Global and blow up solutions for a semilinear heat equation with variable reaction reaction on a general domain

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Abstract

We are concerned with the existence of global and blow-up solutions for the semilinear heat equation with variable exponent\( u_t - \Delta u = h(t)f(u)p(x) \) in \( \Omega \times (0,T) \) with zero Dirichlet boundary condition and initial data in \( C^0(\Omega) \). The scope of our analysis encompasses both bounded and unbounded domains, with \( p(x) \in C(\Omega) \), \( 0 < p - \infty < p(\infty) < p + \), \( h \in C(0,\infty) \), and \( f \in C(0,\infty) \). Our findings have significant implications, as they enhance the blow-up result discovered by Castillo and Loayza in Comput. Math. App. 74(3), 351-359 (2017) when \( f(u) = u \).
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Summary

We are concerned with the existence of global and blow-up solutions for the semilinear heat equation with variable exponent $u_t - \Delta u = h(t)f(u)p(x)$ in $\Omega \times (0, T)$ with zero Dirichlet boundary condition and initial data in $C_0(\Omega)$. The scope of our analysis encompasses both bounded and unbounded domains, with $p(x) \in C(\Omega)$, $0 < p^- \leq p(x) \leq p^+$, $h \in C(0, \infty)$, and $f \in C[0, \infty)$. Our findings have significant implications, as they enhance the blow-up result discovered by Castillo and Loayza in Comput. Math. App. 74(3), 351-359 (2017) when $f(u) = u$.

KEYWORDS:
Semilinear heat equation, Global Solution, Blow up solution, Variable exponent, Arbitrary domain

1 | INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a domain (bounded or unbounded) with smooth boundary $\partial \Omega$. We consider the semilinear parabolic problem

$$\begin{cases}
  u_t - \Delta u = h(t)F(x, u) & \text{in } \Omega \times (0, T), \\
  u = 0 & \text{on } \partial \Omega \times (0, T), \\
  u(0) = u_0 \geq 0 & \text{in } \Omega,
\end{cases} \quad (1)$$

where $F(x, s) = f(s)p(x)$, for $x \in \Omega$, $s \geq 0$, $f \in C[0, \infty)$ is a nondecreasing locally Lipschitz function, $h \in C(0, \infty)$, $p \in C(\Omega)$ is a bounded function such that

$$0 < p^- \leq p(x) \leq p^+ < \infty, \quad (2)$$

for all $x \in \Omega$, with $p^- = \inf_{x \in \Omega} \{p(x)\}$, $p^+ = \sup_{x \in \Omega} \{p(x)\}$, and $u_0 \in C_0(\Omega)$. Here, $C_0(\Omega)$ denotes the closure in $L^\infty(\Omega)$ of infinitely differentiable functions with compact support in $\Omega$. Throughout the work we consider only nonnegative solutions in the sense of $(1)$.

Problem $(1)$ appears in several models of the applied sciences such as electrorheological fluids, thermo-rheological fluids, image processing, chemical reactions, heat transfer and population dynamics. It has been considered for many authors. For example, when $\Omega$ is a bounded domain and $h(t) = 1$, blow up results for problem $(1)$ were obtained in [13] for $F(x, s) = e^{p(x)s}$, and in [21] for $F(x, u) = a(x)u^p(x)$. When $\Omega = \mathbb{R}^N$, Fujita type results were obtained in [12] for $F(x, s) = e^{p(x)s}$, $h(t) = 1$. Specifically, in the last case it was shown that:

- If $p^+ > 1 + 2/N$, then problem $(1)$ possesses global nontrivial solutions.
- If $1 < p^- < p^+ \leq 1 + 2/N$, then all nontrivial solutions to problem $(1)$ blow up in finite time.
- If $p^- < 1 + 2/N < p^+$, then there are functions $p$ such that problem $(1)$ possesses global nontrivial solutions and functions $p$ such that all nontrivial solutions blow up.

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These results were extended for any domain $\Omega$ (bounded or unbounded); see Theorem 1.2 and Remark 1.3 of $[9]$. Specifically, they showed the following result.

**Theorem 1.** Suppose that $F(x,s) = s^p(x)$ for $s \geq 0$.

(i) If $p^+ \leq 1$, then all solutions of problem (1) are global.

(ii) If $p^+ > 1$ and

$$\limsup_{t \to \infty} \|S(t)u_0\|_\infty^{p^+ - 1} \int_0^t h(\sigma) d\sigma = \infty,$$

for every nonnegative $0 \neq u_0 \in C_0(\Omega)$, then every nontrivial solution of problem (1) either blow up in finite time or in infinite time. In the last case, we mean that the solution is global and $\limsup_{t \to \infty} \|u(t)\|_\infty = \infty$.

(iii) If $p^- > 1$ and there exists $w_0 \in C_0(\Omega)$, $w_0 \geq 0$, $w_0 \neq 0$ verifying

$$\int_0^\infty h(\sigma)\|S(t)w_0\|_\infty^{p^- - 1} < \infty,$$

then there exists a constant $\Lambda > 0$, depending on $p^+$ and $p^-$, so that if $0 < \lambda < \Lambda$, then the solution of (1), with initial data $\lambda w_0$, is a nontrivial global solution.

Notice that the conditions (3) and (4) of Theorem 1 are expressed in terms of the asymptotic behavior of $\|S(t)u_0\|_\infty$, where $\{S(t)\}_{t \geq 0}$ denotes the heat semigroup. The first result of this type was given by Meier $[10]$ for problem (1) in the case $F(x,s) = s^p$, $s \geq 0$, $p > 1$. It is important because the conditions are valid for any domain $\Omega$, bounded or unbounded, and because it is sufficient to know the behavior of $\|S(t)u_0\|_\infty$ to decide whether the solution of problem (1) is global or not. For example, we know, in $\mathbb{R}^N$, that $\|S(t)u_0\|_\infty \sim t^{-N/2}$ for $t$ near infinity and $u_0 \in C_0(\mathbb{R}^N)$, $u_0 \neq 0$. Thus, assuming $h = 1$, condition (3) holds if $p^+ < 1 + 2/N$, while condition (4) holds if $p^- > 1 + 2/N$. This coincides with the results obtained in $[11]$. Similar results have been obtained for parabolic coupled system related to problem (1) in $[10]$ and $[12]$.

The main objective of this work is to obtain Meier type results, similar to Theorem 1 for problem (1) considering $F(x,s) = f(s^p(x))$, where $f \in C[0,\infty)$ is a locally Lipschitz and nondecreasing function, and $p \in C(\Omega)$ satisfies condition (2). We also analyze situations where $p(x) < 1$ or $p(x) > 1$ on subdomains of $\Omega$. As a consequence of our results, we improve Theorem 1 (ii) and remove the possibility of the existence of solutions that blow up in infinite time, see Remark 2 (vi).

Our results depend on the conditions:

$$\int_0^\infty \frac{d\sigma}{\min\{f(\sigma)^p, f(\sigma)^{p^-}\}} < \infty,$$

for some $\alpha > 0$ such that $f(\alpha) > 0$, and

$$\int_0^\infty \frac{d\sigma}{\max\{f(\sigma)^p, f(\sigma)^{p^-}\}} = \infty,$$

for all $r > 0$ with $f(r) > 0$.

Note that if $F(x,s) = f(s)$ and $h = 1$, condition (5) turns into

$$\int_0^\infty \frac{d\sigma}{f(\sigma)} < \infty,$$

which is well known as a necessary and sufficient condition for the existence of blow up solutions. Some examples of a function $f$ satisfying condition (7) are $f(u) = u^q$, $f(u) = (1 + u)[\ln(1 + u)]^q$, $f(u) = e^{au} - 1$ for $q > 1$ and $\alpha > 0$.

In our first result we use condition (6) to get global solutions for problem (1).

**Theorem 2.** Assume that condition (6) holds with $p^- < 1$. Then for every $u_0 \in C_0(\Omega)$, $u_0 \geq 0$ there exists a global solution of problem (1).

Moreover, $u$ is a positive if

(i) $f(0) > 0$ or $u_0 \neq 0$ or
Remark 1. Here are some comments about Theorem 2

(i) Condition \( f(0) = 0 \) implies that \( u = 0 \) is a solution of problem (8) and assumption \( \int_0^\tau \frac{d\sigma}{f(\sigma)} < \infty \) guarantees the existence of a positive solution of problem (4).

(ii) The existence of a positive solution of \( \{ \text{with } u_0 = 0, \text{ for } f, s, h = 1, \text{ it was shown in} \} \) considering a subsolution of the form \( u(t) = C t^{1/(1-\tau)} \phi_1 \), for an appropriate constant \( C > 0 \) and \( \phi_1 \) the first eigenfunction of the Laplacian operator on \( H^1_0(\Omega') \). Here, we use the subsolution \( u = \mu(\cdot) \chi_r \) of problem \( u_t - \Delta u = h(t) f(u^\gamma) \) in \( \Omega' \times (0, \tau_1) \). This idea was used firstly in \( \).

(iii) For \( f(s) = s, p(x) = p \in (0, 1) \) constant, \( h = 1 \) and \( \Omega = \mathbb{R}^N \), the function \( u(t) = [(1 - p)t]^{1/(1-\rho)} t > 0 \) is the positive solution of problem \( \) (\( u_0 = 0 \)) which is obtained solving the Cauchy problem: \( x_t = x^\rho, \ x(0) = 0 \), see \( \text{ and } \).

(iv) When \( F(x, s) = s^{p(x)} \), \( s \geq 0 \) and \( 0 < \tau < 1 \) we have

\[
\int_0^\infty \frac{d\sigma}{\max \{ s^{\rho^{-}}, s^{\rho^{+}} \}} = \int_0^1 \frac{d\sigma}{s^{\rho^{-}}} + \int_1^\infty \frac{d\sigma}{s^{\rho^{+}}}
\]

if and only if \( p^+ \leq 1 \). Thus Theorem 2 coincides with Theorem 1(i).

In our second result we use condition (5) to obtain blow up solutions.

Theorem 3. (i) (Global existence) Let \( F : (0, m] \to [0, \infty) \) be defined by \( F(s) = \frac{1}{\delta} \max \{ f(s)^{\rho^-}, f(s)^{\rho^+} \} \) for \( s \in (0, m] \).

Assume that \( F \) is a nondecreasing function and there exists \( v_0 \in C_0(\Omega), 0 \neq v_0 \geq 0, \| v_0 \|_\infty \leq m \) satisfying

\[
\int_0^\infty h(\sigma) F \left( \| S(\sigma)v_0 \|_\infty \right) d\sigma < 1.
\]

Then there exists a constant \( \delta > 0 \) such that for \( u_0 = \delta v_0 \) the solution of problem 1 is a global solution.

(ii) (Nonglobal existence) Assume that \( f(0) = 0 \), condition (5) holds, \( p^- \geq 1 \) and the following assumptions are satisfied:

(a) \( f(s) > 0 \) for all \( s > 0 \), and

\[
f(S(t)v_0) \leq S(t)f(v_0),
\]

for all \( 0 \leq v_0 \in C_0(\Omega) \) and \( t > 0 \).

(b) There exist \( \tau > 0 \) such that

\[
\int_0^\infty \frac{d\sigma}{\min \{ f(\sigma)^{\rho^-}, f(\sigma)^{\rho^+} \}} \leq 2^{-\rho^+} \int_0^\tau h(\sigma) d\sigma.
\]

Then the solution of problem 1 with initial condition \( u_0 \geq 0, u_0 \neq 0 \) blows up in finite time.

Remark 2. Here are some comments about Theorem 3.
(i) If \( f(0) = 0 \) and \( p^- \geq 1 \), then \( \mathcal{F} \) is well defined, since \( f \) is locally Lipschitz, and if we assume additionally that \( f \) is a convex function we have that \( \mathcal{F} \) is nondecreasing.

(ii) Condition \( f(0) = 0 \) is used in inequality \( \text{(5)} \) because the Dirichlet condition on the boundary must be satisfied.

(iii) Constant \( 2^{-p^*} \) in inequality \( \text{(10)} \) appears due to Jensen’s inequality, see Lemma \( \text{(2)} \).

(iv) Condition \( \text{(3)} \) holds for any convex function \( f \) when \( \Omega = \mathbb{R}^N \). This is a consequence of Jensen’s inequality and the representation of the semigroup \( S(t)u_0 = K_t \ast u_0 \), where \( K_t = (4\pi t)^{-N/2} \exp(-|x|^2/(4t)) \) is the heat kernel.

(v) When \( \Omega \) is any domain, condition \( \text{(9)} \) holds for any twice differentiable and convex function with \( f(0) = 0 \). Indeed, if \( v(t) = f(S(t)u_0) \) then
\[
v_t - \Delta v = -f''(S(t)u_0)|\nabla S(t)u_0|^2 \leq 0
\]
in \( \Omega \times (0, \infty) \) and \( v(t) = f(0) = 0 \) on \( \partial \Omega \times (0, \infty) \). Since \( v(0) = f(u_0) \) we conclude by the maximum principle.

(vi) Theorem \( \text{(3)} \) improves Theorem \( \text{(1)} \) ii) if \( p^- > 1 \), \( f(s) = s \) and condition \( \text{(3)} \) holds. Indeed, since \( p^- > 1 \) the condition \( \text{(5)} \) is verified. Thus, it is sufficient to check the condition \( \text{(10)} \). First, note that
\[
\frac{p^+ - p^-}{(p^+ - 1)(p^- - 1)} \|u_0\|_\infty^{p^*-1} + \frac{1}{p^* - 1} \leq \left( \frac{1}{2} \right)^{p^*} \|S(r)u_0\|_\infty^{p^*-1} \int_0^r h(\sigma) d\sigma.
\]
for every \( a > 0 \). From condition \( \text{(3)} \) there exists \( \tau > 0 \) such that
\[
\frac{p^+ - p^-}{(p^+ - 1)(p^- - 1)} \|u_0\|_\infty^{p^*-1} + \frac{1}{p^* - 1} \leq \left( \frac{1}{2} \right)^{p^*} \|S(r)u_0\|_\infty^{p^*-1} + \int_0^r h(\sigma) d\sigma.
\]
Hence,
\[
\int_0^\infty \frac{d\sigma}{\min\{\sigma^{p^*}, \sigma^{p^-}\}} \leq \|S(\tau)u_0\|_\infty^{1-p^*} \left[ \frac{p^+ - p^-}{(p^+ - 1)(p^- - 1)} \|S(\tau)u_0\|_\infty^{p^-} + \frac{1}{p^- - 1} \right]
\]
\[
\leq 2^{-p^*} \int_0^\tau h(\sigma) d\sigma.
\]

By Theorem \( \text{(3)} \) \( u \) blows up in finite time.

In the proof of Theorem \( \text{(3)} \) we adapt the techniques used in\( \text{(13)} \). It is worth noting that in that work, the authors utilized their findings to derive Fujita exponents for the problem \( \text{(1)} \) with \( F(x, u) = (1 + u)(\ln(u + 1))q \) and \( F(x, u) = e^{au} - 1 \). Theorem \( \text{(3)} \) can also be applied to obtain Fujita-type results for problem \( \text{(1)} \) with more complex source terms and on different domains \( \Omega \). This may include the logarithmic function with variable exponent \( [(1 + u)(\ln(u + 1))q]^{p(x)} \) and the exponential with variable exponent \( [e^{au} - 1]^{p(x)} \).

It is important always to be aware that solutions may blow up in a finite time when dealing with large initial data. This was demonstrated in\( \text{(15)} \) Theorem \( \text{3.3} \) using Kaplan’s argument\( \text{(15)} \). Our next Theorem shows how this approach can be modified to present a similar result. We will focus on the scenario where \( h = 1 \) for simplicity.

**Theorem 4.** Suppose that \( p^+ > 1 \), \( h = 1 \) and there exists a bounded subdomain \( \Omega' \subset \Omega \) such that \( p(x) \geq \gamma > 1 \) for all \( x \in \Omega' \). Assume also that \( f \) is a convex function such that \( \int_r^\infty d\sigma f'(\sigma)^\gamma < \infty \) for some \( \tau > 0 \) with \( f(\tau) > 0 \). Then there are solutions of problem \( \text{(1)} \) such that blow up in finite time.

**Remark 3.** Theorem \( \text{(4)} \) for \( f(s) = s \) was established in\( \text{(14)} \) Theorem \( \text{3.3} \).

The rest of the paper is organized as follows. Section 2 is dedicated to analyze the existence of positive global solution and Theorem \( \text{(2)} \) is proved. Blow up for large initial data is shown in Section 3. Section 4 is devoted to the proof of Theorem \( \text{(3)} \).
EXISTENCE AND UNIQUENESS

Solutions of problem (1) are understood in the following sense: given $u_0 \in C_0(\Omega)$, a function $u \in C([0, T), C_0(\mathbb{R}^N))$ is said to be a solution of problem (1) in $(0, T)$ if $u$ is nonnegative and verifies the following equation

$$u(t) = S(t)u_0 + \int_0^t S(t-\sigma)h(\sigma)F(\cdot, u(\sigma))d\sigma$$

for all $t \in (0, T)$, where $F(x, u) = f(u)^p(x)$.

Since $f \in C[0, \infty)$ is a locally Lipschitz function, it is clear that if $p(x) \geq 1$, the nonlinear term $F(x, u)$, for $x \in \Omega$ fixed, is a locally Lipschitz function. Thus, using usual methods it is possible to show the existence of a unique local solution of (1) defined in some interval $[0, T]$. Moreover, this solution can be extended to a maximal interval $[0, T_{\text{max}}]$ and the blow up alternative occurs: either $T_{\text{max}} = +\infty$ (we say that $u$ is a global solution) or $T_{\text{max}} < \infty$ and $\limsup_{t \to T_{\text{max}}} \|u(t)\|_\infty = +\infty$. In the last case, we say that the solution blows up in a finite time, see for example [3, 11, 23] and [9].

When $p(x) < 1$ on some subdomain of $\Omega$, the function $F(x, u)$ is not locally Lipschitz (for $x$ fixed), and we can use an approximation method to find a solution; see problem (12). We give more details in the proof of Theorem 2 below.

The existence of a positive solution of problem (1) for $u_t = 0$ is proved with the aid of the following result given in [16, Lemma 2.1].

**Lemma 1.** There exists a constant $c_\delta$, which depend only on $N$, such that for any $r, \delta > 0$ with $B_{r+2\delta} = B(0, r + 2\delta) \subset \Omega$,

$$S(t)\mathcal{X}_r \geq c_\delta \left( \frac{r}{r + \sqrt{t}} \right)^N \mathcal{X}_{r+\sqrt{t}}$$

for all $0 < t \leq \delta^2$.

**Proof of Theorem 2** Local existence. We use a standard approximation method, see for instance [20]. For every $\varepsilon > 0$, let $F_\varepsilon : \Omega \times [0, \infty) \to [0, \infty)$ be defined by

$$F_\varepsilon(x, s) = \begin{cases} f(s)^p(x) & \text{if } s \geq \varepsilon \text{ or } p(x) \geq 1, \\ f(\varepsilon)^{p(x)} f(s) & \text{if } 0 \leq s < \varepsilon \text{ and } p(x) < 1. \end{cases}$$

Note that since we are assuming $p^- < 1$ there exists a subdomain of $\Omega$ where $p(x) < 1$.

The function $F_\varepsilon(x, \cdot)$ is locally Lipschitz for every $x \in \Omega$. Let $u^\varepsilon$ be a solution of the problem

$$\begin{cases} u_t - \Delta u = h(t)F_\varepsilon(x, u) & \text{in } \Omega \times (0, T), \\ u = \varepsilon & \text{on } \partial \Omega \times (0, T), \\ u(0) = u_0 + \varepsilon & \text{in } \Omega, \end{cases}$$

defined on a maximal interval $[0, T^\varepsilon_{\text{max}}]$. We know that the blow-up alternative occurs, that is, either $T^\varepsilon_{\text{max}} = \infty$ or $T^\varepsilon_{\text{max}} < \infty$ and $\limsup_{t \to T^\varepsilon_{\text{max}}} \|u^\varepsilon(t)\|_\infty = \infty$. Since $u = \varepsilon$ is a subsolution to problem (12), by a comparison principle we conclude that $u^\varepsilon \geq \varepsilon$. Note that if $\varepsilon^* < \varepsilon^2$ then $F_\varepsilon^* (\cdot, u^\varepsilon) = F_{\varepsilon^*} (\cdot, u^\varepsilon)$ and $u^\varepsilon$ is a supersolution to problem (12) (with $\varepsilon = \varepsilon^1$). Hence, by a comparison principle we have $u_t^\varepsilon \leq u^\varepsilon^2$ in $[0, T^\varepsilon_{\text{max}}]$. Thus, we can define $u = \lim_{\varepsilon \to 0^+} u^\varepsilon$ on $[0, T_{\text{max}}^\varepsilon]$ for some $\varepsilon_0 > 0$.

**Global existence.** By the existence part we observe that it is sufficient to show that $T^\varepsilon_{\text{max}} = \infty$ for some $\varepsilon > 0$ sufficiently small. Since $u^\varepsilon$ is a solution of problem (12) and $u^\varepsilon(t) \geq \varepsilon$ we obtain

$$u^\varepsilon(t) = S(t)u_0 + \varepsilon + \int_0^t \int_\mathbb{R}^N h(\sigma)(f'(u^\varepsilon(\sigma)))^{p(x)}d\sigma,$$

for $t \in (0, T^\varepsilon_{\text{max}})$, Hence

$$\|u^\varepsilon(t)\|_\infty \leq \|u_0\|_\infty + \varepsilon + \int_0^t \int_\mathbb{R}^N h(\sigma)(f'(u^\varepsilon(\sigma)))^{p(x)}d\sigma.$$
Thus, 
\[ \| u'(t) \|_\infty \leq \| u_0 \|_\infty + \epsilon + \int_0^t h(\sigma) \max \left\{ [f(\| u'(\sigma) \|_\infty)]^{p'}, \min \left\{ \left[ f(\| u' \|_\infty) \right]^{p'}, 1 \right\} \right\} d\sigma. \]

Set 
\[ \Psi(t) = \| u_0 \|_\infty + \epsilon + \int_0^t h(\sigma) \max \left\{ [f(\| u'(\sigma) \|_\infty)]^{p'}, 1 \right\} d\sigma \]
and 
\[ g_1(t) = \max \left\{ \left[ f(\| u' \|_\infty) \right]^{p'}, 1 \right\}. \]

Then, \( \| u'(t) \|_\infty \leq \Psi(t) \) and 
\[ \Psi'(t) = h(t) \max \left\{ [f(\| u'(t) \|_\infty)]^{p'}, [f(\| u' \|_\infty)]^{p'} \right\} \leq h(t) \max \left\{ \left[ f(\| u' \|_\infty) \right]^{p'}, \left[ f(\Psi(t)) \right]^{p'} \right\}. \]

Fix \( t \in (0, \min\{\epsilon, T_{\max}^c\}) \) such that \( f(\tau) > 0 \) and condition [6] holds. Defining \( H(t) = \int_0^t d\sigma/g_1(\sigma), \) for \( t \geq \tau, \) we obtain \( (H \circ \Psi)'(t) \leq h(t) \) for \( t \in (0, T_{\max}^c). \) Thus, 
\[ \int_r^t \frac{d\sigma}{g_1(\sigma)} \leq \int_r^t \frac{d\sigma}{g_1(\sigma)} \leq \int_0^t h(\sigma)d\sigma + H(\Psi(0)), \]
for \( t \in (0, T_{\max}^c). \) From this inequality, we concluded that \( T_{\max}^c = \infty, \) since \( T_{\max}^c < \infty \) we have that \( \limsup_{t \to T_{\max}^c} \| u'(t) \|_\infty = +\infty, \) which contradicts condition [6].

**Existence of a positive solution.** (i) If \( u_0 \geq 0 \) and \( u_0 \neq 0, \) the result follows from [11] and the strong maximum principle, since \( u(t) \geq S(t)u_0 > 0 \) for \( t > 0. \)

Assume now that \( f(0) > 0. \) Without loss of generality we may assume that \( 0 \in \Omega \) and \( B_{r+\delta} \subset \Omega \) for some \( r > 0 \) and \( \delta > 0, \) where \( B_{r+\delta} = B_{r+\delta}(0). \) Since \( u_0 \) and \( u \) are nonnegative, and \( f \) is nondecreasing, from [11] we have 
\[ u(t) \geq \int_0^t h(\sigma)S(t-\sigma)\|u(\sigma)\|^{p(x)}d\sigma \geq \int_0^t h(\sigma)S(t-\sigma)f(0)^{p(x)}d\sigma \geq \min\{f(0)^{p'}, f(0)^{p'}\} \int_0^t h(\sigma)S(t-\sigma)\chi_r d\sigma, \]
where \( \chi_r = \chi_{B_r}. \) Let \( \phi_{1,r} > 0 \) be the first eigenfunction of the Laplacian operator on \( H^1_0(B_r) \) associated to the first eigenvalue \( \lambda_{1,r} > 0. \) Since \( \chi_r \geq C\phi_{1,r} \) for some constant \( C > 0, \) we have that \( S(t-\sigma)\chi_r \geq Ce^{-((t-\sigma)\lambda_{1,r})}\phi_{1,r}, \) and thus 
\[ u(t) \geq C \min\{f(0)^{p'}, f(0)^{p'}\}e^{-\lambda_{1,r}t}\phi_{1,r} \int_0^t h(\sigma)d\sigma > 0 \]
on \( B_r(0) \times (0, \infty). \)

Using again [11] it is possible to show that \( u(t) \geq S(t-s)u(s) \) for \( t \geq s > 0. \) Thus, since \( 0 \neq u(s) \geq 0, \) by the strong maximum principle, we have that \( u(t) > 0 \) for \( t \geq s > 0. \) Letting \( s \to 0 \) we get the result.

(ii) When \( u_0 = 0, \) from [14] we have that 
\[ \| u'(t) \|_\infty \leq H^{-1} \left( \int_0^t h(\sigma)d\sigma + H(\epsilon) \right), \]
for \( t \in (0, T_{\max}^c). \) Thus, \( f(\| u'(t) \|_\infty) \leq f(\| u'(t) \|_\infty) \leq 1 \) for \( t \in [0, T] \) with \( T = T(\epsilon_0) > 0 \) small and some \( \epsilon_0 > 0. \)

On the other hand, since \( p^- < 1, \) there exists a subdomain \( \Omega' \subset \Omega \) so that \( p(x) \leq \gamma < 1 \) for \( x \in \Omega'. \) Assume that \( 0 \in \Omega' \) and that the ball \( B_{r+\delta} \subset \Omega' \) for some \( r, \delta > 0. \) Since \( \{u'\} \) is nonincreasing in \( \epsilon \) we have that \( f(u'(t)) \leq f(u'(t)) \leq 1 \) for \( 0 < \epsilon \leq \epsilon_0 \) and \( 0 \leq t \leq T. \) Thus, from [15] 
\[ u'(t) \geq \int_0^t h(\sigma)S(t-\sigma) \left\{ [f(u'(\sigma))]^{p(x)} \chi_r \right\} d\sigma \geq \int_0^t h(\sigma)S(t-\sigma) \left\{ [f(u'(\sigma))]^{p'} \chi_r \right\} d\sigma. \]

(15)

It is well known that condition \( \int_0^t \frac{d\sigma}{f(\sigma)^{p'}} < \infty \) assures that the solution \( \mu \) of the Cauchy problem [8] is continuous and positive in some interval \([0, \tau_1]. \) Since \( f(0) = 0 \) and \( \mu(0) = 0, \) it is possible to choose \( \tau_2 \in (0, \tau_1) \) so that \( f(\mu(t)) \leq 1 \) for
\[ t \in (0, \tau_2). \] Thus by Lemma 1

\[
\begin{align*}
\int_0^t h(\sigma)S(t-\sigma)[f(\sigma)]^\nu \chi_\nu d\sigma \\
= \int_0^t h(\sigma)S(t-\sigma)[f(\mu(\sigma))]^\nu \chi_\nu d\sigma \\
= \int_0^t h(\sigma)[f(\mu(\sigma))]^\nu S(t-\sigma)\chi_\nu d\sigma \\
\geq \varepsilon_N \int_0^t h(\sigma)[f(\mu(\sigma))]^\nu \left( \frac{r}{\sqrt{\varepsilon_\sigma + r}} \right)^N \chi_{r+\sqrt{\varepsilon_\sigma}} d\sigma \\
\geq \frac{c_N}{N} \int_0^t h(\sigma)[f(\mu(\sigma))]^\nu \chi_\nu d\sigma \\
= \mu(t)\chi_\nu = u(t),
\end{align*}
\]

(16)

for \( 0 < t < \min\{\tau_2, r^2, \delta^2\} = \tau_3. \)

Subtracting (16) of (15)

\[
\begin{align*}
w(t) - u^*(t) \\
\leq \int_0^t h(\sigma)S(t-\sigma)[f(\nu)]^\nu - [f(u^*(\sigma))]^\nu \chi_\nu d\sigma \\
\leq \gamma \int_0^t h(\sigma)S(t-\sigma)[\theta f(\nu) + (1-\theta)f(u^*)]^\nu \nu (w - u^*)^\nu \chi_\nu d\sigma; \quad \theta \in (0, 1) \\
\leq \gamma \int_0^t h(\sigma)S(t-\sigma)[f(u^*)]^\nu (w - u^*)^\nu \chi_\nu d\sigma \\
\leq \gamma [f(\nu)]^\nu \int_0^t h(\sigma)S(t-\sigma)(w - u^*)^\nu \chi_\nu d\sigma,
\end{align*}
\]

where \( a_+ = \max\{a, 0\} \) for all \( a \in \mathbb{R}. \) Thus,

\[
[w(t) - u^*(t)]_+ \leq p^+[f(\nu)]^\nu \int_0^t h(\sigma)S(t-\sigma)(w - u^*)^\nu \chi_\nu d\sigma,
\]

and

\[
\|[w(t) - u^*(t)]_+ \chi_\nu\|_\infty \leq p^+[f(\nu)]^\nu \int_0^t h(\sigma)\|[w - u^*]_+ \chi_\nu\|_\infty d\sigma.
\]

By Gronwall’s inequality, \( (w(t) - u^*(t))_+ \chi_\nu = 0, \) for \( t \in (0, \tau_3), \) that is, \( w(t) \leq u^*(t) \) on the ball \( B_r \) for \( t \in (0, \tau_3). \) Letting, \( \varepsilon \to 0 \)

we conclude that \( w(t) \leq u(t) \) on \( B_r \times [0, \tau_3). \)

Since \( w \geq 0 \) and \( w \neq 0, \) we can argue as in case (i) to conclude that \( u \) is positive.

### 3 | LARGE INITIAL DATA

For the existence of blow up solutions we need of the following result established in Lemma 3.1.

**Lemma 2.** Let \( \eta \) be a positive measure in \( \Omega \subset \mathbb{R}^N \) such that \( \int_{\Omega} d\eta = 1 \) and let \( f \in L^{p^*}(\Omega, d\eta) \) with \( 1 \leq p^- \leq p(x) \leq p^+ \) for all \( x \in \Omega. \) Then

\[
\int_{\Omega} |f(x)|^{p(x)}d\eta(x) \geq 2^{-p^+} \min \left\{ \left( \int_{\Omega} |f(x)|d\eta(x) \right)^{p^-}, \left( \int_{\Omega} |f(x)|d\eta(x) \right)^{p^+} \right\}.
\]

**Proof of Theorem 4.** Let \( \varphi_1 > 0 \) be the first eigenvalue associated to the first eigenvalue \( \lambda_1 > 0 \) of the Laplacian operator on \( H^1_0(\Omega') \) such that \( \int_{\Omega'} \varphi_1 = 1. \) Let \( \Theta(t) = \int_{\Omega'} u(t)\varphi_1 dx. \) By Lemma 2 and Jensen’s inequality

\[
\begin{align*}
\Theta' + \lambda_1\Theta &\geq \int_{\Omega'} [f(u(t))]^{p(x)}\varphi_1 dx \\
&\geq 2^{-p^+} \min \left\{ \left( \int_{\Omega'} f(u(t))\varphi_1 \right)^{p^-}, \left( \int_{\Omega'} f(u(t))\varphi_1 \right)^{p^+} \right\} \\
&\geq 2^{-p^+} \min \left\{ \int_{\Theta(t)}^{f(u(t))} [f(u(t))]^{p^-}, \int_{\Theta(t)}^{f(u(t))} [f(u(t))]^{p^+} \right\} \\
&\geq 2^{-p^+} f'(\Theta(t)),
\end{align*}
\]

if \( f(\Theta(t)) \geq 1. \) Since \( f' \) is a convex function and \( \int_{\Omega'} \frac{d\eta}{f(x)^{p^+}} < \infty, \) we have that

\[
\lim_{r \to \infty} \frac{f'(r) - f'(0)}{r} = +\infty.
\]

Thus, there exists \( M > 0 \) such that \( \frac{1}{2^{-p^+}} f'(r) - \lambda_1 r > \frac{1}{2^{-p^+}} f'(r) \) for \( r > M. \) Therefore, \( \Theta' > \frac{1}{2^{-p^+}} f'(\Theta) \) whenever \( f(\Theta) \geq 1 \) and \( \Theta > M. \) Taking \( \Theta(0) \) such that \( \Theta(0) > \max\{\Theta, \alpha\}, \) where \( f(\alpha) > 1, \) we have that the solution blows up.
4 | BLOW UP AND GLOBAL EXISTENCE

Proof of Theorem 3 (i) We apply an argument similar to the one used in [24]. Consider \( \delta > 0 \) such that
\[
\delta < \frac{1}{\beta + 1},
\]
where \( \beta > 0 \) satisfies
\[
\int_{0}^{\infty} h(\sigma)F(\|S(\sigma)v_{0}\|_{\infty})d\sigma < \frac{\beta}{\beta + 1},
\]
for some \( v_{0} \in C_{0}(\Omega), v_{0} \geq 0, v_{0} \neq 0 \). Set \( u_{0} = \delta v_{0} \in C_{0}(\Omega) \) and define the sequence \( \{u^{k}\}_{k \geq 0} \) by \( u^{0}(t) = S(t)u_{0} \) and
\[
u^{k}(t) = S(t)u_{0} + \int_{0}^{t} S(t - \sigma)h(\sigma)[f(u^{k-1}(\sigma))]^{p(\sigma)} d\sigma,
\]
for \( k \in \mathbb{N} \) and \( t \geq 0 \).

We claim that
\[
n^{k}(t) \leq (1 + \beta)S(t)u_{0},
\]
for \( k \geq 0 \) and \( t > 0 \). To show this, we use induction on \( k \). Estimate (18) is clear for \( k = 0 \), thus we assume that (18) holds for \( k \). Note that condition (17) implies \( \|(1 + \beta)S(t)u_{0}\|_{\infty} \leq \|S(t)v_{0}\|_{\infty} \leq m \) for \( t > 0 \). Since \( F(0, m) \to [0, \infty) \) and \( f \) are nondecreasing functions, and \( sP(s) = \max\{f(s)^{p'}, f(s)^{p^{*}}\} \) for \( s \in (0, m) \) we have
\[
\begin{align*}
u^{k+1}(t) &= S(t)u_{0} + \int_{0}^{t} S(t - \sigma)h(\sigma)[f(u^{k}(\sigma))]^{p(\sigma)} d\sigma \\
&\leq S(t)u_{0} + \int_{0}^{t} h(\sigma)S(t - \sigma)[f((1 + \beta)S(\sigma)u_{0})]^{p(\sigma)} d\sigma \\
&\leq S(t)u_{0} + \int_{0}^{t} h(\sigma)S(t - \sigma)[f(S(\sigma)v_{0})]^{p(\sigma)} d\sigma \\
&\leq S(t)u_{0} + \int_{0}^{t} h(\sigma)S(t - \sigma)\max\{[f(S(\sigma)v_{0})]^{p'}, [f(S(\sigma)v_{0})]^{p^{*}}\} d\sigma \\
&= S(t)u_{0} + \int_{0}^{t} h(\sigma)S(t - \sigma)F(S(\sigma)v_{0})S(\sigma)v_{0} d\sigma \\
&\leq S(t)u_{0} + S(t)v_{0} \int_{0}^{t} h(\sigma)F(\|S(\sigma)v_{0}\|_{\infty}) d\sigma \\
&\leq S(t)u_{0} + (1 + \beta)S(t)u_{0} \frac{\beta}{\beta + 1} = (1 + \beta)S(t)u_{0}.
\end{align*}
\]
Hence, claim (18) holds for \( k + 1 \).

On the other hand, using again induction on \( k \), it is possible to that \( u^{k+1} \leq u^{k} \) for all \( k \in \mathbb{N} \). Thus, from monotone convergence theorem and estimate (18), we conclude that \( u = \lim u_{n} \) is a global solution of (1).

Proof of Theorem 3 (ii) We argue by contradiction and assume that there exists a global solution \( u \in C([0, \infty), C_{0}(\Omega)) \) of problem (1) with initial condition \( u_{0} \neq 0 \), that is
\[
u(t) = S(t)u_{0} + \int_{0}^{t} S(t - \sigma)h(\sigma)[f(u(\sigma))]^{p(\sigma)} d\sigma,
\]
for \( t \geq 0 \). Let \( 0 < t < s \). Then,

\[
S(s-t)u(t) = S(s)u_0 + \int_0^t h(\sigma)S(s-\sigma)[f(u(\sigma))]^{p(x)} \, d\sigma. \tag{19}
\]

Set \( \Phi(t) = S(s)u_0 + \int_0^t h(\sigma)S(s-\sigma)[f(u(\sigma))]^{p(x)} \, d\sigma \), for \( t \in [0,s] \). Then

\[
\Phi'(t) = h(t)S(s-t)[f(u(t))]^{p(x)},
\]

and from Lemma 2

\[
S(s-t)[f(u(t))]^{p(x)} = \int_{\Omega} K_\Omega(x,y;s-t)[f(u(t,y))]^{p(y)} \, dy \geq 2^{-p^*} \min \left\{ \frac{\|S(s-t)f(u(t))\|^{p^*}}{\delta(s-t,x)^{p^*-1}}, \frac{\|S(s-t)f(u(t))\|^{p^*}}{\delta(x-s,t)^{p^*-1}} \right\},
\]

where \( K_\Omega \) is the Dirichlet heat kernel on \( \Omega \) and \( a(s-t,x) = \int_{\Omega} K_\Omega(x,y;s-t) \, dy \). Since \( K_\Omega(x,y;s-t) \leq K_{\Omega}(x,y,s-t) \), we conclude that \( a(s-t,x) \leq 1 \). Thus, since \( p^* \geq 1 \), \( f \) is nondecreasing, inequality (9) and (19) we obtain

\[
\Phi'(t) \geq 2^{-p^*} h(t) \min \left\{ \|S(s-t)f(u(t))\|^{p^*}, \|S(s-t)f(u(t))\|^{p^*} \right\} \geq 2^{-p^*} h(t) \min \left\{ \|f(S(s-t)u(t))\|^{p^*}, \|f(S(s-t)u(t))\|^{p^*} \right\} = 2^{-p^*} h(t) \min \left\{ \|f(\Phi(t))\|^{p^*}, \|f(\Phi(t))\|^{p^*} \right\}. \tag{20}
\]

Set \( g_2(t) = \min\{\|f(t)\|^{p^*}, \|f(t)\|^{p^*} \} \) for all \( t \geq 0 \). Then, by (20) we have \( \Phi'(t) \geq 2^{-p^*} h(t)g_2(\Phi(t)) \). Defining \( G(t) = \int_t^{+\infty} \frac{d\sigma}{g_2(\sigma)} \) for \( t > 0 \) we obtain \( G(\Phi(t))' = -\frac{\Phi(t)}{g_2(\Phi(t))} \leq -2^{-p^*} h(t) \), for \( 0 < t < s \). Note that condition (5) guarantees that the function \( G \) is well defined.

Integrating, from 0 to \( s \), we obtain

\[
-G(S(s)u_0) \leq \int_{G(\Phi(s))}^{\int_{G(\Phi(0))}} \frac{d\sigma}{g_2(\sigma)} = G(\Phi(s)) - G(\Phi(0)) \leq -2^{-p^*} \int_0^s h(\sigma) \, d\sigma
\]

which is equivalent to \( 2^{-p^*} \int_0^s h(\sigma) \, d\sigma \leq G(\|S(s)u_0\|) \). Since \( G \) is decreasing and the left hand does not depend on \( x \), we conclude that

\[
2^{-p^*} \int_0^s h(\sigma) \, d\sigma \leq G(\|S(s)u_0\|_{\infty}),
\]

which contradicts condition (10).

5 | CONCLUSIONS

We deal with the parabolic problem \( u_t - \Delta u = h(t)F(x,u) \) in \( \Omega \times (0,T) \), where \( \Omega \) is a smooth domain (bounded or unbounded), \( F(x,u) = f(u)^{p(x)} \), with \( f \in C[0,\infty) \) non-decreasing, \( h \in C(0,\infty) \) and \( p \in C(\Omega) \) with \( 0 < p^- \leq p(x) \leq p^+ \). We assume that \( u_0 \in C_0(\Omega), u_0 \geq 0 \) and consider only non-negative solutions.

Under the assumption \( \int_{\Omega} \frac{d\sigma}{\max\{f(\sigma)^{p^*},f(\sigma)^{p^+}\}} = \infty \) we show that all the solutions non-negative are global. Moreover, we establish some conditions to get positive solutions in the case that \( u_0 = 0 \), extending the results of the classical case \( F(x,t) = f(t) \) with \( 0 < q < 1 \). When \( \int_{\Omega} \frac{d\sigma}{\min\{f(\sigma)^{p^*},f(\sigma)^{p^+}\}} < \infty \) we obtain blow up solutions and we use this result to improve a result established in [9].

Global existence is obtained for small initial data assuming that \( \int_0^\infty h(\sigma)F(\|S(\sigma)v_0\|_{\infty}) \, d\sigma < 1 \) for some \( v_0 \in C_0(\Omega), v_0 \neq 0 \), where \( F(\sigma) = \max\{f(\sigma)^{p^*},f(\sigma)^{p^+}\} / \sigma \) defined on a small interval \((0,m)\).

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Conflict of interest

This work does not have any conflicts of interest.

