation of undesired spurious modes launched into the test point dipole at the center of the spherical Proca source/receiver system in Fig. 2.

V. DIRECTIVE RADIATION CHARACTERISTICS OF CLASSICAL PROCA SOURCE SYSTEMS

A. Derivation of the Basic Radiation Formulas of Proca Array Systems

We utilize here the previous main result (Theorem 3) to derive general radiation pattern formulas for classical Proca antennas, i.e., classical sources embedded into Proca MTMs. The term classical source means that the radiated fields will be treated using classical field theory (no quantization is applied.) the theory of quantum Proca antennas will be developed elsewhere.

Let the radiating point source current be given by the formula

$$\mathbf{J}(\mathbf{k}, \omega) = \hat{\alpha}_s e^{i \mathbf{k} \cdot \mathbf{x}} \exp \{2 \pi i \delta(\omega - \omega_s)\},$$  \hspace{1cm} (15)

which is the Fourier transform of (8). Here, \( \mathbf{x}_s \) is the location of the point source; the direction of the infinitesimal dipole is specified by the unit vector \( \hat{\alpha}_s \); the sinusoidal excitation (circular) frequency is \( \omega_s \); finally, the frequency-dependent complex-valued quantity \( J_0 = J_0(\omega_s) \) gives the excitation current density amplitude.

The angular radiated power intensity of a point source is given by [8]

$$U_{rad}(\varphi, \theta; \omega_s) = |J_0|^2 \Gamma \omega_s \left[ \omega_s^2 - \omega_{ph}^2 \right]^{\frac{3}{2}}$$  \hspace{1cm} (16)

where

$$\Gamma := \frac{1}{8 \varepsilon_0 \pi^2 c^3},$$  \hspace{1cm} (17)

where \( \varepsilon_0 \) is the vacuum electric permittivity. The cutoff frequency \( \omega_{ph} \) is the photon mass frequency, which is defined by:

$$\omega_{ph} = m_{ph} c^2 = mc.$$  \hspace{1cm} (18)

In [8], this quantity was interpreted as the quantum Planck “internal clock” frequency of a bosonic particle with mass \( m_{ph} \) and energy \( m_{ph} c^2 \). A similar idea of such internal clock was proposed earlier for different applications, e.g., see [49].

We would like now to move beyond the point-source scenario discussed above so far by considering source array systems, i.e., Proca sources comprised of multiple point-dipole source. Our key result of Proca source array system is the following theorem:

**Theorem 4.** Let the source current be a sum of discrete point sources located at positions \( \mathbf{x}_n \in \mathbb{R}^3, n = 1, 2, \ldots, N \), as follows:

$$\mathbf{J}(\mathbf{x}, t) = \sum_{n=1}^{N} \hat{\alpha}_{sn} J_n(\omega_s) \delta(\mathbf{x} - \mathbf{x}_n) e^{-i \omega_s t},$$  \hspace{1cm} (19)

Then the total radiation power intensity profile is given by

$$U_{rad}(\theta, \varphi) = \Gamma \omega_s \left[ \omega_s^2 - \omega_{ph}^2 \right]^{\frac{3}{2}} \sum_{n=1}^{N} J_n \times \exp \left\{ \frac{-i}{c} \left( \omega_s^2 - \omega_{ph}^2 \right)^{\frac{1}{2}} x_n \cos \varphi \sin \theta \right\} \times \exp \left\{ \frac{-i}{c} \left( \omega_s^2 - \omega_{ph}^2 \right)^{\frac{1}{2}} y_n \cos \varphi \sin \theta \right\} \times \exp \left\{ \frac{-i}{c} \left( \omega_s^2 - \omega_{ph}^2 \right)^{\frac{1}{2}} z_n \cos \theta \right\},$$  \hspace{1cm} (20)

where

$$x_n := \hat{x} \cdot \mathbf{x}_n, \; y_n := \hat{y} \cdot \mathbf{x}_n, \; z_n := \hat{z} \cdot \mathbf{x}_n, \; n = 1, 2, \ldots, N,$$  \hspace{1cm} (21)

**Proof.** The spherical coordinates form of \( \hat{k} \) is given by

$$\hat{k} = \hat{k}(\Omega) = \hat{x} \cos \varphi \sin \theta + \hat{y} \cos \varphi \sin \theta + \hat{z} \cos \theta.$$  \hspace{1cm} (22)

On the other hand, the wavenumber of a Proca wave must satisfy the dispersion relation (12), i.e., we have:

$$k(\omega_s) = \frac{1}{c} \left( \omega_s^2 - \omega_{ph}^2 \right)^{\frac{1}{2}}.$$  \hspace{1cm} (23)

Using \( \mathbf{k} = \hat{k} \hat{k} \), taking the Fourier transform of (19) with the help of (1) (as we did to obtain (15)), and finally applying (16) to each radiation term in the point source expansion, the expression (20) is obtained. Note that by Theorem 3, the individual polarizations of each element \( \hat{\alpha}_n \) do not play any role in the final expression because the total radiated power (or energy) in a Proca MTM constitutes a PIR system.

The expression (20) is the general formula characterizing the radiation pattern of a generic Proca source array where the various radiating elements can be located at arbitrary locations in the 3-dimensional space. It is similar to the exact scalar array factor expression in classical antenna theory [14] but with the crucial difference that the dispersion relation (23) is nonlinear, in contrast to the case of radiation in free space where the dispersion law is linear. The transmission of information using massive photons is expected then to lead to nontrivial phenomena such as wavepacket spreading and others.

In particular, we highlight a key difference between Proca sources and classical scalar antennas radiating in free space. Note that (20) is valid only when the following condition is satisfied:

$$\hbar \omega_s > \hbar \omega_{ph}.$$  \hspace{1cm} (24)

That is, the source operating frequency must exceed the Proca wave cutoff frequency \( \omega_{ph} \). On the other hand, in classical source theory (Maxwell’s wave, or EM waves in free space), the corresponding condition is simply \( \omega_s > 0 \). Therefore, Proca source systems possess an inherent minimum energy threshold below which photon excitation (or classical radiation) cannot take place.
As a concrete example, consider a linear uniformly spaced array of discrete sources positioned along the z-axis, with interelement spacing of \(d\). The expression (20) reduces in this case to

\[
U_{\text{rad}}(\theta, \zeta) = \Gamma \omega_{\text{ph}}^2 \zeta \left[ \frac{\zeta^2 - 1}{\zeta^2} \right]^\frac{1}{2} \times \left| \sum_{n=1}^{N} J_n \exp \left\{ -i \frac{2 \pi n d}{\lambda_{\text{ph}}} \left[ \zeta^2 - 1 \right]^{\frac{1}{2}} \cos \theta \right\} \right|^2,
\]

where we introduced the dimensionless frequency

\[
\zeta := \frac{\omega_s}{\omega_{\text{ph}}} = \frac{\lambda_{\text{ph}}}{\lambda}.
\]

Here, the photon mass wavelength is defined by

\[
\lambda_{\text{ph}} := \frac{2 \pi c}{\omega_{\text{ph}}} = \frac{2 \pi}{m}.
\]

A Proca source effectively radiates into the exterior region only if \(\zeta > 1\), i.e., when

\[
\lambda_{\text{ph}} > \lambda.
\]

In other words, at the same operating frequency \(\omega_s\), a massive photon antenna appears to be electrically smaller than a massless photon antenna. The two quantities \(\lambda_{\text{ph}}\) and \(\omega_{\text{ph}}\) are fundamental in the theory of Proca source systems and will appear repeatedly in what follows. The dimensionless parameter \(\zeta\) measures the depth of how far the system is operating above its intrinsic radiation threshold, or, equivalently, how the electrical size is reduced.

Let us move now to a detailed quantitative investigation of how the Proca antenna directs massive photons in the exterior region. To achieve this, we borrow the concept of source directivity from classical antenna theory. The directivity of the radiating system is given by the expression [1], [10], [14], [50]:

\[
D(\theta, \varphi; \zeta) := 4 \pi \frac{U_{\text{rad}}(\theta, \varphi; \zeta)}{P_{\text{rad}}(\theta, \zeta)}
\]

(29)

Here, the total net radiated power through a full angular span is given by:

\[
P_{\text{rad}}(\theta, \zeta) = \int_{4\pi} d\Omega U_{\text{rad}}(\Omega, \zeta) = \int_{0}^{\pi} d\varphi \int_{0}^{\pi} d\theta \sin \theta U_{\text{rad}}(\theta, \zeta).\]

(30)

From (29), the linear array of the Proca type whose radiation power pattern is given by (25) will generate a directivity pattern with the form:

\[
D(\theta; \zeta) = \frac{4 \pi}{\int_{4\pi} d\Omega} \left| \sum_{n=1}^{N} J_n(\zeta) \exp \left\{ -i \frac{2 \pi n d c}{\lambda_{\text{ph}}} \left[ \zeta^2 - 1 \right]^{\frac{1}{2}} \cos \theta \right\} \right|^2.
\]

(31)

We note how the frequency-dependent multiplicative Proca factor \(\Gamma \omega_{\text{ph}}^2 \zeta \left[ \frac{\zeta^2 - 1}{\zeta^2} \right]^\frac{1}{2}\) cancels out in the directivity expression (31) but the general form of the directivity profile \(D(\theta; \zeta)\) is still a strong function of frequency as will be illustrated with numerical examples below.

For completeness, we also state the corresponding radiation law for classical scalar sources where radiated photons are massless \((m_{\text{ph}} = 0)\). In such case, (20) reduces to

\[
U_{\text{rad}}(\theta, \zeta) = \omega_{\text{ph}}^2 \Gamma \sum_{n=1}^{N} J_n \exp \left\{ -i \frac{\omega_s}{c} x_n \cos \varphi \sin \theta \right\} \times \exp \left\{ -i \frac{\omega_s}{c} z_n \cos \theta \right\} \exp \left\{ -i \frac{\omega_s}{c} z_n \cos \theta \right\}.
\]

(32)

For the case of the uniformly-spaced linear array, the expression (32) becomes

\[
U_{\text{rad}}^1(\theta, \varphi) = \omega_{\text{ph}}^2 \Gamma \sum_{n=1}^{N} J_n(\omega_s) \exp \left\{ -i \frac{2 \pi n d_{\text{cl}}}{\lambda} \cos \theta \right\}.
\]

(33)

\(\lambda = 2 \pi c/\omega_s\) is the free-space wavelength and \(d_{\text{cl}}\) is the interelement spacing of the classical array. The classical array directivity is then

\[
D_{\text{cl}}(\theta; \zeta) = 4 \pi \frac{\sum_{n=1}^{N} J_n(\omega_s) \exp \left\{ -i \frac{2 \pi n d_{\text{cl}}}{\lambda} \cos \theta \right\}}{\int_{4\pi} d\Omega \left| \sum_{n=1}^{N} J_n(\omega_s) \exp \left\{ -i \frac{2 \pi n d_{\text{cl}}}{\lambda} \cos \theta \right\} \right|^2}.
\]

(34)

In (34), we have denoted the interelement spacing in the free space array by \(d_{\text{cl}}\) instead of the \(d\) used in the Proca array expression (20) because of the inherent difference in spatial scale due to (28). In general, one would like to tune interelement spacings in massless and massive source systems such that similar directive and power profiles can be obtained (cf. Sec. V-C.)

To facilitate analysis at multiple frequencies (frequency sweep control), we define the maximum directivity, or simply the directivity of a Proca source systems, as follows:

\[
D_0(\zeta) := \max_{\theta \in [0, \pi], \varphi \in [0, 2\pi]} D(\theta, \varphi; \zeta),
\]

(35)

and similarly for the classical array case. In this way, the angular information of the directivity pattern can be eliminated in order to simplify the analysis of the frequency-dependent properties of the source as usually done in antenna theory [14].

C. Basic Examples and Illustrations

Next, we illustrate the above general theory with a series of examples.
Example 1. It is interesting to note that by discarding the frequency-dependant factors $\omega, \Gamma$ and $\zeta^2$ in (25) and (33), then the purely directive radiation characteristics (at a single frequency) of both the classical and Proca systems become qualitatively identical provided the following condition is satisfied:

$$\frac{d}{dcl} = \frac{\lambda_{ph}}{\lambda} [\zeta^2 - 1]^{-\frac{1}{2}} = \zeta [\zeta^2 - 1]^{-\frac{1}{2}}. \tag{36}$$

In other words, the directivities of Proca and classical arrays are similar provided the interelement spacings are properly adjusted according to the scaling law (36). The existence of nonzero photon mass then leads to the physical realizability of exact scalar arrays but these arrays have directive properties similar to classical (massless photon) arrays with proper adjustment of interelement spacing at each frequency. Imposing the condition $d/dcl = 1$ implies the relation $\zeta^2 (\zeta^2 - 1) = 1$, which has a single real solution $\zeta = 1.2720$. Therefore, at this particular frequency, the interelement spacing design parameters in the two physically distinct cases massive and massless photon arrays can be identical while giving rise to qualitatively similar directive properties. At values other than $\zeta = 1.2720$, the radiation characteristics are distinct.

Example 2. Consider $m_{ph} = 1.054 \times 10^{-36}$ kg which is a typical massive photon value, e.g., see [4], [51], [52]. The corresponding photon mass wavelength is $\lambda_{ph} = 2.09 \mu m$ while the photon mass frequency is $\omega_{ph} = 9.02 \times 10^{15}$ rad/s. The details of the photon mass parameters depend on the actual design of the corresponding Proca MTM, see [8] for extensive discussion. The normalized frequency $\zeta = 1.2720$ corresponds to operating array source wavelength of $\lambda = 1.6431 \mu m$. In this case, an interelement spacing of $d/\lambda_{ph} = 0.5$ is equivalent to $d = 1.0450 \mu m$.

Example 3. (6-element Radiation Pattern at Single Frequency) A 6-element array of point sources is shown in Fig. 3, comprised of exact equally spaced infinitesimal dipole sources, each with the form (8). The polarization of each point source $\hat{e}_s$ is ignored since by Theorem 3, the total radiated power due to each isolated dipole is perfectly isotropic. However, when each source is excited differently, the net radiated power obviously may exhibit strong directive properties due to interference processes [10], [50]. In what follows, we compare the Proca version of the array system with the “classical array” case, where the term “classical” here means a similar array of uniform linear discrete array but with the Proca MTM replaced by free space. In Fig. 4, we show the directivity radiation patterns of the two cases computed by means of the formulas (31) and (34) for the case $d/\lambda_{ph} = 0.5$. The phases and amplitudes of the excitation current are assumed to be uniformly distributed. We have selected a realization of these excitations with values given in the caption of Fig. 4. As expected, the Proca system’s directive angular pattern is different from the free space case, with some Proca’s radiation peaks shifted (e.g., at $\theta = 0.6\pi$), or drastically attenuated (e.g., at $\theta = 0.15\pi$), or completely destroyed (e.g., at $\theta = 0.3\pi$). This behaviour is consistent with effective modification of the electrical length $d/\lambda$ between the elements as often observed in classical antenna theory [14]. Here, such modification of the effective spacing $d/\lambda$ is due to the nonzero effective mass of the photon in Proca arrays, which is accounted for by the presence of the factor $\zeta \sqrt{\zeta^2 - 1}$ in (31).

Example 4. We investigate the frequency response behaviour of Proca source systems. Since the Proca radiation formula (25) differs from the corresponding classical expression (32) by the phase factor $\sqrt{\zeta^2 - 1}$, it is imperative to compare the corresponding behaviour of the two radiating source array types at different values of the normalized frequency $\zeta$. The total radiated power of a 6-element linear uniform Proca array can be computed by means of (25) and (30). In general, the spectral dependence of the source current excitation amplitude $|J_0|$ is important to ensure convergence at high frequencies. Let the source spectral profile be written as

$$|J_0|^2 = \frac{b}{\omega s^2}, \tag{37}$$

where $r \geq 1$ is a positive integer and $b > 0$ is a dimensional constant. In Fig. 5, we plot $P_{rad}(\zeta)$ when $d/\lambda_{ph} = 0.5$. For $r < 2$, it can be shown that the total radiated power diverges. For $r = 2$, the radiated power oscillates around a constant level even at very high frequencies. For $r > 2$, $P_{rad}(\zeta)$ begins.
Fig. 5: The total radiated power $P_{\text{rad}}$ as a function of the normalized frequency $\zeta$ for the 6-element array of Fig. 3. The radiated power is normalized with respect to $g := \Gamma \omega_\text{ph}^2$, while we set $b = 1$. The spectral source amplitude profile $|J_0| \propto \omega^{-r}$ is varied for $r = 2, 3, 4$, where $\omega$ is the source frequency $\omega_s \propto \zeta$. The amplitude and phase values of the various array element excitations $J_n$, $n = 1, \ldots, 6$, are drawn from a uniform random distribution.

Fig. 6: A 6-element discrete scalar source array. The normalized frequency is $\zeta = 1.95$ with interelement spacing of $d/\lambda_{\text{ph}} = 0.5$ for the Proca system and $d_{\text{cl}}/\Delta = 0.5$ for the classical array. The random current excitations values are $0.0266 + 0.160 i, 0.0090 - 0.1190 i, -0.0160 + 0.4981 i, -0.9591 - 0.0359 i, -0.1071 - 0.3231 i, 0.4531 - 0.3705 i$, all normalized.

to decline with high frequency excitations. Total convergence of power in the time domain can be attained only if $r \geq 2$.

**Example 5. (6-element Radiation Power at Multiple Frequencies)** Consider again the 6-element array of point sources as shown in Fig. 3. A set of random amplitude and phase excitations are selected from a uniform distribution, with numerical values shown in the caption of Fig. 6, where the Proca source radiation pattern directivity is computed by means of the expression (35) for various values of the normalized frequency $\zeta$. The classical array’s corresponding directivity is computed using the same formula, but now while (34) is used to estimate the angular distribution of the radiation directivity profile. In order to compare the Proca array’s directivity with the classical array’s, we compute $D_0(\zeta)$ for the latter case under the constraint $d_{\text{cl}}/\lambda = 0.5$, which is the classical counterpart of the massive photon array scenario when $d/\lambda_{\text{ph}} = 0.5$ as used above. In other words, for the classical array case (radiation by scalar sources in free space), we change the interelement spacing $d_{\text{cl}}$ at each frequency $\zeta$ in order to keep the electrical interelement distance ratio $d_{\text{cl}}/\lambda$ fixed. Fig. 5 provides a comparison between the two systems’ maximal directivity $D_0$ as function of the normalized frequency $\zeta$. It can be seen that the Proca array’s directivity oscillates around the classical limit, with even some frequency bands at which the Proca directivity can be significantly higher than the classical case. It is interesting to note that such enhancement of directivity in nonlocal source systems was observed earlier even in numerical examples involving weakly nonlocal metamaterials, e.g., see [38]. The current example further confirms this trend and suggests a possible practical application of nonlocal media such as Proca MTMs in the field of antenna technology.

VI. CONCLUSION

We conducted an extensive theoretical investigation into the fundamental structure of classical electromagnetic radiation in Proca metamaterials, which provides a model for massive electromagnetism, i.e., radiating source systems emitting massive photons. We first formulated and proved a general result stating that a point source in a Proca metamaterial domain possesses a perfectly isotropic (PIR) power radiation intensity profile. The theorem was then used to construct the radiation intensity formulas for more generic arrays of point sources, which can approximate arbitrary continuous Proca source distributions. Afterwards, a concept of source directivity pattern was borrowed from classical antenna theory and reformulated for Proca systems. The directive radiation characteristics of basic examples of linear uniform arrays of Proca sources were investigated with several examples, and a comparison with classical scalar antennas radiating massless photons in free space is made. Proca source systems require a minimum energy (frequency) threshold below which they cannot radiate, regardless to the amount of power injected into the input port. This behaviour, similar to waveguides operating below cutoff frequency, does not exist in massless photon antennas (classical antenna theory). Moreover, Proca antennas correspond to exact scalar radiation theory, in contrast to classical (massless) electromagnetism where scalar fields are never exactly realizable. The nonlinear dispersion relation of massive photons also introduces new frequency (or wavelength) dependent effects in the phase and amplitude excitations of each individual point source element in the radiating Proca source system.

**REFERENCES**
