State Estimators for Discrete-Time Descriptor Linear Systems with Mixed Uncertainties and State Constraints

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Abstract

This paper presents two novel mixed-uncertainty state estimators for discrete-time descriptor linear systems, namely linear time-varying mixed-uncertainty filter (LTVMF) and linear time-invariant mixed-uncertainty filter (LTIMF). The former estimator is based on the minimum-variance approach, from which quadratic and explicit formulations are derived and addressed to LTI and LTV systems. Both formulations incorporate the knowledge of state linear constraints, such as equality (in the descriptor form) and inequality, to mitigate precision and accuracy issues related to initialization and evolution of the state estimates. The explicit version is developed to reduce the computational burden of quadratic solvers, which is based on a particularity of the state inequality constraints. The LTIMF algorithm is based on the mixed $H_2 / H_\infty$ criterion motivated by performing low-cost computations. This speed benefit is originated from a reachability analysis involving constant design matrices. Both LTVMF and LTIMF algorithms solve state-estimation problems in which the uncertainties are combined to yield the so-called mixed-uncertainty vector, which is composed by set-bounded uncertainties, characterized by constrained zonotopes, and stochastic uncertainties, characterized by Gaussian random vectors. As mixed-uncertainty vectors imply biobjective optimization problems, we innovatively present multiobjective arguments to justify the choice of the solution on the Pareto-optimal front. According to these arguments, LTVMF is introduced with a cost normalization, which enables the combination of beyond minimum-variance approaches. Likewise, the mixed $H_2 / H_\infty$ criterion of LTIMF is introduced with slack variables to improve the quality of the state estimates. In order to discuss the advantages and drawbacks, the state estimators are tested in two numerical examples.
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KEYWORDS:
Mixed-uncertainty state estimators, Discrete-time linear systems, Gaussian random vectors, Constrained zonotopes, Pareto-optimal front, State constraints.

1 INTRODUCTION

State estimators are often designed assuming a unique type of uncertainty. For instance, Kalman and particle filters are based on random vectors, while set-based estimators are based on deterministic sets such as ellipsoids, intervals, zonotopes, or polytopes. A great challenge is to combine both approaches to better accord with some state-estimation problems. For instance, in a
double integrator process (trajectory estimation) is affected by white noise, and the measurements are coarsely quantized through rounding to nearest integers. In\textsuperscript{13}, the dynamics of a Van der Pol oscillator are affected by an unknown input but with known bounds, and the measurements are influenced by white noise. In\textsuperscript{14}, networked sensor systems are modeled with five uncertainties, being common considering two types: modeling uncertainty (including norm-bounded and random parameter uncertainties) and network-induced uncertainty. Accordingly with this topic, different solutions have been proposed\textsuperscript{15,16} to extend the Kalman filter to incorporate interval parameters under the Bayesian and minimum-variance perspectives. In\textsuperscript{9,10}, the stochastic and set-based approaches are combined to improve the accuracy of the algorithms. Likewise\textsuperscript{11,12}, merge stochastic and set-bounded uncertainties to design unified algorithms. Unlike the aforementioned approaches, there exist works\textsuperscript{17,18,19,20,21} that address robustness to stochastic filtering without assuming set-bounded noises. In this case, the robustness comes from the use of Huber’s function to bound both a posteriori covariance matrix and prediction error\textsuperscript{19} or of linear equality constraints to compensate the model mismatching\textsuperscript{14}.

When it comes to combining uncertainties in state estimators, three research fields are highlighted next. The first one consists of switching between (or combining) different algorithms given set-bounded uncertainties\textsuperscript{9,10,15,16}. When the switching mode is chosen, an algorithm is run at a time; when the combination mode is chosen, all algorithms are run at once. These methodologies are appealing to mitigate the weaknesses of the individual algorithms, related to computational issues, precision or accuracy. The second field consists of defining hybrid uncertainties from compact sets\textsuperscript{17,18,19,20,21}. In other words, the bounds of compact sets are also used to define random vectors, whose result is a hybrid method containing both confidence and robustness senses. This technique is useful to accelerate fault diagnosis and to formulate control strategies with probabilistic constraints that imply large attraction domains. In this case, the robustness of each method relies on the preset confidence level. Finally, the third field consists of merging different representations for uncertainties into a same estimator\textsuperscript{17,18,19,20,21}. Since both stochastic and set-bounded uncertainties are present in the third case, the state-estimation problem becomes even more challenging. In this work, we refer to this scenario as mixed uncertainty. Advantageously, these algorithms allow to combine different information about dynamics and measurements without the need of setting a unique type of uncertainty. For instance, Kalman-based filters would require that all uncertainties (including those non-Gaussian) were approximated by Gaussian random vectors (GRVs), while set-based algorithms would demand that all uncertainties (including those whose bounds are not exactly known) were represented by sets. Moreover, stochastic and set-theoretic tools can be employed to individually manipulate the corresponding uncertainties.

Due to closeness between shape and covariance matrices\textsuperscript{22}, the first mixing of uncertainties has involved ellipsoids and GRVs\textsuperscript{23,24}. Afterwards, new developments have been addressed to zonotopes to mitigate conservatism caused by Minkowski sum of ellipsoids\textsuperscript{25,26}. Recently, constrained zonotopes (CZs)\textsuperscript{27} have been applied to mixed-uncertainty filtering in\textsuperscript{12} to approximate the parameter mapping by a convex polytope, whose asymmetry is efficiently propagated over Minkowski sum and affine transformation. However, this work has not provided either a method to merge CZs with GRVs or a strategy to incorporate state equality and inequality constraints. In this paper, we propose novel mixed-uncertainty algorithms combining CZs and GRVs, with CZs being additionally used to intersect feasible sets.

Here, mixed uncertainties are treated in the interval observers framework, where state forecasts and measurement errors are combined by means of a Kalman-like gain to yield a posteriori state estimates\textsuperscript{28}. The justification is that such uncertainties are defined as the summation of real values. That is, the mixing is defined in the Euclidean space and, then, elements from sets and realizations from random vectors are taken into account. Therefore, mixed-uncertainty filters should comply with the superposition principle.

If two types of uncertainty are assumed for dynamical systems, then two different objectives must be pursued. As these objectives are conflicting, they characterize biobjective problems, which are commonly solved in terms of monobjective ones. Rigorously, optimization problems with more than one objective should be studied in light of the Pareto curve, which relates solutions from the variable space to the objective space. Our main interest is to obtain global optima, since they yield optimal costs on the Pareto-optimal front. So far, prior works on mixed uncertainty have solved convex monobjective problems using the weighted-sum approach, but the choice of the scalarization parameter is let free to the user. Here, we additionally employ the minimum-distance approach. By exploiting the convexity of our problems, we present a procedure to take a solution from the Pareto-optimal front. Innovatively, we provide a multiobjective interpretation for the solution choice.

The novel state estimators here proposed are called linear time-varying mixed-uncertainty filter (LTVMF) and linear time-invariant mixed-uncertainty filter (LIMF). The first one is based on a minimum-variance formulation and is addressed to LTV descriptor systems. The second algorithm is based on mixed $H_2/H_{\infty}$ formulation and is addressed for LTI descriptor systems. Recall that the descriptor representation is a general form of embedding linear equality constraints on the states, such that state space is a special case. LTVMF extends the (non-mixed) zonotopic estimator proposed in\textsuperscript{23} by: (i) replacing zonotopes with...
CZs, (ii) introducing GRVs, and (iii) enforcing state linear inequality constraints in addition to the equality constraints defined by the descriptor representation. We highlight that the existing mixed-uncertainty algorithms do not enforce state inequality constraints. In turn, LTIMF extends the algorithm of the descriptor representation. We highlight that the existing mixed-uncertainty algorithms by replacing zonotopes with CZs, (ii) introducing descriptor analysis, and (iii) merging CZs with GRVs. Similar to the extended algorithm, our novel filters employ additional design matrices (degrees of freedom) to potentiate the reduction of objective functions, resulting in better solutions than those with arbitrarily chosen matrices.

In order to better introduce our new methods, two state-estimation problems are formulated in Section 2. Preliminary results are reviewed and extended to accord with CZs in Section 3. The LTMF and LTIMF algorithms are presented in Sections 4 and 5, respectively, and illustrated in Section 6 using two numerical examples. Section 7 concludes the paper.

Notation

The sets of natural, positive natural, positive integer, and real numbers are, respectively, denoted as $\mathbb{N}$, $\mathbb{N}_+$, $\mathbb{Z}_+$, and $\mathbb{R}$. An $(n \times 1)$-dimensional vector and an $(n \times m)$-dimensional matrix are, respectively, denoted as $b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times m}$. An $(n \times m)$-dimensional zero matrix, an $(n \times m)$-dimensional one matrix, and an $(n \times n)$-dimensional identity matrix are, respectively, denoted as $0_{n \times n}$, $1_{n \times n}$, and $I_n$. The transpose, the absolute value, the rank, the vectorization, and the reshape of a matrix are, respectively, denoted as $(\cdot)^\top$, $|\cdot|$, rank$(\cdot)$, vec$(\cdot)$, and reshape$(\cdot)$. The Cholesky decomposition, the trace, and the inverse of a square matrix are, respectively, denoted as $(\cdot)^{1/2}$, tr$(\cdot)$, and $(\cdot)^{-1}$. The diagonal matrix obtained from a vector, as well as the vector obtained from the diagonal of a square matrix, is denoted as diag$(\cdot)$. The 1th row and the 2th column of a matrix are, respectively, denoted as $(\cdot)_{1,1}$ and $(\cdot)_{1,2}$. A suboptimal solution is denoted as $(\hat{\cdot})$. The optimal solution of an optimization problem is denoted as $(\hat{\cdot})$. A punctual estimate is denoted as $(\hat{\cdot})$. The matrices $D^i$ and $D^e$, as well as the vectors $d^i$ and $d^e$, compose, respectively, the state linear inequality and equality constraints. An $(n \times n)$ negative-definite matrix and an $(n \times n)$ positive-definite matrix are, respectively, denoted as $(\cdot) < 0_{n \times n}$ and $(\cdot) > 0_{n \times n}$.

2 | PROBLEM FORMULATION

Consider the discrete-time linear time-varying (LTV) descriptor system

\begin{align}
E_k x_k &= A_{k-1} x_{k-1} + B_{k-1} u_{k-1} + w_{k-1}, \\
y_k &= C_k x_k + v_k,
\end{align}

where $E_k \in \mathbb{R}^{n \times n}$, $A_{k-1} \in \mathbb{R}^{n \times n}$, $B_{k-1} \in \mathbb{R}^{n \times p}$, and $C_k \in \mathbb{R}^{m \times n}$ are known matrices, $u_{k-1} \in \mathbb{R}^p$ is the known input vector, $y_k \in \mathbb{R}^m$ is the measured output vector, and $x_k \in \mathbb{R}^n$ is the state vector to be estimated over $k \in \mathbb{Z}_+$. We assume that

\[
\text{rank} \left( \begin{bmatrix} E_k & C_k \end{bmatrix} \right) = n.
\]

Additionally, we assume that the states $x_k$ satisfy

\[
D_k^i x_k \leq d_k^i,
\]

where $D_k^i \in \mathbb{R}^{n \times n}$ and $d_k^i \in \mathbb{R}^n$ compose inequality state constraints. No assumption about the rank of $D_k^i$ is required. In Figure 1 we illustrate some possible sets originated from (5).

Although it is not explicit, the descriptor representation is an appealing form to couple dynamics (or difference equations) with algebraic constraints (or conservation laws). Then, in order to cover state-space cases with equality constraints $D_k^e x_k = d_k^e$, we suggest Subsection 2.1 to obtain an equivalent descriptor representation.

Boldfaced capital letters characterize random vectors, while calligraphic letters characterize set such as CZs. Also, the terms defined by $c$, $z$, and $g$ denote center, an element of a CZ, and a realization of a GRV, respectively.

The terms $w_{k-1} = z_{w_{k-1}}^w + g_{w_{k-1}}^w \in \mathbb{R}^n$ and $v_k = z_{v_k}^\epsilon + g_{v_k}^\epsilon \in \mathbb{R}^m$ correspond to the process and measurement noises, respectively. The set-bounded parcels $z_{w_{k-1}}^w$ and $z_{v_k}^\epsilon$ are characterized by the zero-center CZs $W_{k-1}$ and $V_k$, respectively, whose generator matrices $G_{w_{k-1}}^w$ and $G_k^\epsilon$ have full rank. The stochastic parcels $g_{w_{k-1}}^w$ and $g_{v_k}^\epsilon$ are realizations of independent stochastic processes, and characterized by the zero-mean GRVs $W_{k-1}$ and $V_k$, respectively, with positive-definite covariance matrices $Q_{k-1} > 0_{n \times n}$ and $R_k > 0_{m \times m}$.

In turn, the initial state vector $x_0 = c_0^w + z_0^w + g_0^\epsilon$ is considered mixed, where $c_0^w$ is its center, $z_0^w$ is the set-bounded term represented by the zero-center CZ $X_0$ whose generator matrix satisfies $\text{rank} \left( G_0^w \right) \leq n$, and $g_0^\epsilon$ is the stochastic term represented
by the zero-mean GRV $X_0$ with covariance matrix $P_0^{ss} > 0_{nx}$. Moreover, we assume that $x_k^0$ is uncorrelated to $x_{k-1}^w$ and $x_k^x$. Therefore, the state vector $x_k$ is given by the combination of $c_k^x$, $a_k$, and $X_k$.

Two state-estimation problems are defined next. The first one is based on minimum-variance formulation, whose solution is given by LTVMF, while the second one is based on a mixed $H_2/H_{\infty}$ formulation, yielding LTIMF.

**Problem 1.** Consider the dynamical system in (1)-(2), for which the input $u_{k-1}$, the measurements $y_{k-1}$ and $y_k$, and the inequality constraints (5) are known. Given GRVs $W_{k-1}$, $V_k$, and $X_0$, CZs $W_{k-1}$, $V_k$, and $X_0$, and the center estimate $\hat{c}_0^x$, the goal is to determine the a posteriori state estimates given by $\hat{c}_k^x$, $\hat{a}_k$, and $\hat{X}_k$, for $k \in \mathbb{Z}_+$, based on minimum-variance criterion.

**Problem 2.** Assume that the dynamical system in (1)-(2) is LTI. Consider that the input $u_{k-1}$, the measurements $y_{k-1}$ and $y_k$, and the inequality constraints (5) are known. Given GRVs $W$, $V$, and $X_0$, CZs $W$, $V$, and $X_0$, and the center estimate $\hat{c}_0^x$, the goal is to determine the a posteriori state estimates given by $\hat{c}_k^x$, $\hat{a}_k$, and $\hat{X}_k$, for $k \in \mathbb{Z}_+$, based on a mixed $H_2/H_{\infty}$ criterion.

### 3 | PRELIMINARIES

In this section, some concepts and results about GRV, CZ, and the mixed-uncertainty vector are revisited and extended, when necessary.

#### 3.1 | Random Vectors

Let $X$ be an $n$-dimensional random vector. The mean and covariance matrix of $X$ are given by $\hat{x} = E[X]$ and $P^{xx} = \text{cov}(X, X) \triangleq E[(X - \hat{x})(X - \hat{x})^\top]$, respectively, where $E[\cdot]$ is the expected value operator. A GRV $X$ is characterized by its Gaussian probability density function $p(x)$, which is completely defined by the mean $\hat{x}$ and covariance matrix $P^{xx}$. Therefore, a GRV $X$ can be abbreviated by $X \sim \mathcal{N}(\hat{x}, P^{xx})$.

A random variable with chi-square distribution for $n$ degrees of freedom is defined as

$$X^x \triangleq (X - \hat{x})^\top P^{xx}^{-1}(X - \hat{x}),$$

where realization $x^x$ satisfies $x^x \geq 0$. The chi-square random variable $X^x$ can be abbreviated by $X^x \sim \chi^2(n)$, where $E[X^x] = n$ and $\text{cov}(X^x, X^x) = 2n$.

The cumulative distribution function $c(x)$ of a given probability density function $p(x)$ is defined as

$$c(x) \triangleq \int_{-\infty}^{x} \ldots \int_{-\infty}^{x_j} p(\theta_1, \ldots, \theta_n) d\theta_1 \ldots d\theta_n,$$

with $0 \leq c(x) \leq 1$, $\forall x \in \mathbb{R}^n$.

The affine transformation and sum operations of uncorrelated GRVs are computed as

$$LX + m \sim \mathcal{N}(L\hat{x} + m, LP^{xx}L^\top),$$

$$X + W \sim \mathcal{N}(\hat{x} + \hat{w}, P^{xx} + P^{ww}),$$

where $L \in \mathbb{R}^{b \times n}$, $m \in \mathbb{R}^b$, and $W \sim \mathcal{N}(\hat{w}, P^{ww})$. 

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**FIGURE 1** Illustrations of sets in 2D. A halfspace is an unbounded set with one of its directions bounded by a hyperplane. The nonempty intersection between two parallel halfspaces yields a strip, but the most general intersection yields a polyhedron. When intersection yields compact sets, polyhedra are called polytopes.
3.2 Constrained Zonotopes

An n-dimensional interval $[x] \subset \mathbb{R}^n$ is defined as a box $[x] \triangleq [\xi^L, \xi^U]$, where $\xi^L \in \mathbb{R}^n$ and $\xi^U \in \mathbb{R}^n$ are the known lower and upper boundaries, respectively. An n-dimensional unitary box $B^n \triangleq [-1, 1]^n$ is defined as $n$ unitary intervals. The midpoint and radius of a box $[x]$ are defined as $\text{mid}([x]) \triangleq \frac{1}{2} (\xi^L + \xi^U)$ and $\text{rad}([x]) \triangleq \frac{1}{2} (\xi^U - \xi^L)$, respectively.

The Minkowski sum of two sets is defined as $Z_1 \oplus Z_2 \triangleq \{ z_1 + z_2 : z_1 \in Z_1, z_2 \in Z_2 \}$. Given a set $Z \subset \mathbb{R}^n$ and a matrix $L \in \mathbb{R}^{m \times n}$, the linear transformation of $Z$ is defined as $LZ = \{ Lz : z \in Z \}$.

The constrained unitary box is defined as $B(A^n, b^n) \triangleq \{ \xi : \xi \in B^n, A^n \xi = b^n \}$, where $A^n \in \mathbb{R}^{n \times n}$ and $b^n \in \mathbb{R}^n$ represent linear equality constraints on the unitary box $B^n$. Thereby, the CZ $X$ is defined as the affine transformation of the constrained unitary box $B(A^n, b^n)$, that is,

$$X \triangleq \{ G^x, c^x, A^n, b^n \} = G^n B(A^n, b^n) \oplus c^n,$$

where $G^x \in \mathbb{R}^{n \times n}$ and $c^x \in \mathbb{R}^n$ are called generator matrix and center, respectively. Each column of $G^x$ is called generator and denoted as $G^x_{:, j}$, for $j = 1, \ldots, n_g$.

Zonotopes are special cases of CZs with no constraints (given by $A^n$ and $b^n$) on the unitary box $B^n$, then $B(A^n, b^n) \subset B^n$. Using CZs rather than zonotopes over operations leads to smaller sets, since CZs allow to exactly propagate asymmetric sets with the similar computational advantage of zonotopes. Here, to rewrite a zonotope as a CZ, we have $X = \{ G^x, c^x \}$. If the generator matrix $G^x$ is square and invertible, then the zonotope is called parallelotope.

Let $m \in \mathbb{R}^n, L \in \mathbb{R}^{m \times n}, \mathcal{X} = \{ G^x, c^x, A^n, b^n \} \subset \mathbb{R}^n, \mathcal{W} = \{ G^w, c^w, A^w, b^w \} \subset \mathbb{R}^p, \text{and } M \in \mathbb{R}^{p \times n}$. The affine transformation, sum, and generalized intersection operations of CZs are computed as

$$LX \oplus m = \{ LG^x, (LC^x + m), A^n, b^n \},$$

$$X \oplus \mathcal{Y} = \left\{ \begin{array}{ccc} [G^x, G^y], (c^x + c^y), & [A^n, 0_{n \times n_g}] & [b^n] \\ 0_{n_g \times n_g} & A^y & \end{array} \right\},$$

$$X \cap M \mathcal{W} = \left\{ \begin{array}{ccc} [G^x, 0_{n_g \times n_g}], c^x, & [A^n, 0_{n \times n_g}] & [b^n] \\ 0_{n_g \times n_g} & A^w & \end{array} \right\} + \left\{ \begin{array}{ccc} MG^x & -G^w & \end{array} \right\},$$

where $M$ is a matrix that relates the elements from $X$ to the elements from $\mathcal{W}$.

Since the number of constraints and generators of a CZ increases over operations, such as the Minkowski sum $\mathcal{X} \oplus \mathcal{Y}$ and intersection $\mathcal{X} \cap \mathcal{W}$, it is necessary to define a procedure to limit such growth. The order-reduction algorithm for CZs is then presented in[23], and abbreviated here as

$$\mathcal{X}^l = \text{CZR}(\mathcal{X}, \varphi_c, \varphi_g),$$

where $\varphi_c$ and $\varphi_g$ are the desired maximum constraint and generator orders, respectively. The reduction algorithm for CZs requires another algorithm to reduce the number of generators. For completeness, we use the zonotope order reduction proposed by[23].

In order to obtain the smallest box $\square \mathcal{X}$ containing the CZ $\mathcal{X}$, we can use linear programming solver,[44] property 1. The box $\square \mathcal{X}$ is called interval hull, and it allows to obtain the following equivalence for a series of sets $\mathcal{X}_i$, $i = 1, \ldots, n,[10]$

$$\square \left( \bigoplus_{i=1}^n \mathcal{X}_i \right) = \bigoplus_{i=1}^n \square \mathcal{X}_i,$$

where

$$\bigoplus_{i=1}^n \mathcal{X}_i \triangleq \mathcal{X}_1 \oplus \cdots \oplus \mathcal{X}_n.$$

According to[23], the constrained unitary box $B(A^n, b^n)$ can be exactly computed applying the equality linear constraints $A^n$ and $b^n$ to a box smaller than the unitary box. To achieve that, the constrained unitary box $B(A^n, b^n)$ is rewritten in terms of its interval hull $\square B(A^n, b^n)$ such that $B(A^n, b^n) \subseteq \square B(A^n, b^n) \subseteq B^n$. To obtain $\square B(A^n, b^n)$, we can use the rescaling step proposed by[23], solving linear programs or an iterative method.
FIGURE 2 Diagram to illustrate the mixed-uncertainty vector in 2D. The vector $c^x$ ($\bullet$) represents the center of the mixed-uncertainty vector $x$. The realization $g^x$ ($\ast$) of a GRV (red graph) has a probability $p(g^x)$ related to; for brevity, the realizations of this GRV are illustrated into a box containing a larger 99.9% confidence level. The element $z^x$ ($\times$) corresponds to a point into a CZ (blue graph). The summation of these parcels yields the mixed-uncertainty vector $x$. This example illustrates a case in which the center of CZ does not belong to the own set because $G^x \xi \neq 0_{2 \times 1}$ for $\xi \in B(A^x, b^x)$ ³. For this reason, applications as linearization and state feedback may demand a punctual estimate more suitable than $c^x$ to improve reliability and accuracy.

3.3 Mixed-Uncertainty Vector

In ²⁴ some results related to the mixed-uncertainty vector are presented with zonotopes. In this subsection, we extend those results to CZs. In order to combine a GRV and a CZ, the mixed-uncertainty vector is defined next and is illustrated in Figure 2.

**Definition 1.** Consider the CZ $X_k = \{G^x_k, 0_{n \times 1}, A^x_k, b^x_k\}$ and the GRV $X_k \sim \mathcal{N}(0_{n \times 1}, P^{xx})$. Given the elements $z^x_k$ from $X_k$ and realizations $g^x_k$ from $X_k$, the mixed-uncertainty vector $x_k$ is defined as

$$x_k \triangleq c^x_k + z^x_k + g^x_k,$$

where $c^x_k \in \mathbb{R}^n$ is the mixed center of $x_k$.

**Remark 1.** In Definition 1, the center $c^x_k$ is differed from the terms $z^x_k$ and $g^x_k$ to make explicit its influence during the design of state estimators in Sections 4 and 5.

As the support of GRVs is unbounded, we review the confidence ellipsoid to approximate GRVs by a bounded set.

**Definition 2.** Consider the GRV $X \sim \mathcal{N}(\hat{x}, P^{xx})$, where $P^{xx} > 0_{n \times n}$, and the significance level, or type I error, $\alpha \in [0, 1]$. The confidence ellipsoid is defined as

$$\mathcal{E} \triangleq \{x \in \mathbb{R}^n : (x - \hat{x})^T(\zeta P^{xx})^{-1}(x - \hat{x}) \leq 1\},$$

where $\zeta \geq 0$ is the greatest value for the chi-square random variable with $n$ degrees of freedom ⁴, taken from the cumulative distribution function ⁵ with confidence level of $(1 - \alpha)$, such that the probability $p(x \in \mathcal{E}) = (1 - \alpha)$ is satisfied.

Next, we define the confidence constrained zonotope, in order to merge a CZ with a GRV, given a significance level $\alpha$, in a CZ. After, we define another confidence set, called confidence box. These sets are useful to sketch states and detect faults over time, but we highlight that CZs are more precise than boxes whereas boxes have lower complexity. For completeness, these definitions are exemplified in Example 1.
Definition 3. Let the mixed center \( c^x \), the CZ \( \mathcal{X} \), and the GRV \( \mathcal{X} \) with covariance \( P^{xx} \) be characterizations of the mixed-uncertainty vector \( x \). Let also the confidence ellipsoid \( \mathcal{E} \) be defined as (13). Then, the confidence CZ \( \mathcal{X}^a \) is defined as
\[
\mathcal{X}^a \triangleq c^x \oplus \mathcal{X} \oplus P,
\]
(14)
where \( P = \{G^x, 0_{nx1}\} \) is the smallest parallelotope containing \( \mathcal{E} \), whose generator matrix is computed as
\[
G^x = (\zeta P^{xx})^{1/2},
\]
(15)
with \((\zeta P^{xx})^{1/2}\) being the lower triangular matrix obtained by Cholesky decomposition, and \( \zeta \geq 0 \) being defined in (13).

Definition 4. Let the mixed center \( c^x \), the CZ \( \mathcal{X} \), and the GRV \( \mathcal{X} \) with covariance \( P^{xx} \) be characterizations of the mixed-uncertainty vector \( x \). Let also the confidence ellipsoid \( \mathcal{E} \) be defined as (13). Then, the confidence box \( I^a \) is defined as
\[
I^a \triangleq c^x \oplus \square\mathcal{X} \oplus \square\mathcal{E},
\]
(16)
where \( \square\mathcal{X} = [\zeta^{Lx}_i, \zeta^{Ux}_i] \) is the interval hull of \( \mathcal{X} \), whose bounds are computed as
\[
\zeta^{Lx}_i = \min \left\{ G^x_i, \xi : \xi \in B(A^x, b^x) \right\}, \quad i = 1, \ldots, n,
\]
(17)
\[
\zeta^{Ux}_i = \max \left\{ G^x_i, \xi : \xi \in B(A^x, b^x) \right\}, \quad i = 1, \ldots, n,
\]
(18)
\( G^x_i \) represents the \( i \)th row of \( G^x \), \( \square\mathcal{E} = [\zeta^{Lg}, \zeta^{Ug}] \) is the interval hull of \( \mathcal{E} \), whose bounds are computed as
\[
\zeta^{Lg} = -r^g, \quad \zeta^{Ug} = r^g,
\]
(19)
\( r^g = \sqrt{\zeta \text{diag}(P^{xx})} \),
(20)
with \( \zeta \geq 0 \) being defined in (13), and \( \text{diag}(P^{xx}) \) being a vector containing the diagonal elements of the matrix \( P^{xx} \).

Example 1. Consider a mixed-uncertainty vector \( x \) characterized by center \( c^x = [1, 3]^T \), CZ \( \mathcal{X} = \left\{ \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 0.2_{x1}, 0.5, 0.1, 1.5 \end{bmatrix}, 1 \right\} \), and GRV \( \mathcal{X} \sim \mathcal{N}(0_{2x1}, P^{xx}) \) with covariance \( P^{xx} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \). Consider also a significance level \( \alpha = 0.0027 \). Next, we show how Definitions 3 and 4 use such information to yield the sets sketched in Figure 3. Given \( X \) and \( \alpha \), as in Definition 3 compute the confidence ellipsoid \( \mathcal{E} \) and, thereby, the parallelotope \( P \), whose Minkowski sum with the mixed center \( c^x \) and the CZ \( \mathcal{X} \) yields the confidence CZ \( \mathcal{X}^a \) in (14). Definition 4 computes the separate interval hull of \( \mathcal{E} \) and \( \mathcal{X} \), whose Minkowski sum with the center \( c^x \) yields the confidence box \( I^a \) in (16). This result is more precise than computing the direct interval hull of \( \mathcal{X}^a \) given by Definition 3 since \( \square\mathcal{E} \leq \square P \).
Remark 2. The metric C3 proposed by\cite{33} can be used to calculate a punctual estimate $\hat{x} \in \mathcal{X}^a$, where $\mathcal{X}^a$ is the confidence CZ with center $c^a$ given by Definition\cite{3} Specifically, we decide $\hat{x} = c^a$, if $c^a \in \mathcal{X}^a$; otherwise, $\hat{x}$ is chosen as the closest point in $\mathcal{X}^a$ to $c^a$ (in 1-norm sense).

Remark 3. In Definition\cite{4} we consider the confidence box without starting from $\mathcal{X}^a$ given by Definition\cite{3} This is motivated by the fact that $[\hat{x}^{\alpha}]$ can be analytically computed and the outerapproximation of ellipsoids by parallelotopes yields larger interval hulls. Then, besides reducing the computational burden by solving smaller linear programs and by avoiding Cholesky decomposition, we obtain smaller interval hulls with Definition\cite{4}. Further complexity reduction can be obtained if $\mathcal{X}$ is a zonotope with center $\hat{x}$ and generator matrix $G^\alpha$. In this case, its interval hull is analytical and readily computed as $[\xi^{\alpha L}, \xi^{\alpha U}] = [\hat{x} - r^\alpha, \hat{x} + r^\alpha]$, where $r^\alpha = |G^\alpha| \mathbf{1}_{n \times 1}$. According to this latter methodology, we can also reduce the computational burden of $[\hat{x}^{\alpha}]$ for CZs at the cost of slightly enlarging the exact solution. First, we eliminate all constraints of a CZ $\mathcal{X}$ by applying CZR, yielding a zonotope $\hat{\mathcal{X}} \supset \mathcal{X}$. After, we use analytical expressions to obtain the interval hull of $\hat{\mathcal{X}}$.

4 | MIXED-UNCERTAINTY FILTER FOR LTV SYSTEMS

4.1 | Estimator Design

4.1.1 | Original Problem

Consider the descriptor LTV system given by (1)-(2). As rank $\left[ \begin{matrix} E_k \\ C_k \end{matrix} \right] = n$, there exists a full-rank matrix $[T_k \ N_k]$ such that

$$T_k E_k + N_k C_k = I_n,$$  \hfill (22)

where $T_k \in \mathbb{R}^{m \times n}$ and $N_k \in \mathbb{R}^{m \times n}$ are design matrices, and $I_n$ is the $(n \times n)$-dimensional identity matrix. Post-multiplying (22) by $x_k$ and using (1)-(2), we obtain

$$x_k = T_k A_{k-1} x_{k-1} + T_k B_{k-1} u_{k-1} + N_k y_k + T_k z^w_{k-1} + T_k g^w_{k-1} - N_k z^y_k - N_k g^y_k.$$  \hfill (23)

Besides adding degrees of freedom by means of $T_k$ and $N_k$\cite{29,30}, the equality (22) is useful to obtain a non-descriptor representation for (1)-(2). According to (2) and (23), we formulate the following center estimate of the so-called LTVMF:

$$\hat{\epsilon}^x_k = T_k A_{k-1} \hat{\epsilon}^x_{k-1} + T_k B_{k-1} u_{k-1} + K_{k-1} (y_{k-1} - C_{k-1} \hat{\epsilon}^x_{k-1}) + N_k y_k.$$  \hfill (24)

The matrix $K$ in (24) is presented at $k - 1$ because it takes $y_{k-1}$ into account, but LTVMF is categorized as filter because matrices $T_k$ and $N_k$ introduce $y_k$. Specifically, if we choose $T_k = I_n$ and $N_k = 0_{n \times m}$, then the estimator would be categorized as predictor.

Next, we present two lemmas. The first addresses the state estimation error, while the second characterizes the mixed state vector $x_k$.

Lemma 1. Let $\hat{\epsilon}^x$ given by (24) be the state estimate for the system (1)-(2). Given Definition\cite{1} and the estimation error $e_k \triangleq x_k - \hat{\epsilon}^x_k$, we obtain the following constrained zonotopic $\hat{\epsilon}^x_k$ and Gaussian $\hat{g}^x_k$ terms:

$$\hat{\epsilon}^x_k = \tilde{A}_{k-1} \hat{\epsilon}^x_{k-1} + T_k z^w_{k-1} - K_{k-1} z^y_{k-1} - N_k z^y_k,$$  \hfill (25)

$$\hat{g}^x_k = \tilde{A}_{k-1} \hat{g}^x_{k-1} + T_k g^w_{k-1} - K_{k-1} g^y_{k-1} - N_k g^y_k,$$  \hfill (26)

where

$$\tilde{A}_{k-1} \triangleq T_k A_{k-1} - K_{k-1} C_{k-1}.$$  \hfill (27)

Proof. According to $\hat{\epsilon}^x_k$ in (24), the estimation error is given by

$$e_k = x_k - \hat{\epsilon}^x_k = T_k A_{k-1} (x_{k-1} - \hat{\epsilon}^x_{k-1}) + T_k z^w_{k-1} + T_k g^w_{k-1} - N_k z^y_k - N_k g^y_k - K_{k-1} y_{k-1} + K_{k-1} C_{k-1} \hat{\epsilon}^x_{k-1},$$

where $x_k$ is given by (23). After making explicit $y_{k-1} = C_{k-1} x_{k-1} + z^y_{k-1} + g^y_{k-1}$, we obtain

$$e_k = (T_k A_{k-1} - K_{k-1} C_{k-1}) (x_{k-1} - \hat{\epsilon}^x_{k-1}) + T_k z^w_{k-1} + T_k g^w_{k-1} - K_{k-1} z^y_{k-1} - K_{k-1} g^y_{k-1} + K_{k-1} C_{k-1} \hat{\epsilon}^x_{k-1},$$
By considering the trace criterion, a new challenge arises about approximating CZs by zonotopes. This approximation is required to keep algebraic connections between covariance and covariation matrices, and thereby, to formulate quadratic programs. According to 23, we identify three possible strategies: (i) either to disregard the equality constraints given by $\hat{A}_k$ and $\hat{b}_k^z$, (ii) or to apply CZR to completely remove equality constraints, or (iii) to replace $B(\hat{A}_k, \hat{b}_k^z)$ by its interval hull. In order to verify implications of each choice, we present the following illustrative example.
Example 2. Consider the CZ shown in Figure 4. This set is overapproximated by zonotopes through three methods, from which: (i) returns the most conservative set; (ii) returns the most precise set; and (iii) returns the intermediate precision set.

According to Example 2, we realize that (ii) would be the most appealing strategy to approximate the CZ $\hat{\mathcal{X}}_k$ given by (28), by a zonotope $\mathcal{X}_k$, since more precise sets could be obtained. However, (ii) would require the knowledge of the design matrices $T_k$, $N_k$, and $K_{k-1}$ (so far unknown because the zonotopic approximation is here a requirement to address optimal solutions to such matrices) to execute the order reduction, then it is disregarded. Among the remaining options, (iii) is indeed the most recommended methodology, because affine operations do not alter the original equality constraints of CZs (known).

Then, the following proposition is presented to make minimum-trace formulations involving CZs.

Proposition 1. Let $[\zeta^L, \zeta^U] \triangleq \Box B(\hat{\mathcal{X}}_k, \hat{b}_k)$ be the interval hull of the constrained unitary box $B(\hat{\mathcal{X}}_k, \hat{b}_k)$ of the CZ $\mathcal{X}_k$, given by (28), where

$$[\zeta^L, \zeta^U] = \begin{bmatrix} \zeta^L & \zeta^U \end{bmatrix} = \begin{bmatrix} \Box B(\hat{\mathcal{X}}_{k-1}, \hat{b}_{k-1}) & \Box B(\hat{\mathcal{X}}_{k-1}, \hat{b}_{k-1}) & \Box B(\hat{\mathcal{X}}_{k-1}, \hat{b}_{k-1}) & \Box B(\hat{\mathcal{X}}_{k-1}, \hat{b}_{k-1}) \end{bmatrix}. \tag{36}$$

By employing the enclosure (36), the zonotope $\mathcal{X}_k \supset \hat{\mathcal{X}}_k$ is obtained as follows:

$$\mathcal{X}_k = \{ \tilde{G}_k \tilde{c}_k \} = \tilde{G}_k [\zeta^L, \zeta^U], \tag{37}$$

where

$$\tilde{G}_k = \frac{1}{2} \hat{G}_k \text{diag} (\zeta^U - \zeta^L) = \begin{bmatrix} \hat{A}_{k-1} \hat{G}_{k-1}^x & T_k \hat{G}_{k-1}^w & -K_{k-1} \hat{G}_{k-1}^v - N_k \hat{G}_{k-1}^v \end{bmatrix}, \tag{38}$$

$$\tilde{c}_k = \frac{1}{2} \hat{c}_k \hat{G}_k \text{diag} (\zeta^L + \zeta^U), \tag{39}$$

with $\hat{G}_k = \frac{1}{2} \hat{G}_k \text{diag} (\zeta^U - \zeta^L)$, $\hat{G}_{k-1}^x$ $\hat{G}_{k-1}^w$ $\hat{G}_{k-1}^v$ $\hat{c}_k$ and $\hat{G}_k$ $\hat{c}_k$ $\hat{G}_{k-1}^v$ $\hat{G}_{k-1}^v$ $\hat{G}_{k-1}^v$ $\hat{G}_{k-1}^v$ $\hat{G}_{k-1}^v$ $\hat{G}_{k-1}^v$.

Proof. Due to the block-diagonal characteristic of $\hat{A}_k$, given by (31), each inner constraint block can be individually evaluated to yield the box $[\zeta^L, \zeta^U]$ in (36) Algorithm. To obtain the zonotope $\mathcal{X}_k$ in (37), we rewrite the box $[\zeta^L, \zeta^U]$ in terms of affine arithmetic, where the generator matrix $\hat{G}_k$ in (38) and the center $\tilde{c}_k$ in (39) are made explicit in the standard format of (7) with null constraints. Since $B(\hat{\mathcal{X}}_k, \hat{b}_k) \subset \Box B(\hat{\mathcal{X}}_k, \hat{b}_k)$, we ensure that $\mathcal{X}_k \subset \mathcal{X}_k$. \hfill \blacksquare

Next, we present how to compute the design matrices $T_k$, $N_k$, and $K_{k-1}$ given a quadratic optimization problem involving the cost $J_k$ given by (33), in which state equality and inequality constraints are transmitted to the desired matrices.
Theorem 1. Consider the cost function \( J_k^{\text{eg}} \), the known weight parameter \( \eta \in [0, 1] \), and the corresponding inequality \( (3) \) and equality \( (22) \) constraints. Let \( z = \text{vec} \left( [I_k \ N_k \ K_{k-1}] \right) \in \mathbb{R}^{(n^2+2nm)} \) be the variable vector, where \( \text{vec}(\cdot) \) is the vectorization operator that stacks up columns of a matrix starting from the first one to the last one, and \( M \in \mathbb{R}^{(n^2+2nm)\times(n^2+2nm)} \) be the vectorized transpose matrix such that \( \left[ \text{vec} \left( T_k^T \right)^T \text{vec} \left( N_k^T \right)^T \text{vec} \left( K_{k-1}^T \right)^T \right]^T = Mz \). Let \( \otimes \) be the Kronecker product, and reshape \( (z, n, m) \) be the operator that takes \( nm \) specific elements of \( z \) to build a matrix with dimension \( n \times m \). Then, minimizing \( (33) \) subject to \( (3) \) and \( (22) \) is equivalent to solving the quadratic optimization problem

\[
\begin{align*}
\min & \quad \frac{1}{2} z^T H z \\
\text{s.t.} & \quad \tilde{A} z = \tilde{b}, \quad \tilde{D} z \preceq d_k^1,
\end{align*}
\]

where

\[
H = 2M^T \tilde{P} M, \\
\tilde{P} = \begin{bmatrix}
I_n \otimes P_{k|k-1}^{eg} & 0_{n^2 \times nm} & -I_n \otimes A_{k-1} P_{k-1}^{eg} C_k^T \\
0_{nm \times mm} & I_n \otimes R_{k-1}^{eg} & 0_{nm \times mm} \\
-I_n \otimes C_{k-1} P_{k-1}^{eg} A_k^T & 0_{nm \times mm} & I_n \otimes S_{k-1}^{eg}
\end{bmatrix},
\]

\[
\tilde{A} = \begin{bmatrix}
E_k^T & C_k^T & 0_{nm \times mm}
\end{bmatrix},
\]

\[
\tilde{b} = \text{vec} \left( I_n \right),
\]

\[
\tilde{D} = \tilde{x}^T \otimes D_k^1,
\]

\[
\tilde{x} = \begin{bmatrix}
(A_{k-1} C_k x_k - B_{k-1} u_{k-1})^T & y_k^T & (y_{k-1} - C_{k-1} C_k x_k)^T
\end{bmatrix}^T,
\]

\[
P_{k|k-1}^{eg} = (1 - \eta) \tilde{G}_{k-1}^x \left( \tilde{G}_{k-1}^x \right)^T + \eta \tilde{P}_{k-1}^x,
\]

\[
Q_{k|k-1}^{eg} = (1 - \eta) \tilde{G}_{k-1}^y \left( \tilde{G}_{k-1}^y \right)^T + \eta \tilde{Q}_{k-1},
\]

\[
P_{k|k-1}^{eg} = A_{k-1} P_{k-1}^{eg} A_k^T + \tilde{G}_{k-1}^x,
\]

\[
R_{k|k-1}^{eg} = (1 - \eta) \tilde{G}_{k-1}^y \left( \tilde{G}_{k-1}^y \right)^T + \eta R_{k-1},
\]

\[
R_{k|k-1}^{eg} = (1 - \eta) \tilde{G}_{k}^y \left( \tilde{G}_{k}^y \right)^T + \eta R_{k},
\]

\[
S_{k|k-1}^{eg} = C_k P_{k|k-1}^{eg} C_k^T + R_{k|k-1}^{eg},
\]

with optimal solution \( \tilde{z} \), from which we obtain the optimal matrices

\[
\tilde{T}_k = \text{reshape} \left( \tilde{z}_{1:n^2}^T, n, n \right),
\]

\[
\tilde{N}_k = \text{reshape} \left( \tilde{z}_{(n^2+1):n(n^2+nm)}^T, n, m \right),
\]

\[
\tilde{K}_{k-1} = \text{reshape} \left( \tilde{z}_{(n^2+nm+1):n(n^2+2nm)}^T, n, m \right).
\]

Proof. The original problem to be solved is to minimize the cost function \( J_k^{\text{eg}} \), given by \( (33) \), subject to equality \( (22) \) and inequality \( (3) \) constraints. To achieve that, we aim at a standard quadratic formulation with variables \( z = \text{vec} \left( [T_k \ N_k \ K_{k-1}] \right) \).

First, \( [T_k \ N_k \ K_{k-1}] \left[ E_k^T \ C_k^T \ 0_{nm \times mm} \right]^T = I_n \) is rewritten as \( \tilde{A} z = \tilde{b} \) using the equivalence \( \text{vec}(ABC) = (C^T \otimes A)\text{vec}(B) \) which results in \( (33) \) and \( (44) \). Second, the inequality constraints \( (3) \) are applied to the center estimate \( \hat{C}_k \) given by \( (24) \), yielding \( D_k \left[ T_k \ N_k \ K_{k-1} \right] \tilde{x} \preceq d_k^1 \), where \( \tilde{x} \) is given by \( (46) \). Applying the relation \( \text{vec}(ABC) = (C^T \otimes A)\text{vec}(B) \) once again, we reach \( (45) \) such that \( \tilde{D} z \preceq d_k^1 \).

Third, by expanding \( J_k^{\text{eg}} \), using the relation \( \text{tr}(A) + \text{tr}(B) = \text{tr}(A + B) \) we obtain

\[
\begin{align*}
\text{tr} \left( P_{k|k-1}^{eg} \right) &= \text{tr} \left( (T_k A_{k-1} - K_{k-1} C_{k-1} - P_{k|k-1}^{eg} T_k A_{k-1} - K_{k-1} C_{k-1})^T \right) \\
&+ T_k P_{k|k-1}^{eg} T_k^T + K_{k-1} R_{k-1}^{eg} K_{k-1}^T + N_{k} R_{k}^{eg} N_{k}^T \\
&= \text{tr} \left( T_k P_{k|k-1}^{eg} T_k^T \right) + \text{tr} \left( N_{k} R_{k}^{eg} N_{k}^T \right) + \text{tr} \left( K_{k-1} S_{k-1}^{eg} K_{k-1}^T \right) \\
&+ \text{tr} \left( -K_{k-1} C_{k-1} P_{k|k-1}^{eg} A_{k-1}^T T_k^T \right) + \text{tr} \left( -T_k A_{k-1} P_{k|k-1}^{eg} C_{k-1}^T K_{k-1}^T \right),
\end{align*}
\]
whose matrices $P_{k-1}^g$, $Q_{k-1}^g$, $P^g_{k|k-1}$, $R_{k-1}^g$, $R_k^g$, and $S^g_k$ are given by (47)-(52). According to the equivalence $\text{tr}(ABC) = \text{vec}(A^T)(I \otimes B)\text{vec}(C)$, each prior trace can be rewritten as

$$\text{tr} \left( T_k P^g_{k|k-1} T_k^\top \right) = \text{vec}(T_k)^\top \left( I_n \otimes P^g_{k|k-1} \right) \text{vec}(T_k),$$

$$\text{tr} \left( N_k R^g_k N_k^\top \right) = \text{vec}(N_k)^\top \left( I_n \otimes R^g_k \right) \text{vec}(N_k),$$

$$\text{tr} \left( K_{k-1} S^g_{k-1} K_{k-1}^\top \right) = \text{vec}(K_{k-1}^\top)^\top \left( I_n \otimes S^g_{k-1} \right) \text{vec}(K_{k-1}),$$

$$\text{tr} \left( -K_{k-1} C_k P^g_{k|k-1} A_{k-1}^\top T_k^\top \right) = \text{vec}(K_{k-1}^\top)^\top \left( -I_n \otimes C_{k-1} P^g_{k-1} A_{k-1} \right) \text{vec}(T_k),$$

$$\text{tr} \left( -T_k A_{k-1} P^g_{k|k-1} C_{k-1}^\top K_{k-1}^\top \right) = \text{vec}(T_k)^\top \left( -I_n \otimes A_{k-1} P^g_{k|k-1} C_{k-1} \right) \text{vec}(K_{k-1}).$$

Therefore, $J^g_k$ is equivalent to

$$\text{tr} \left( P^g_k \right) = \begin{bmatrix} \text{vec}(T_k)^\top & \text{vec}(N_k)^\top & \text{vec}(K_{k-1}) \end{bmatrix}^\top \tilde{P} \begin{bmatrix} \text{vec}(T_k) \\ \text{vec}(N_k) \\ \text{vec}(K_{k-1}) \end{bmatrix},$$

where $\tilde{P}$ is a symmetric and square matrix given by (42). In order to make explicit $z$ in $J^g_k$, we take a block-diagonal matrix $M = \text{blkdiag}(M_1, M_2, M_3)$ such that $\text{vec}(T_k^\top) = M_1 \text{vec}(T_k)$, $\text{vec}(N_k^\top) = M_2 \text{vec}(N_k)$, and $\text{vec}(K_{k-1}^\top) = M_3 \text{vec}(K_{k-1})$. Finally, the format of (40) is achieved after embedding the variable transformation in the matrix $H$, given by (41). Once the quadratic optimization problem (40) is solved, the design matrices (53)-(55) are obtained by applying a reshape in elements from $\tilde{z}$, which is an inverse procedure to vectorization.

**Remark 5.** Given $H$ in (41), the convexity of the quadratic problem presented in Theorem 2 relies on $\tilde{P}$ (42) being positive definite. To achieve that, a sufficient condition is that generator and covariance matrices of noises $w_{k-1}$, $v_{k-1}$, and $v_k$ have full rank. However, if $H$ is positive semidefinite, then numerical problems may occur. In this case, it is possible to add a matrix $\delta I_{(n^2 + 2nm)}$ to (41), with $\delta > 0$ being the smallest possible value, only to mitigate the singularity of $H$.

### 4.1.2 Normalized Problem

In Theorem 2, we show how to compute matrices $T_k$, $N_k$, and $K_{k-1}$ from $J^g_k$ in (53) for a previously chosen value of the weight parameter $\eta$. In this subsection, we show how to produce normalized solutions $\tilde{T}_k$, $\tilde{N}_k$, and $\tilde{K}_{k-1}$ with multiobjective meaning.

Next, two definitions are presented. The first one uses the weighted-sum strategy to normalize costs. The second definition states a cost function that evaluates the distance between two points in the objective space.

**Definition 5.** Consider the monobjective function $J^g_k(K_k) = (1 - \eta)J^L_k + \eta J^U_k$, for a chosen weighting parameter $\eta \in [0, 1]$, with $K_k$ being the gain that minimizes $J^g_k$. The application of $\eta = 0$ yields the limits $J^L_k$ and $J^U_k$, whereas the application of $\eta = 1$ yields the limits $J^L_k$ and $J^U_k$. By assuming these limits are bounded, the costs $J^L_k$ and $J^U_k$ may be normalized as

$$\tilde{J}^L_k(\tilde{K}_k) \triangleq \frac{J^L_k - J^L_k}{J^U_k - J^L_k},$$

$$\tilde{J}^U_k(\tilde{K}_k) \triangleq \frac{J^U_k - J^L_k}{J^U_k - J^L_k},$$

where $\tilde{K}_k$ is the corresponding matrix of normalized costs in $[0, 1]$.

**Definition 6.** Consider two functions $J^L_k$ and $J^U_k$. Let $\mu_{1,k}$, $\mu_{2,k} \in \mathbb{R}$ be a reference objective to be reached. Then, the minimum-distance objective function is defined as

$$J^\text{md}_{k}(K_k) \triangleq \left[ |\mu_{1,k} - J^L_k|^r + |\mu_{2,k} - J^U_k|^r \right]^{\frac{1}{r}},$$

where $K_k$ is the gain that minimizes $J^\text{md}_{k}$, and $r \in \mathbb{N}_+$. Equation (58) depicts an appealing case in multiobjective optimization problems, in which we pursue a point on the Pareto-optimal front, which denotes global solutions.

Many works involving mixed uncertainties apply $\eta = 0.5$ to unconstrained problems with non-normalized costs to obtain a solution from the Pareto-optimal front. In doing so, the authors are assuming that the uncertainties are well balanced.
Therefore, we intuitively expect that the obtained solution is around the middle point of the Pareto curve. Actually, this expected result is justified by Definition 5 which provides a methodology to compare global solutions. The following lemma states when the weighted-sum solution achieves the minimum distance.

**Lemma 3.** Consider \( r = 1, \tilde{\eta} = 0.5, \tilde{\mu}_{1,k} = \tilde{\mu}_{2,k} = 0 \), and the normalized costs \( \tilde{J}^z_k \) and \( \tilde{J}^g_k \) given by (56) and (57), respectively, with \( \tilde{K}_k \) being the variable matrix. Assume that there are no inequality constraints given by (3). Then, minimizing

\[
\tilde{J}^{eg}_k (\tilde{K}_k) = (1 - \tilde{\eta})J^z_k + \tilde{\eta}J^g_k
\]

is equivalent to minimizing

\[
\tilde{J}^{md}_k (\tilde{K}_k) = \left[ |\tilde{\mu}_{1,k} - \tilde{J}^z_k|^r + |\tilde{\mu}_{2,k} - \tilde{J}^g_k|^r \right] ^{\frac{1}{r}}.
\]

In this case, \( \tilde{\eta} = 0.5 \) implies the best tradeoff between the partial costs \( \tilde{J}^z_k \) and \( \tilde{J}^g_k \) in 1-norm sense.

**Proof.** Given two costs \( J^z_k \in [J^z_{1,k}, J^z_{2,k}] \) and \( J^g_k \in [J^g_{1,k}, J^g_{2,k}] \), their normalization as in Definition 5 yields \( \tilde{J}^z_k, \tilde{J}^g_k \in [0, 1] \). Therefore, we can define the normalized monobjective functions \( \tilde{J}^{eg}_k \) and \( \tilde{J}^{md}_k \) given by (59) and (60), respectively. In order to relate these functions, we choose the norm \( r = 1 \), the scalar parameter \( \tilde{\eta} = 0.5 \), and the reference objective \( \tilde{\mu}_{1,k} = 0 = \tilde{\mu}_{2,k} \). Since the normalized partial costs are nonnegative, we have \( \tilde{J}^z_k = 0.5 \tilde{J}^{md}_k \). Then, the functions \( \tilde{J}^{eg}_k \) and \( \tilde{J}^{md}_k \) assume different values under the same variable matrix \( \tilde{K}_k \).

However, there exist cases in which \( \tilde{K}_k \) is the same global solution for both functions, such as those involving unconstrained or equality-constrained problems. The unconstrained case can be verified doing \( \partial \tilde{J}^{eg}_k / \partial \tilde{K}_k = \partial \tilde{J}^{md}_k / \partial \tilde{K}_k = 0 \). The equality-constrained case can be verified as follows. By employing a Lagrange function \( L(\cdot) \), the equality constraints \( h(\tilde{K}_k) = 0 \) are transmitted to the normalized functions as a common additive parcel, that is,

\[
\tilde{J}^{eg+}_k = \tilde{J}^{eg}_k + L \left( h \left( \tilde{K}_k \right), \Lambda_k \right),
\]

\[
\tilde{J}^{md+}_k = \tilde{J}^{md}_k + L \left( h \left( \tilde{K}_k \right), \Lambda_k \right),
\]

where \( \Lambda_k \) denotes the Lagrange multipliers. By computing the first derivatives and equalling them to zero, we have

\[
\partial \tilde{J}^{eg+}_k / \partial \tilde{K}_k = \partial \left( \tilde{J}^{eg}_k + L \left( h \left( \tilde{K}_k \right), \Lambda_k \right) \right) / \partial \tilde{K}_k = \partial \left( 0.5 \tilde{J}^{md}_k \right) / \partial \tilde{K}_k + \partial L \left( h \left( \tilde{K}_k \right), \Lambda_k \right) / \partial \tilde{K}_k = 0,
\]

\[
\partial \tilde{J}^{md+}_k / \partial \tilde{K}_k = \partial \tilde{J}^{md}_k / \partial \tilde{K}_k + \partial L \left( h \left( \tilde{K}_k \right), \Lambda_k \right) / \partial \tilde{K}_k = 0.
\]

Subtracting \( \partial \tilde{J}^{eg+}_k / \partial \tilde{K}_k \) from \( \partial \tilde{J}^{md+}_k / \partial \tilde{K}_k = 0 \), we reach the same result than the unconstrained case. Then, the two depicted cases show that \( \tilde{\eta} = 0.5 \) provides the best tradeoff between the partial costs \( \tilde{J}^z_k \) and \( \tilde{J}^g_k \) in 1-norm sense.

Finally, whenever there exist inequality constraints on \( \tilde{K}_k \), we cannot equal the derivatives to zero. Then, weighted sum does not imply minimum distance anymore. ■

Thanks to the convexity of the quadratic problem (40) in Theorem 1 we guarantee that the obtained solution \( \tilde{z} \) yields the partial costs \( J^z_k \) and \( J^g_k \), given by (34) and (35), respectively, on the Pareto-optimal front. Conversely, the optimization problem contains inequality constraints and, according to Lemma 3, Theorem 1 must be reformulated in terms of Definition 6 to yield the final solutions with a multiobjective perspective. This result is presented in the following theorem for the normalized cost function

\[
\tilde{J}^{md}_k = \tilde{J}^z_k + \tilde{J}^g_k \tag{61}
\]

where \( \tilde{J}^z_k \) and \( \tilde{J}^g_k \) are the partial costs \( J^z_k \) (34) and \( J^g_k \) (35) normalized as in Definition 5.

**Theorem 2.** Consider the cost function (61), and the corresponding inequality (3) and equality (22) constraints. Let \( z = \text{vec} \left( [T_k \mathcal{N}_k \mathcal{K}_{k-1}^{-1}] \right) \in \mathbb{R}^{(n^2+2mn)} \) be the variable vector, and let \( M \in \mathbb{R}^{(n^2+2mn) \times (n^2+2mn)} \) be the vectorized transpose matrix such that \( \text{vec} \left( \tilde{T}^T_k \right)^\top \text{vec} \left( \mathcal{N}^T_k \right)^\top \text{vec} \left( \mathcal{K}^T_{k-1} \right) = M z \). Then, minimizing (61) subject to (3) and (22) is equivalent to solving the quadratic optimization problem

\[
\min \quad \frac{1}{2} z^\top H z
\]

s.t. \( \bar{A} z = \bar{b}, \ ar{D} z \leq \bar{d} \).
where $H$, $A$, $b$, and $D$ are given by (41), (43), (44), and (45), respectively, assuming that

$$P_{k-1}^g = \frac{1}{J_{k}^{Uz} - J_{k}^{Lz}} \bar{G}_k^x (\bar{G}_k^{x})^\top + \frac{1}{J_{k}^{Ug} - J_{k}^{Lg}} \bar{P}_k^x,$$

$$Q_{k-1}^g = \frac{1}{J_{k}^{Uz} - J_{k}^{Lz}} \bar{G}_k^w (\bar{G}_k^{w})^\top + \frac{1}{J_{k}^{Ug} - J_{k}^{Lg}} Q_k,$$

$$R_{k-1}^g = \frac{1}{J_{k}^{Uz} - J_{k}^{Lz}} \bar{G}_k^v (\bar{G}_k^{v})^\top + \frac{1}{J_{k}^{Ug} - J_{k}^{Lg}} R_k,$$

$$R_{k}^g = \frac{1}{J_{k}^{Uz} - J_{k}^{Lz}} \bar{G}_k^v (\bar{G}_k^{v})^\top + \frac{1}{J_{k}^{Ug} - J_{k}^{Lg}} R_k.$$

Thereby, we obtain the optimal vector $\tilde{z}$, and consequently, the optimal matrices

$$\bar{T}_k = \text{reshape} (\tilde{z}_{1:5}; n, n),$$

$$\bar{N}_k = \text{reshape} (\tilde{z}_{(5n+1): (5n+nm)}; n, m),$$

$$\bar{K}_{k-1} = \text{reshape} (\tilde{z}_{(5n+nm+1): (5n+2nm)}; n, m).$$

**Proof.** By making explicit the costs $J_k^f$ and $J_k^g$, given by (34) and (35) respectively, in (61), we have

$$\bar{J}_{k}^{md} = \frac{1}{J_{k}^{Uz} - J_{k}^{Lz}} J_k^f + \frac{1}{J_{k}^{Ug} - J_{k}^{Lg}} J_k^g - \frac{1}{J_{k}^{Uz} - J_{k}^{Lz}} J_k^f - \frac{1}{J_{k}^{Ug} - J_{k}^{Lg}} J_k^g.$$

As the last two terms of $\bar{J}_{k}^{md}$ are constant, they can be removed during the optimization process. In so doing, we obtain a cost similar to $J_k^g$ in Theorem 1 whose difference is the replacement of $(1 - \eta)$ by $\frac{1}{J_{k}^{Uz} - J_{k}^{Lz}}$, and of $\eta$ by $\frac{1}{J_{k}^{Ug} - J_{k}^{Lg}}$. The development and the solution of the new optimization problem follow the prior modified theorem.

### 4.1.3 Algorithm

The LTVMF algorithm solves Problem 1 by means of Theorems 1 and 2. In short, the first theorem is used to normalize costs whereas the second theorem provides normalized solutions with multiobjective meaning.

Regarding the CZ $\hat{X}_k$ in (28), note that its number of constraints and generators is larger than the initial values $n_h$ and $n_g$, respectively. In order to mitigate such growth over iterations, we here apply CZR to $\hat{X}_{k-1}$, yielding

$$\mathcal{X}_{k-1}^{\perp} = \text{CZR} (\hat{X}_{k-1}, \varphi_c, \varphi_g) = \{ G_{k-1}^{lx}, c_{k-1}^{lx}, A_{k-1}^{lx}, b_{k-1}^{lx} \},$$

with $\varphi_c$ constraints and $\varphi_g$ generators, and employ the resulting CZ $\mathcal{X}_{k-1}^{\perp}$ instead of $\hat{X}_{k-1}$. As $e_{k-1}^{lx}$ may be nonzero, we shift it to the center estimate $\hat{e}_{k-1}^{x}$ to obtain a conventional zero-center CZ. Next, LTVMF is described in Algorithm 1 and illustrated in Figure 5.

**Algorithm 1** $[\hat{e}_{k-1}^{x}, \hat{e}_{k-1}^{v}, \hat{X}_k, \mathcal{X}_k, I_k^g]$ = LTVMF($\hat{e}_{k-1}^{x}, \hat{X}_k, \mathcal{X}_k, I_k^g$, $C_{k-1}, C_k, Y_{k-1}, V_{k-1}, \varphi_c, \varphi_g, \alpha$)

1: Compute $\mathcal{X}_{k-1}^{\perp}$ given by (69)
2: Apply Proposition 1 to obtain the generator matrices $\bar{G}_k^{x}, \bar{G}_k^{w}, \bar{G}_k^{v}$, and $\bar{G}_k^{v}$
3: Given $\eta = 0$, compute the covariance matrices $P_{k-1}^g, \bar{Q}_k^g, P_{k-1}^g, R_{k-1}^g, R_k^g$, and $S_{k-1}^g$ given by (47)–(52)
4: Apply Theorem 1 to compute the matrices $T_k, N_k,$ and $K_{k-1}$
5: Compute $\bar{G}_k^x$ and $\bar{P}_k^x$ using (38) and (32), respectively, to determine $J_k^{Lz}$ in (34) and $J_k^{Lg}$ in (35)
6: Execute the steps (3)–(5) with $\eta = 1$ to compute $J_k^{Uz}$ in (34) and $J_k^{Ug}$ in (35)
7: Apply Theorem 2 to compute the final matrices $\bar{T}_k, \bar{N}_k,$ and $\bar{K}_{k-1}$
8: Determine the mixed state estimates $\hat{e}_{k}^{x}$, $\hat{e}_{k}^{v}$, $\hat{X}_k$, and $\mathcal{X}_k$, where $\hat{X}_k$ is given by (27), (28), (30)–(31) and $\hat{X}_k$ is given by (29), (32)
9: Compute the confidence CZ $\mathcal{X}_{k}^{\perp}$ and the confidence box $T_k^g$ using (14) and (16), respectively
Particular Case with Polytopes

Lagrange functions of Theorems 1 and 2 by disregarding inequality constraints. To achieve that, the equality constraints (22) are transmitted to enforce state constraints.

\[ M \]

since quadratic programming solvers can be replaced by such estimates using inequality constraints (3) for the polytopic case. This procedure aims at reducing the computational burden, since quadratic programming solvers can be replaced by explicit solutions as in [29]. For the unconstrained cases, we just do not enforce state constraints.

The following corollaries address the main results, where the design matrices are obtained explicitly. They are special cases of Theorems 1 and 2 by disregarding inequality constraints. To achieve that, the equality constraints (22) are transmitted to the cost functions \( J^g_k \) and \( J^md_k \), given by (33) and (61) respectively, through the Lagrange multipliers \( \Lambda_k \in \mathbb{R}^{n \times n} \), yielding the Lagrange functions

\[
J^q_k(T_k, N_k, K_{k-1}, \Lambda_k) = J^g_k + \text{tr} \left( \Lambda_k \left( E_k^T T_k^T + C_k^T N_k^T - I_n \right) \right) ,
\]

\[
J^q_k(T_k, \tilde{N}_k, \tilde{K}_{k-1}, \tilde{\Lambda}_k) = J^md_k + \text{tr} \left( \tilde{\Lambda}_k \left( E_k^T \tilde{T}_k^T + C_k^T \tilde{N}_k^T - I_n \right) \right) .
\]

**Corollary 1.** Let \( \Lambda_k \in \mathbb{R}^{n \times n} \) be the Lagrange multipliers and \( \Psi = [T_k N_k K_{k-1}] \in \mathbb{R}^{n \times (n+2m)} \) the variable matrix. Then, minimizing (33) subject to (22) is equivalent to minimizing \( J^q_k \) in (70), from which we obtain the optimal matrices

\[
\tilde{\Lambda}_k = -\left( \Pi_k^T \Sigma_k^{-1} \Pi_k \right)^{-1} ,
\]

\[
\Psi = -\tilde{\Lambda}_k \Pi_k^T \Sigma_k^{-1} ,
\]

where

\[
\Pi_k = \left[ E_k^T C_k^T 0_{m \times m} \right]^T,
\]

\[
\Sigma_k = 2 \left[ \begin{array}{cccc}
P_k^g & 0_{m \times m} & -A_{k-1} P_{k-1}^g C_k^T & 0_{m \times m} \\
0_{m \times m} & R_k^g & 0_{m \times m} & S_{k-1}^g \\
-C_{k-1}^T A_{k-1}^T & 0_{m \times m} & S_{k-1}^g & 0_{m \times m} \\
\end{array} \right],
\]

and the covariance matrices \( P_{k-1}^g, P_{k|k-1}^g, R_k^g, \) and \( S_{k-1}^g \) are given by (47), (49), (51), and (52), respectively. Thereby, the design matrices are made explicit as follows

\[
\tilde{T}_k = \Psi_{1; n, 1:n}^1 ,
\]

\[
\tilde{N}_k = \Psi_{1; n(n+1): (n+m)}^1 ,
\]

\[
\tilde{K}_{k-1} = \Psi_{1; n(n+m+1): (n+2m)}^1 .
\]

**Proof.** The unconstrained optimization problem to be solved is to minimize \( J^q_k \) in (70). Recall the relations \( \partial \text{tr}(M K^T N)/\partial K = M^T N \) and \( \partial \text{tr}(M K N K^T O)/\partial K = N K^T O M + N^T K^T M O \). Taking the partial derivative of \( J^q_k \) with respect to \( T_k, N_k, K_{k-1} \),
$K_{k-1}$, and $\Lambda_k \in \mathbb{R}^{n \times n}$ yields

$$\frac{\partial J_k}{\partial T_k} = 2T_k P_{k|k-1}^{PZ} - 2K_{k-1} C_{k-1} P_{k-1}^{PZ} A_{k-1}^T + \Lambda_k,$$

$$\frac{\partial J_k}{\partial N_k} = 2N_k R_{k|k}^{PZ} + \Lambda_k C_k^T,$$

$$\frac{\partial J_k}{\partial K_{k-1}} = 2K_{k-1} S_{k-1}^{PZ} - 2T_k A_{k-1} P_{k-1}^{PZ} C_{k-1}^T,$$

$$\frac{\partial J_k}{\partial \Lambda_k} = T_k E_k + N_k C_k - I_n.$$

where $P_{k|k-1}^{PZ}$, $R_{k|k-1}^{PZ}$, and $S_{k-1}^{PZ}$ are given by (47), (49), (51), and (52), respectively. Doing each partial derivative equals to zero yields the following linear equation system:

$$\Psi \Sigma_k + \Lambda_k \Pi_k^T = 0_{n \times (n+2m)},$$

$$\Psi \Pi_k = I_n,$$

where $\Psi = [T_k \ N_k \ K_{k-1}]$ is the variable matrix, and $\Pi_k$ and $\Sigma_k$ are given by (74) and (75), respectively.

By convenience, we can rewrite $\Sigma_k$ as follows

$$\Sigma_k = \Xi_k + \Gamma_k,$$

where

$$\Xi_k = 2 \begin{bmatrix} Q_{k|k-1}^{PZ} & 0_{n \times m} & 0_{n \times m} \\ 0_{m \times m} & R_k^{PZ} & 0_{m \times m} \\ 0_{m \times m} & 0_{m \times m} & R_{k|k-1}^{PZ} \end{bmatrix},$$

$$\Gamma_k = 2 \begin{bmatrix} A_{k-1} & 0_{m \times m} \\ 0_{m \times m} & P_{k-1}^{PZ} \\ -C_{k-1} & 0_{m \times m} \end{bmatrix}.$$

Since $Q_{k|k-1}^{PZ} > 0_{n \times m}$, $R_k^{PZ} > 0_{m \times m}$, and $R_{k|k-1}^{PZ} > 0_{m \times m}$, for all $k \geq 1$, we have $\Xi_k > 0_{(n+2m) \times (n+2m)}$. In turn, $\Gamma_k \geq 0_{(n+2m) \times (n+2m)}$ and, thereby, is invertible. Thus from (83), $\hat{\Psi}$ is uniquely determined by $-\hat{\Lambda}_k \Pi_k^T \Sigma_k^{-1} \hat{\Pi}_k$ (73). After substituting this result in (84), we obtain $\hat{\Lambda}_k \left(-\Pi_k^T \Sigma_k^{-1} \Pi_k\right) = I_n$. Since $\Pi_k$ has full rank, $-\Pi_k^T \Sigma_k^{-1} \Pi_k$ is invertible. Then, $\hat{\Lambda}_k$ is uniquely determined by (72). Finally, the desired matrices $\hat{T}_k$ (76), $\hat{N}_k$ (77), and $\hat{K}_{k-1}$ (78) are made explicit.

**Corollary 2.** Let $\hat{\Lambda}_k \in \mathbb{R}^{n \times n}$ be the Lagrange multipliers and $\hat{\Psi} = \begin{bmatrix} \hat{T}_k \ \hat{N}_k \ \hat{K}_{k-1} \end{bmatrix} \in \mathbb{R}^{n \times (n+2m)}$ the variable matrix. Then, minimizing (61) subject to (22) is equivalent to minimizing $J_k^{eq}$ in (77), from which we obtain the optimal matrices

$$\hat{\Lambda}_k = -\left(\Pi_k^T \Sigma_k^{-1} \Pi_k\right)^{-1},$$

$$\hat{\Psi} = -\hat{\Lambda}_k \Pi_k^T \Sigma_k^{-1},$$

where

$$\Pi_k = \begin{bmatrix} E_k^T & C_k^T & 0_{n \times m} \end{bmatrix}^T,$$

$$\Sigma_k = 2 \begin{bmatrix} P_{k|k-1}^{PZ} & 0_{n \times m} & -A_{k-1} P_{k-1}^{PZ} C_{k-1}^T \\ 0_{m \times m} & R_k^{PZ} & 0_{m \times m} \\ -C_{k-1} P_{k-1}^{PZ} A_{k-1}^T & 0_{m \times m} & S_{k-1}^{PZ} \end{bmatrix},$$

and the covariance matrices $P_{k|k-1}^{PZ}$, $P_{k|k-1}^{PZ}$, $R_k^{PZ}$, and $S_{k-1}^{PZ}$ are given by (62), (49), (65), and (52), respectively. Thereby, the design matrices are made explicit as follows

$$\hat{T}_k = \hat{\Psi}_{1:n;1:n},$$

$$\hat{N}_k = \hat{\Psi}_{1:n;(n+1):(n+m)},$$

$$\hat{K}_{k-1} = \hat{\Psi}_{1:n;(n+m+1):(n+2m)}.$$
Proof. By making explicit the costs $J^z_k$ (34) and $J^g_k$ (35) in (71), we have

$$
\hat{J}^q_k = \frac{1}{J^u_k - J^z_k} J^z_k + \frac{1}{J^u_k - J^g_k} J^g_k + \text{tr} \left( \hat{A}_k \left( F_k^T \hat{N}_k + C_k^T \hat{N}_k^T - 1_n \right) \right) - \frac{1}{J^u_k - J^z_k} J^L_z - \frac{1}{J^u_k - J^g_k} J^L_g.
$$

As the last two terms of $\hat{J}^q_k$ are constant, they can be removed during the optimization process. In so doing, we obtain a cost similar to $J^{\text{eq}}_k$ in Corollary 1 whose difference is the replacement of $(1 - \eta)$ by $\frac{1}{J^{z_k} - J^g_k}$, and of $\eta$ by $\frac{1}{J^{z_k} - J^g_k}$. The development and the solution of the new optimization problem follow the prior modified corollary.

Remark 6. Due to the fact that both $\hat{A}_k$ (72) and $\Psi$ (73) are uniquely determined, we guarantee the uniqueness of solution for the linear equation system (83)–(84).

So far, we show how to compute the design matrices $\hat{T}_k$, $\hat{N}_k$, and $\hat{K}_{k-1}$, which are employed to compute the first state estimates $\hat{c}_k$ (24), $\hat{\lambda}_k$ (28), and $\hat{X}_k$ (29). Now, we discuss how to enforce inequality constraints (3) by means of an additional step. To achieve that, $D_k x_k \leq d_k^\parallel$ in (3) is rewritten as the CZ $\hat{\lambda}_k^F$.

First, we replace the zero center of the CZ $\hat{\lambda}_k^+$ by the center $c_k^x$ of the CZ $\hat{\lambda}_k^F$, obtaining

$$\hat{\lambda}_k^+ = \left\{ \hat{G}_k, c_k^x, \hat{\lambda}_k^F, \hat{X}_k^x \right\}. \tag{91}$$

After, using (10) yields

$$\hat{\lambda}_k^{++} = \hat{\lambda}_k^+ \cap_{h_k} \hat{\lambda}_k^F. \tag{92}$$

Soon after, we cancel the shift $c_k^x$ of $\hat{\lambda}_k^{++}$, yielding the desired zero-center CZ $\hat{\lambda}_k^x$. Second, we correct the center $\hat{c}_k$ and covariance $\hat{P}_k^{xx}$ estimates through some stochastic approach, similar to (38) such as interval-constrained unscented transformation (ICUT), probability density function truncation, and estimate projection, for instance; see [33,39] for in-depth discussions. Here, we employ ICUT [33] Subsection 5.1.1 with the spreading parameter $\kappa = 0$ to obtain

$$\left[ \hat{c}_k^{\text{st}}, \hat{P}_k^{\text{st}} \right] = \text{ICUT} \left( \hat{c}_k, \hat{P}_k^{xx}, x_k^{\text{min}}, x_k^{\text{max}}, \kappa \right), \tag{93}$$

where

$$\left[ x_k^{\text{min}}, x_k^{\text{max}} \right] = \Box x_k^F. \tag{94}$$

Finally, the estimates $\hat{c}_k^{\text{st}}$, $\hat{\lambda}_k^{++}$, and $\hat{P}_k^{\text{st}}$ are used to find the confidence CZ $\hat{\lambda}_k^c$ (14) and the confidence box $T_k^c$ (16). The precision of these sets can still be improved by means of intersection with $\lambda_k^F$ yielding the corresponding constrained sets

$$\hat{\lambda}_k^c \triangleq \hat{\lambda}_k^c \cap_{h_k} \lambda_k^F, \tag{95}$$

$$T_k^c \triangleq \Box \{ T_k^c \cap_{h_k} \lambda_k^F \}. \tag{96}$$

Next, the explicit LTVMF algorithm (LTVMF-E) is described in Algorithm 2 and illustrated in Figure 6.

Algorithm 2 \([\hat{c}_k^{\text{st}}, \hat{\lambda}_k^+, \hat{X}_k^{x}, \lambda_k^F, T_k^c] = \text{LTVMF-E} \left( \hat{c}_k^{\text{st}}, \hat{\lambda}_k^+, \hat{X}_k^{x}, \lambda_k^F, E_k, A_{k-1}, B_{k-1}, u_{k-1}, W_{k-1}, W_{k-1}, V_{k-1}, \nu, \kappa \right)\),

1: Apply the steps 1–8 of Algorithm 1 with Corollaries 1 and 2 (instead of Theorems 1 and 2) to compute the mixed state estimates $c_k^{\text{st}}$ (24), $\lambda_k^+$ (28), and $\hat{X}_k$ (29).
2: Compute the CZ $\hat{\lambda}_k^x$ given by (92).
3: Annul the center of $\hat{\lambda}_k^{++}$ to obtain the zero-center CZ $\hat{\lambda}_k^x$.
4: Compute $c_k^{\text{st}}$ and $\hat{P}_k^{\text{st}}$ using (92).
5: Let the mixed state estimates be given by $\hat{c}_k^{\text{st}} = c_k^{\text{st}}$, $\hat{\lambda}_k = \hat{\lambda}_k$, and $\hat{X}_k \sim \mathcal{N} \left( 0_{n_x}, \hat{P}_k^{\text{st}} \right)$
6: Compute the constrained confidence CZ $\lambda_k^c$ and the constrained confidence box $T_k^c$ given by (95) and (96), respectively.

Remark 7. The correction of $\hat{\lambda}_k^+$ (28) via intersection with $\lambda_k^F$ requires a translation over Euclidean space since $\hat{\lambda}_k$ does not contain the whole state information. Here, we opt by (92) because the center reference does not depend on state estimates. Other possible choices of translation are: (i) to replace the center $c_k^x$ of $\lambda_k^F$ by the zero center of $\hat{\lambda}_k$; or (ii) to shift $\hat{\lambda}_k$ to the center estimate $\hat{c}_k^x$. Besides altering the center of specific sets, each choice can imply different solutions.
FIGURE 6 The LTVMF-E algorithm is illustrated by a diagram, whose cyan boxes highlight its main items. Given mixed uncertainties at  \( k \), LTVMF-E explicitly solves a minimum-variance problem containing equality constraints and normalized costs, whose solutions are the design matrices  \( \hat{T}_{k+1} \),  \( \hat{N}_{k+1} \), and  \( \hat{K}_{k} \). These matrices are used to yield the first mixed state estimates at  \( k + 1 \). In the presence of polytopic constraints, intersection (for CZ) and some stochastic technique (for both center and GRV) are applied to correct the prior state estimates. Finally, the new state estimates are used to compute a constrained confidence CZ  \( \mathcal{X}_{k+1}^{c} \) and a constrained confidence box  \( I_{k+1}^{c} \) that merge the mixed estimates in compact sets.

5  |  MIXED-UNCERTAINTY FILTER FOR LTI SYSTEMS

5.1  |  Estimator Design

Consider the LTI version of (1)-(2) given by:

\[
E x_k = A x_{k-1} + B u_{k-1} + z_w^k + \xi_{k-1}^w,
\]

\[
y_k = C x_k + z_e^k + g_k,
\]

where  \( E \in \mathbb{R}^{m \times n} \),  \( A \in \mathbb{R}^{m \times n} \),  \( B \in \mathbb{R}^{m \times p} \), and  \( C \in \mathbb{R}^{m \times n} \) are known constant matrices. Without loss of generality, the process and measurement noises are expressed in terms of their linear mapping, that is,  \( z_w^k = G_w^e \eta_{k-1}^w \) and  \( g_k = F^e \eta_k^e \), where  \( G_w^e \in \mathbb{R}^{m \times n} \) and  \( G^v \in \mathbb{R}^{m \times m} \) correspond to generator matrices of known constant CZs  \( \mathcal{W} = \{ G_w^e, 0_{nx1}, A^w, b^w \} \) and  \( \mathcal{V} = \{ G^v, 0_{mx1}, A^v, b^v \} \), respectively, with  \( \xi_{k-1}^w \in B(A^w, b^w) \) and  \( \xi_k \in B(A^v, b^v) \), while  \( F_w^e \in \mathbb{R}^{nxm} \) and  \( F^v \in \mathbb{R}^{mxm} \) define covariance matrices  \( Q = F_w^e (F_w^e)^T \) and  \( R = F^v (F^v)^T \) of constant GRVs  \( \mathcal{W} \sim \mathcal{N}(0_{nx1}, Q) \) and  \( \mathcal{V} \sim \mathcal{N}(0_{mx1}, R) \), respectively, and  \( \eta_{k-1}^w \in \mathbb{R}^n \) and  \( \eta_k^v \in \mathbb{R}^m \) are realizations from normalized GRVs with zero mean and identity covariance.

For the system (97)-(98), (22) is reduced to

\[
T E + NC = I_n.
\]

Following similar steps to (24)-(27), we propose the center estimate

\[
\hat{x}_k^c = \hat{A} \hat{x}_{k-1}^c + TBu_{k-1} + Ky_{k-1} + Ny_k
\]

with error terms

\[
\hat{z}_k^c = \hat{A} \hat{z}_{k-1}^c + TG_w^e \eta_{k-1}^w - KG^v \xi_{k-1}^v - NG^v \xi_{k-1}^v,
\]

\[
\hat{g}_k^c = \hat{A} \hat{g}_{k-1}^c + TF^v \eta_{k-1}^v - K F^v \eta_{k-1}^v - NF^v \eta_{k-1}^v,
\]

where

\[
\hat{A} = TA - KC.
\]

For convenience, the error terms (101) and (102) are rewritten as the error system

\[
e_k^a = \tilde{A} e_{k-1}^a + \tilde{B} d_{k-1}^a,
\]

\[
z_k^a = e_k^a,
\]

where  \( s = \{ z, g \} \),  \( e_k^a = s_k^a \),  \( \tilde{B} = [TG_w^e - KG^v - NG^v] \in \mathbb{R}^{n \times (n + 2m)} \),  \( d_{k-1}^a = \left[ (\xi_{k-1}^w)^T, (\xi_{k-1}^v)^T, (\xi_{k-1}^e)^T \right]^T \in \mathbb{R}^{n + 2m} \),

\[
\tilde{B} = [TF^v - K F^v - N F^v] \in \mathbb{R}^{m \times (n + 2m)} \), and  \( d_{k-1}^g = \left[ (\eta_{k-1}^w)^T, (\eta_{k-1}^v)^T, (\eta_{k-1}^e)^T \right]^T \in \mathbb{R}^{m + 2m} \).

We formulate a mixed  \( H_2/H_\infty \) problem to obtain the constant matrices  \( T \),  \( N \), and  \( K \). Now, we justify why normalized realizations  \( \xi \) and elements  \( \eta \) in  \( B(A, b) \) are considered. Both  \( H_2 \) and  \( H_\infty \) norms in the time domain work on  \( l_2 \) norm, thus all
signals evaluated must be bounded in magnitude. On one hand, the constrained zonotopic terms \( z_{k-1}^w \), \( z_{k-1}^g \), and \( z_k^g \) are bounded by CZs, therefore \( H_\infty \) norm is here employed to ensure robustness. \( H_\infty \) norm is not modified by realizations of bounded inputs, but it is influenced by linear mapping of variables; therefore, we decide to make explicit all linear transformations \( \tilde{B}^g \) on \( d_{k-1}^g \). On the other hand, the Gaussian terms \( g_{k-1}^w \), \( g_{k-1}^g \), and \( g_k^g \) can assume any value in the Euclidean space, then system norms cannot be directly used. According to \( H_\infty \) this case is discussed for continuous-time systems, where discrete-time systems are approached in \( \tilde{B}^w \). In order to obtain identity power spectral density for \( g^w \) and \( g^g \), we can use Cholesky decomposition, yielding \( \tilde{B}^g \) and \( d_k^g \). Then, we employ \( H_\infty \) norm to minimize the worst-case elements of \( \tilde{z}_k^w \) \( \gamma \), and \( H_2 \) norm to minimize the steady-state variance of \( \tilde{z}_k^g \) \( \delta \).

Solving an optimization problem containing both \( H_2 \) and \( H_\infty \) norms is not a simple task. One of the difficulties is to obtain the extreme points of the Pareto-optimal front, since the non-minimized norm may reach a too large value. It means that the weighted-sum approach is not suitable to scale the biobjective optimization problem. Commonly, this type of problem is solved using the epsilon-constraint or minimum-distance approach. Furthermore, an iterative process on the upper bound of the square \( H_\infty \) norm can be performed to adjust the final solution. Independently of which strategy is employed, we solve convex optimization problems and, thereby, any solution is on the Pareto-optimal front. Here, we are interested in finding a good tradeoff between the objectives as in LTVMF. Therefore, we formulate a constrained optimization problem in which the summation of the square \( H_2 \) \((\psi > 0)\) and \( H_\infty \) \((\gamma > 0)\) norms is minimized. According to Definition \( \delta \) this procedure is equivalent to setting \( r = 1 \) and \( \mu_1 = 0 = \mu_2 \), and assuming nonnegative costs (to remove the absolute value). Next, the constrained optimization problem is presented in terms of linear matrix inequalities.

**Theorem 3.** Consider the variables \( P_1 = P_1^T > 0_{n\times n}, P_2 = P_2^T > 0_{n\times n}, M \in \mathbb{R}^{n \times n} \) an invertible matrix, \( H = H^T > 0_{n(n+2m) \times n(n+2m)}, Y_1 \in \mathbb{R}^{n \times (n+m)}, Y_2 \in \mathbb{R}^{n \times m}, \psi > 0, \) and \( \gamma > 0, \) where \( \psi \) is the square \( H_2 \) norm of the Gaussian error system \((104)-(105), \) for \( s = g, \) and \( \gamma \) is the square \( H_\infty \) norm of the constrained zonotopic error system \((104)-(105), \) for \( s = z. \) Solving the constrained optimization problem

\[
\begin{align*}
\min \quad & (\psi + \gamma) \\
\text{s.t.} \quad & \begin{bmatrix} I_n - P_1 & \Omega_1^T \\ \Omega_1 & P_1 - M - M^T \end{bmatrix} < 0_{2n \times 2n}, \\
& \begin{bmatrix} -H & \Omega_2^T \\ \Omega_2 & P_1 - M - M^T \end{bmatrix} < 0_{2(n+m) \times 2(n+m)}, \\
& \begin{bmatrix} I_n - P_2 & 0_{n \times (n^w + 2n^g)} \\ 0_{n \times (n^w + 2n^g)} & -\gamma I_{n^g \times n^g} \end{bmatrix} \begin{bmatrix} \Omega_1^T \\ \Omega_3 \end{bmatrix} < 0_{n \times \bar{n}}, \\
& \Omega_4 = M^T \Theta^1 \alpha_1 A + Y_1 \Psi \alpha_1 A - Y_2 C, \\
& \Omega_5 = \left[(M^T \Theta^1 \alpha_1 + Y_1 \Psi \alpha_1) F^w - Y_2 F^v - (M^T \Theta^1 \alpha_2 + Y_1 \Psi \alpha_2) F^v \right], \\
& \Omega_6 = \left[(M^T \Theta^1 \alpha_1 + Y_1 \Psi \alpha_1) G^w - Y_2 G^v - (M^T \Theta^1 \alpha_2 + Y_1 \Psi \alpha_2) G^v \right], \\
& \Theta = \begin{bmatrix} E \\ C \end{bmatrix}, \quad \alpha_1 = \begin{bmatrix} I_n \\ 0_{n \times m} \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0_{n \times m} \\ I_m \end{bmatrix}, \quad \Psi = I_{n(n+m)} - \Theta \Theta^\dagger, \\
& \Theta^\dagger = (\Theta^\top \Theta)^{-1} \Theta^\top,
\end{align*}
\]

we obtain the optimal matrices

\[
\begin{align*}
\hat{T} &= \Theta^1 \alpha_1 + (\hat{M}^\top)^{-1} \hat{Y}_1 \Psi \alpha_1, \\
\hat{N} &= \Theta^1 \alpha_2 + (\hat{M}^\top)^{-1} \hat{Y}_1 \Psi \alpha_2, \\
\hat{K} &= (\hat{M}^\top)^{-1} \hat{Y}_2.
\end{align*}
\]
Proof. Based on (111), the square $H_2$ norm of the error system (104)-(105) with $s = g$ is computed as
\[
\min \psi \\
\text{s.t.} \quad \bar{A}^T P_1 \bar{A} - P_1 + I_n < 0_{n \times n}, \quad (\bar{B}^e)^T P_1 \bar{B}^e < H, \quad \text{tr}(H) < \psi. \tag{118}
\]

Applying Schur’s complement to the inequality constraint $\bar{A}^T P_1 \bar{A} - P_1 + I_n < 0_{n \times n}$, we have
\[
\begin{bmatrix}
I_n - P_1 & (T \bar{A} - K \bar{C})^T \\
(T \bar{A} - K \bar{C}) & -P_1^{-1}
\end{bmatrix} < 0_{2n \times 2n}. \tag{119}
\]

In order to let the Lyapunov matrix $P_1$ free over linearization, we apply a congruence transformation to (119) with $\varpi = \text{blkdiag} (I_n, M)$, and $M$ a slack variable. Post- and pre-multiplying (119) by $\varpi^T$ and $\varpi$, respectively, we obtain
\[
\begin{bmatrix}
I_n - P_1 & (T \bar{A} - K \bar{C})^T M \\
M^T (T \bar{A} - K \bar{C}) & -M^T P_1^{-1} M
\end{bmatrix} < 0_{2n \times 2n}. \tag{120}
\]

To treat the nonlinearity regarding $P_1$, we make use of $-M^T P_1^{-1} M < P_1 - M - M^T \triangleright_2 \Psi$ Theorem 1.

Since $T$ and $N$ are related by (99), we make explicit such dependency using (111) Equation (17), yielding
\[
T = \Theta^T a_1 + S \Psi a_1, \tag{121}
\]
\[
N = \Theta^T a_2 + S \Psi a_2, \tag{122}
\]
where $S \in \mathbb{R}^{n \times (n+m)}$, and matrices $\Theta \in \mathbb{R}^{(n+m) \times n}$, $\Theta^T \in \mathbb{R}^{n \times (n+m)}$, $a_1 \in \mathbb{R}^{(n+m) \times n}$, $a_2 \in \mathbb{R}^{(n+m) \times m}$, and $\Psi \in \mathbb{R}^{(n+m) \times (n+m)}$ are given by (113), (114). As the product $M^T (T \bar{A} - K \bar{C})$ is nonlinear, we define the linearizing transformations $Y_1 = M^T S$ and $Y_2 = M^T K$ to become (120) the linear matrix inequality (107).

Now, by applying Schur’s complement to the inequality constraint $(\bar{B}^e)^T P_1 \bar{B}^e < H$, we have
\[
\begin{bmatrix}
-H (\bar{B}^e)^T \\
\bar{B}^e - P_1^{-1}
\end{bmatrix} < 0_{2(n+m)2(n+m)}. \tag{123}
\]

Following similar steps to (120) to let $P_1$ free, we reach
\[
\begin{bmatrix}
-H (\bar{B}^e)^T \\
M^T \bar{B}^e - P_1 - M - M^T
\end{bmatrix} < 0_{2(n+m)2(n+m)}. \tag{124}
\]

As the product $M^T \bar{B}^e$ is nonlinear, we use the linearizations $Y_1$ and $Y_2$ to become (124) the linear matrix inequality (108).

Based on (115) Lemma 5.2, the square $H_\infty$ norm of the error system (104)-(105) with $s = z$ is computed as
\[
\min_{\gamma} \\
\text{s.t.} \quad \left[ \begin{array}{c} \bar{A}^T P_2 \bar{A} + I_n - P_2 \\
(\bar{B}^e)^T P_2 \bar{A} - \bar{B}^e \end{array} \right] < 0_{(n+\phi) \times (n+\phi)}, \tag{125}
\]
where $\phi = n^g + 2n^s$. The inequality constraint (125) can be rewritten as
\[
\begin{bmatrix}
I_n - P_2 & 0_{n \times n} - \gamma I_{\phi} \\
0_{n \times n} & -\gamma I_{\phi}
\end{bmatrix} + \left[ \begin{array}{c} \bar{A}^T \bar{B}^e \\
(\bar{B}^e)^T P_2 \bar{A}
\end{array} \right] < 0_{(n+\phi) \times (n+\phi)}. \tag{125}
\]

Applying Schur’s complement, we obtain
\[
\begin{bmatrix}
I_n - P_2 & 0_{n \times n} \gamma I_{\phi} \\
0_{n \times n} & -\gamma I_{\phi}
\end{bmatrix} \left[ \begin{array}{c} \bar{A}^T \bar{B}^e \\
(\bar{B}^e)^T P_2 \bar{A}
\end{array} \right] + \left[ \begin{array}{c} \bar{A}^T \bar{B}^e \\
(\bar{B}^e)^T P_2 \bar{A}
\end{array} \right] < 0_{(n+\phi) \times (n+\phi)}. \tag{125}
\]

Following similar steps to (120) to let the Lyapunov matrix $P_2$ free, we reach
\[
\begin{bmatrix}
I_n - P_2 & 0_{n \times n} \gamma I_{\phi} \\
0_{n \times n} & -\gamma I_{\phi}
\end{bmatrix} \left[ \begin{array}{c} \bar{A}^T M \\
(\bar{B}^e)^T M
\end{array} \right] + \left[ \begin{array}{c} \bar{A}^T M \\
(\bar{B}^e)^T M
\end{array} \right] < 0_{(n+\phi) \times (n+\phi)}. \tag{125}
\]
This matrix inequality is equivalent to the linear matrix inequality (109) by linearizing $M^T B^\gamma$ with $\Omega_1$ (112) (analogously to $\Omega_2$ (111)). Finally, by minimizing the summation of the square $H_2(\varphi)$ and $H_\infty(\varphi)$ norms, the constrained optimization problem (106)-(109) is finished, whose outputs are the design matrices $T_2$ (115), $N$ (116), and $K$ (117).

Remark 8. In Theorem 3, we consider different matrices $P_1$ and $P_2$. This procedure requires to add a slack variable $M$ to match the mixed $H_2/H_\infty$ formulation. In doing so, we obtain a stronger formulation than with $P_1 = P_2$ because there exist more degrees of freedom to reach a better Pareto-optimal front.

Given the matrices (115)-(117), we show how to compute a confidence zonotope $X^a_k$ from (14) and a confidence box $T^a_k$ from (16) at each time step. First, the center estimate $\hat{z}_k^a$ is directly given by (100). Second, the GRV $\hat{X}_k \sim N\left(0_{nx1}, \hat{P}^{xx}_k\right)$ is obtained through

$$ \hat{P}^{xx}_k = \hat{\Lambda} \hat{P}^{xx}_{k-1} \hat{\Lambda}^T + TQT^T + KRK^T + NRN^T. $$

(126)

Third, we employ reachable sets analysis to efficiently compute the interval hull of the CZ

$$ \hat{A}_k \hat{A}_{k-1} \oplus TYW \oplus (\neg K) \oplus (\neg N) = \{ \hat{G}_k 0_{nx1}, \hat{A}_k^0 \hat{b}_k^0 \}, $$

(127)

without storing increasing-order matrices and vectors over iterations. Unlike (28), CZR is not required anymore. On the one hand, the one-step ahead return value reduces Czs at the cost of both larger reachable set and complexity (due to the order-reduction algorithms). On the other hand, the reachable set analysis here exploited provides interval hull of exact Czs (with no order reduction) through low-cost computations.

Consider the constrained zonotopic term $\hat{z}_k^a$ in (101). From $\hat{z}_k^a$ to $\hat{z}_k^x$, the reachability analysis leads to

$$ \hat{z}_k^x = \hat{A}_k \hat{z}_k^{xx} + \sum_{i=0}^{k-1} \hat{A}_i T W^i \hat{z}^{xx}_{k-1-i} - \hat{A}_i K G^i x^{\gamma}_0 - N G^i x^{\gamma}_1 - \hat{A}_i N G^i x^{\gamma}_k - N G^i x^{\gamma}_2, $$

Note that there exist common terms between $\hat{z}_k^{xx}$ and $\hat{z}_k^{\gamma}_x$. After combining them, we reach the following recursive form:

$$ \hat{z}_k^x = \hat{A} k \hat{z}_k^{xx} + \sum_{i=0}^{k-1} \hat{A}_i T W^i \hat{z}^{xx}_{k-1-i} - \hat{A}_i K G^i x^{\gamma}_0 - N G^i x^{\gamma}_1 - \sum_{i=0}^{k-2} \hat{A}_i (\hat{A} N + K) G^i x^{\gamma}_{k-1-i}, \forall k \geq 2. $$

(128)

Replacing all realizations $\xi$ by constrained unitary boxes, we obtain

$$ \hat{X}_k = \{ \hat{A} k \hat{z}_k^{xx} \} \oplus \bigoplus_{i=0}^{k-1} \{ \hat{A}_i T W^i \hat{z}^{xx}_{k-1-i} \} - \{ \hat{A}_i K G^i \hat{z}^{\gamma}_0 \} - \{ \hat{A}_i N G^i \hat{z}^{\gamma}_1 \} - \{ \hat{A}_i N G^i \hat{z}^{\gamma}_k \} - \{ \hat{A}_i N G^i \hat{z}^{\gamma}_2 \}, \forall k \geq 2. $$

(129)

From (111), we have

$$ [X_k] = \{ \hat{A} k \hat{z}_k^{xx} \} \oplus \bigoplus_{i=0}^{k-1} \{ \hat{A}_i T W^i \hat{z}^{xx}_{k-1-i} \} - \{ \hat{A}_i K G^i \hat{z}^{\gamma}_0 \} - \{ \hat{A}_i N G^i \hat{z}^{\gamma}_1 \} - \{ \hat{A}_i N G^i \hat{z}^{\gamma}_k \} - \{ \hat{A}_i N G^i \hat{z}^{\gamma}_2 \}, \forall k \geq 2. $$

(130)

According to (130), we should store some products and summations within auxiliary variables ($\Pi$ and $\zeta$) to aid the next iterations.

Finally, given $\hat{c}_k^z$ in (100), $\hat{P}^{xx}_k$ in (126), and $[X_k] = \{ \text{diag } (\text{rad } (\Gamma X_k)) , \text{mid } (\chi X_k) \}$ in (130), we apply Definitions 3 and 4 to yield both confidence zonotope $X^a_k$ and confidence box $T^a_k$: $X^z_k$ is a zonotope because this is result of the Minkowski sum between a parallelotope $P_k$ and a box $\Pi X_k$.

5.2 \ Algorithm

The LTIMF algorithm solves Problem 2. As the reachability analysis requires that all state evolution ($\hat{c}_k^z, \hat{A}_k$, and $\hat{X}_k$) is traceable since $k = 1$, inequality constraints (3) cannot be enforced during the state estimation. In fact, this is an intrinsic limitation of
In this section, we experiment the LTVMF and LTIMF algorithms. In order to fairly compare our algorithms with literature results, we implement the mixed-uncertainty filter proposed in [22] called confidence set-membership state estimation (CSMSE). For comparison purposes, we compute three performance indexes, namely: (i) mean processing time ($T^{\text{CPU}}$), given by $T^{\text{CPU}} \triangleq$
Setup

1 We assume that the noise terms are Gaussian distributions. Therefore, we decided to compensate the lack of the descriptor by filters. However, numerical errors are imparted to the obtained representation, making unlikely the satisfaction of the equality constraint with respect to center, covariance, and covariation. Thereby, we set the parameters \( \varphi_c = 3 \), \( \varphi_w = 8 \), and \( \alpha = 0.0027 \). On one hand, the CSMSE algorithm solves unconstrained optimization problems, then its simulation is here executed with the dynamic equations of the compartmental system. This is a fair decision because \( x_k \) already satisfies the state constraints.

On the other hand, the LTIMF and LTVMF algorithms solve constrained optimization problems. To achieve a descriptor representation from a state-space representation, we employed the methodology discussed in Subsection 2.1 and simulated the filters. However, numerical errors are imparted to the obtained representation, making unlikely the satisfaction of the equality constraint with respect to center, covariance, and covariation. Therefore, we decided to compensate the lack of the descriptor by
augmenting the measurement model with the equality constraint such that
\[
\begin{bmatrix}
y_k \\
d^e_k
\end{bmatrix} = \begin{bmatrix} C \\ D^e \end{bmatrix} x_k + \begin{bmatrix} v_k \\ 0 \end{bmatrix},
\]
\[
\hat{y}_k = \tilde{C} x_k + \hat{u}_k.
\]
After, we set the descriptor matrix \( E = I_3 \) and the augmented variables \( \tilde{V} = \{ G^\circ, 0_{3 \times 1} \} \) and \( \tilde{V} \sim \mathcal{N} \left( 0_{3 \times 1}, F^\circ \left( F^\circ \right)^T \right) \), where \( G^\circ = \text{blkdiag} \left( 0.5 I_2, 10^{-6} \right) \) and \( F^\circ = \text{blkdiag} \left( F^\circ, 10^{-3} \right) \). Note that the matrices were augmented with nonzero terms to mitigate numerical issues such as Cholesky decomposition and matrix inversion. Accordingly, we add \( 10^{-8} I_{27} \) to matrix \( H \) in (41) to complete the positive definiteness. Therefore, both quadratic and explicit versions of LTVMF can be employed. By solving the optimization problem in Theorem 3, we obtain the following design matrices for LTIMF:
\[
T = \begin{bmatrix} 1.0511 & 0.0802 & 0.0804 \\ 0.0087 & 1.0014 & 0.0045 \\ 0.0257 & 0.0040 & 1.0006 \end{bmatrix}, \quad N = \begin{bmatrix} -0.0292 & 0.0002 & -0.0805 \\ -0.0042 & 0.0031 & -0.0046 \\ -0.0251 & -0.0034 & -0.0007 \end{bmatrix}, \quad K = \begin{bmatrix} 0.0281 & 0.0004 & 0.0723 \\ -0.0032 & 0.0029 & 0.0167 \\ -0.0251 & -0.0034 & 0.0430 \end{bmatrix}.
\]

In order to experiment the different algorithms proposed in this paper, we execute two scenarios of state estimation. The first one corresponds to running LTVMF (Algorithm 1) and LTIMF (Algorithm 3) with the original halfspace constraints \( x_k \geq 0_{3 \times 1} \). In this case, LTVMF is carried out with its quadratic version, while LTIMF does not enforce any state inequality constraint. In the second scenario, LTVMF (Algorithm 2) and LTIMF (Algorithm 3) enforce the polytopic constraints 
\[
\mathcal{X}^F = \{ \text{diag}(\text{rad}([\Lambda])), \text{mid}([\Lambda]) \}, \quad \text{where } \left[ \begin{array}{c} 0 \\ 5 \\ 10 \\ 5 \end{array} \right] \text{ was defined by simulation. In this case, } \text{LTVMF is carried out with its explicit version, called LTVMF-E, while LTIMF enforces such polytopic constraints as in Remark 9 being then called LTIMF-P. For both cases, CSMSE returns the same solution since it cannot enforce state constraints.}
\]

6.1.2 Discussion

The LTVMF, LTIMF, and CSMSE algorithms solve minimum-variance problems. Then, in comparison with the stochastic framework\[13\], the solution of those problems brings up implications to estimates of center, and covariance and covariation matrices. To test the satisfaction of equality constraint, we verify if \( D^e \hat{c}_k = d^e_k \), \( D^e \hat{P}_{kk} = 0_{1 \times 3} \), and \( D^e \hat{G}_k^x \left( \hat{G}_k^x \right)^T = 0_{1 \times 3} \), with \( \hat{G}_k^x \) being the generator matrix of the zonotope used to define the minimum-variance problem, as in Theorems 1 and 2. These tests are originated from \( D^e \left( \hat{z}^x_k + \hat{g}^x_k \right) = 0 \) assuming that \( D^e \hat{c}_k = d^e \) and \( \hat{z}^x_k \) and \( \hat{g}^x_k \) are uncorrelated. Thereby, we extract the relations
\[
D^e \hat{g}^x_k \left( \hat{g}^x_k \right)^T + D^e \hat{z}^x_k \left( \hat{z}^x_k \right)^T + D^e \hat{y}^x_k \left( \hat{y}^x_k \right)^T = 0_{1 \times 3},
\]
whose application of the expected value operator yields \( D^e \hat{P}_{kk} = 0_{1 \times 3} \) and \( D^e \hat{G}_k^x \left( \hat{G}_k^x \right)^T = 0_{1 \times 3} \), respectively. However, since zonotopes participate only on the optimization problem, we only expect that the center estimates comply with the equality constraint.

In Figure 8, we illustrate that each algorithm returns different solutions. In terms of precision, CSMSE is the most conservative method. The justification is that, although the dynamic equations of the compartmental system are compatible with the equality constraints, both covariance \( \hat{P}_{00}^x \) and covariation \( \hat{G}_0^x \left( \hat{G}_0^x \right)^T \) matrices were not initialized in \( D^e x_0 = d^e \). The correct initialization of the covariation matrix (consistent with minimum variance) is not trivial since we do not have the explicit link between covariation matrix and the corresponding CZ, only the inverted procedure is known and makes sense. Therefore, CSMSE cannot by itself guarantee that its state estimates are consistent with equality constraints. This also justifies why CSMSE is the most inaccurate method, as shown by Table 1.

The prior discussion motivates the use of constrained algorithms to correct the initial evolution of state estimates. As illustrated by Figure 8 and Table 1, our mixed-uncertainty filters reach better precision and accuracy than CSMSE. Moreover, we guarantee that \( D^e \hat{c}_k = d^e_k \) for all \( k \in Z_+ \). Among the proposed algorithms, LTVMF returns the most precise results (graph (a)). It occurs because the state nonnegativity constraints are optimally incorporated in the formulated quadratic programs. Then, the LTVMF is motivated for LTV descriptor systems whose dynamics satisfy convex polyhedral constraints and linear equality constraints, implying better precision and accuracy. In turn, LTVMF-E yields larger solutions (graph (b)) since the intersection between CZs (executed in the feasibility step) here attributes few correction of trajectory. That is, the improvement of precision may be deteriorated due to the order reductions and the modification of the reachable set (similar to introducing transient after each intersection). However, Table 1 highlights computational benefits with LTVMF-E. This algorithm is motivated for LTV descriptor systems whose main requirement is the reduction of computational cost.
FIGURE 8 Time results of state estimation for the compartmental system regarding one separate simulation. (a) and (b) depict the same true unmeasured state $x_{3,k}$ (black solid line) involved by different confidence boxes. These estimates are computed by CSMSE (cyan color), LTVMF (blue color), and LTIMF (red color). In (a), halfspace constraints $x_{i,k} \geq 0$, $i = 1, 2, 3$ are used to execute LTVMF with its quadratic version. In (b), polytopic constraints $0 \leq x_{i,k} \leq 5$ are incorporated by LTVMF (through its explicit version) and LTIMF.

We also highlight the relevance of LTIMF for the present case study. Table[1] shows that both LTIMF and LTIMF-P demand the smallest $T_{CPU}$ and result in a competitive accuracy with LTVMF. These advantages motivate the use of LTIMF in LTI descriptor systems. Analogously to LTVMF-E, the LTIMF-P here brings up few improvement of precision for LTIMF at the cost of larger $T_{CPU}$. Conversely, LTIMF-P is motivated in the presence of polytopic constraints to refine the confidence sets. Remember that numerical corrections were applied to the quadratic programs of LTVMF, therefore there exists a loss of precision and accuracy (including LTVMF-E).

6.2 LTV System

6.2.1 Setup

Consider a three-state LTV descriptor system[29] of the form (1)-(2), where

$$
E_k = \begin{bmatrix}
1 + 0.2 \sin(0.1k) & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad A_k = \begin{bmatrix}
0.6 - 0.1 \sin(0.1k) & 0 & 0.3 \\
-0.2 \sin(0.2k) & 0.4 & 0 \\
0 & 1 & 1
\end{bmatrix}, \quad B_k = \begin{bmatrix}
1 \\
\sin(0.1k) \\
0
\end{bmatrix}, \quad C_k = \begin{bmatrix}
0 & 0 & 1 + 0.2 \sin(0.2k) & 0.5
\end{bmatrix},
$$

$F^w = 0.0053I_3, G^w = 0.02I_3, F^v = 0.0017, G^v = 0.005$, and $u_k = 0.5 \sin(0.15k)$. The realizations $z^w_{k-1}$ and $z^v_k$ are taken from uniform distributions, while the realizations $u^w_{k-1}$ and $g^v_k$ are taken from GRVs. This example illustrates the application of LTVMF in LTV descriptor systems.

To simulate the states, we set $x_0 = [0.1 \ 0 \ 0]^T$, $k_f = 80$, $k_0 = 4$, and $m_s = 1$. To estimate the states, we set $\hat{c}_0 = 0_{3x1}$, $\hat{x}_0 = \{0.1I_3, 0_{3x1}, 1_{3x3}, 1\}$, $\beta = 0.0007I_3$, $\varphi_c = 3$, $\varphi_s = 8$, and $\alpha = 0.0027$, such that $x_0 \in \mathcal{X}_0^w$.

Since there are no inequality constraints [3], only the explicit version of LTVMF (Algorithm 2) is executed. In this case, we are going to obtain the same state estimates than the quadratic version (Algorithm 1) with smaller $T_{CPU}$. 

FIGURE 9 State estimation for the three-state LTV system. In (a), the true states (black solid line) and the punctual estimates computed by LTVMF (blue dashed-dotted line) are sketched in 3D along with some confidence boxes. The green boxes make explicit the initial and final sets. In (b), the maximum radius of confidence boxes computed by LTVMF is presented over time.

6.2.2 Discussion

In Figure 9a, the punctual estimate $\hat{x}_k \in \mathcal{X}_k^a$ and the true state $x_k$ evolve over a specific region in the Euclidean space, which is related to linear equality constraint embedded in the descriptor representation. In order to illustrate the mixing of the state estimates, we also sketch some confidence boxes $\mathcal{I}_k^a$. Regarding the estimate $\hat{x}_k$, we now rise a brief discussion. Due to the guaranteed characteristic of CZs (to include the true states $x_k$) and to the a posteriori center estimate $\hat{c}_k$, it satisfies all state constraints, we expect in most cases that $\hat{x}_k = \hat{c}_k^a$; this occurs in Figure 9a. It is noteworthy that by assuming that $\hat{c}_k^a$ is close enough to $x_k$, the confidence CZ $\mathcal{X}_k^a$ may include its center $\hat{c}_k^a$, resulting in $\hat{x}_k = \hat{c}_k^a$. However, since $\hat{c}_k^a \in \mathcal{X}_k^a$ cannot be enforced over the optimization problems, the use of Remark 2 is required to yield punctual estimates $\hat{x}_k \in \mathcal{X}_k^a$. Therefore, $\hat{x}_k$ cannot have the same properties than $\hat{c}_k^a$, although it is the best approximation of $\hat{c}_k^a$ (in $1$-norm sense) that tries to comply with the state constraints. This discussion includes both quadratic and explicit versions of LTVMF.

Figure 9b illustrates that the maximum radius of $\mathcal{I}_k^a$ varies over time. This behavior is expected because the system is LTV. The peak value around $k = 50$ is justified by the growth of volume verified in $\hat{\mathcal{X}}_k$ given by (28), which is due to the small values of $\varphi_c$ and $\varphi_g$. Besides, the trace minimization does not necessarily imply volume minimization.

7 CONCLUDING REMARKS

This paper proposes two mixed-uncertainty state estimators for discrete-time descriptor linear systems. The descriptor representation is employed to generalize the use of the algorithms since it implicitly embeds linear equality state constraints. The novel filters introduce results with the class of constrained zonotopes, state linear inequality constraints, and multiobjective interpretations on the Pareto-optimal front.

The algorithm denoted as LTVMF is formulated under minimum-variance criterion, but it is able to work on others cost criteria due to the proposed cost normalization. For polyhedral constraints, LTVMF is addressed to a quadratic optimization problem, where specific solvers are required to yield online solutions. For polytopic constraints, an explicit solution is alternatively proposed to reduce computational burden.

The algorithm named LTIMF is formulated under a mixed $H_2/H_{\infty}$ criterion. This filter is addressed to LTI systems, having as the main advantage its low-cost computations. This gain of speed is originated from the reachability analysis with constant matrices, which is also the reason of LTIMF not enforcing inequality constraints. The prior drawback can be attenuated with polytopic constraints, since intersection can be applied posteriorly.

The state estimators here proposed are compared to a mixed-uncertainty algorithm of the literature called CSMSE[12]. This state estimator is also based on GRVs and CZs but cannot incorporate state constraints and requires approximating GRVs by
confidence ellipsoids to combine Gaussian and set-based costs in an unconstrained optimization problem. Over two numerical examples, we illustrate the advantages of our proposals with a multiobjective perspective.

According to the cost normalization here proposed, different combinations of objective function will be intended in future works. This study may be useful to increase the precision with respect to set-bounded uncertainties. In addition, other forms of merging stochastic and set-based uncertainties will be investigated.

References


**TABLE 1** Results of $T^{CPU}$, RMSE, and $r^{□}$ for the compartmental system. The suffix “P” denotes that polytopic constraints were enforced by LTIMF.

<table>
<thead>
<tr>
<th>Algorithms</th>
<th>$T^{CPU}$</th>
<th>RMSE$_1$</th>
<th>RMSE$_2$</th>
<th>RMSE$_3$</th>
<th>$r^{□}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CSMSE</td>
<td>6.30 ms</td>
<td>0.217</td>
<td>0.0683</td>
<td>6.16</td>
<td>39.2</td>
</tr>
<tr>
<td>LTVMF</td>
<td>25.2 ms</td>
<td>0.0546</td>
<td>0.0385</td>
<td>0.0664</td>
<td>0.233</td>
</tr>
<tr>
<td>LTVMF-E</td>
<td>12.5 ms</td>
<td>0.133</td>
<td>0.140</td>
<td>0.201</td>
<td>1.30</td>
</tr>
<tr>
<td>LTIMF</td>
<td>3.30 ms</td>
<td>0.0538</td>
<td>0.0296</td>
<td>0.0614</td>
<td>0.737</td>
</tr>
<tr>
<td>LTIMF-P</td>
<td>4.40 ms</td>
<td>0.0538</td>
<td>0.0296</td>
<td>0.0614</td>
<td>0.734</td>
</tr>
</tbody>
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