Riemann boundary value problems on the Archimedean spiral

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Abstract

This paper studies the Riemann boundary value problems on the Archimedean spiral. We characterized the functions which are intergrable on the Archimedean spiral. We also study the asymptotic behaviors of Cauchy-type integral and Cauchy principal value integral on Archimedean spiral at the origin and infinity. At the end, we discuss the Riemann boundary value problems for sectionally holomorphic functions with the Archimedean spiral as their jump curve and obtain the explicit form.
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Keywords Archimedean spiral · generalized principle part · Plemelj formula · Riemann boundary value problem

Mathematics Subject Classification (2010) MSC 30E25

1 Introduction

Boundary value problems of analytic functions (Riemann-Hilbert problem) on finite curves are discussed in some classical textbooks, [5,9,10]. There are also some results that extend them to a variety of functions and special curves, [2, 3,7]. Riemann boundary value problems are the most fundamental, i.e., many
boundary value problems of holomorphic functions can be transformed into
Riemann boundary value problems, [9,10].

In the last two decades, there has been renewed interest in applications of
Riemann boundary value problems (RBVP), especially, the Riemann-Hilbert
technique for the asymptotic behaviour of orthogonal polynomials, [4,6,8].
These applications are involved in the RBVP on infinite curves, which are not
completely solved. There are few articles discuss the RBVP on infinite curves,
in which the RBVP on infinite curves was transformed into the ones on finite
curves. However, the the transformed RBVP are not equivalent to the original
ones. In fact, dealing with the RBVP on infinite curves is much harder. What is
more, it is difficult to obtain the global asymptotic behaviour of the orthogonal
polynomials on \( \mathbb{C} \) by the transformed ones since the conformal mappings are
very complicated and not global. In [11], the authors directly study the RBVP
on the positive real axis. They generalize the concept of both the principal
part of a holomorphic function at the origin and infinity on \( \mathbb{C} \) which is cut by
the positive real axis.

In this paper, we discuss the RBVP on the Archimedean spiral \( L = \{ \theta e^{i \theta}, \theta > 0 \} \). We also introduce the definitions of the generalized principal
parts and generalized orders at the origin and infinity of the holomorphic
functions on \( \mathbb{C} \) which is cut by the Archimedean Spiral. At the end, we state
the corresponding RBVP and give the solutions.

2 Preliminary

In this paper, let \( L \) be the Archimedean spiral and \( \widehat{L_aL_b} \) be the arc from \( L_a \) to
\( L_b \) on \( L \), where \( L_a = ae^{ia} \) is the point on the Archimedean spiral with length
\( a \) and \( a \in [0, \infty) \). In particular, \( L_0 = 0 \) and \( L_\infty = \infty \). Without confusion, we
do not distinguish between \( L_0 \) and \( 0 \), \( L_\infty \) and \( \infty \), respectively.

**Definition 21** Let \( f(t) \) be given on \( L, \Delta > 0 \) and \( 0 < \mu \leq 1 \). If \( f \in H^\mu(\widehat{L_0L_\Delta}) \), denote by \( f \in H^\mu(\infty) \). And if there exist positive constants \( M \) such that
\[
|f(t') - f(t'')| \leq M \left| \frac{1}{t'} - \frac{1}{t''} \right|^{\mu},
\]
for \( t', t'' \in L \setminus \widehat{L_0L_\Delta} \). Then \( f \) is said to belong to \( \hat{H}^\mu(\infty) \), or briefly \( \hat{H}(\infty) \) if not concerning \( \mu \). Moreover, if for any finite sub-arc \( L' \) of \( L \), \( f \in H^\mu(L') \).
Then \( f \) is said to belong to \( \hat{H}^\mu(L) \), or \( \hat{H}(L) \). If \( f \in \hat{H}^\mu(\infty) \) and \( f(\infty) = 0 \),
denote by \( f \in \hat{H}^\mu_0(\infty) \) or briefly \( \hat{H}_0(\infty) \).

**Definition 22** If \( f \in H^\mu(\widehat{L_aL_b}) \) for any \( 0 < a < b \), denoted by \( f \in H^\mu_0(L) \) or
briefly \( f \in H_0(L) \). Moreover, if there exists \( \delta > 0 \) such that \( f \in H^\mu(\widehat{L_0L_\delta}) \),
denote by \( f \in H^\mu(0) \) or briefly \( f \in H(0) \).
Definition 23 Let $f$ be defined on the $L \setminus L_0 L_\Delta$, where $\Delta > 0$. If for any $t', t'' \in L \setminus L_0 L_\Delta$ we have
\[
|f(t') - f(t'')| \leq \frac{M'|t' - t''|^\mu}{\max\{|t'|^\nu, |t''|^\nu\}},
\]
where $M, 0 < \mu \leq 1 < \nu$ are definite constants, denote by $f \in \bar{H}^\mu_\nu(\infty)$.

Definition 24 If there exists $\Delta > 0$ and $f^*$ is a bounded function such that
\[
f(t) = \frac{f^*(t)}{t^\nu}, \quad |t| \geq \Delta,
\]
denote by $f \in O^\nu(\infty)$. Moreover, if $f^* \in H^\mu(\infty)$, denote by $f \in H^\mu_\nu(\infty)$.

If there exists $\delta > 0$ and $f^*$ is a bounded function such that
\[
f(t) = \frac{f^*(t)}{t^\delta}, \quad |t| \leq \delta,
\]
denoted by $f \in O^\delta(0)$. Moreover, if $f^* \in H^\mu(0)$, denote by $f \in H^\mu_\nu(0)$.

Sometimes, we also need to consider the following types of functions:
\[
f(t) = \frac{f^*(t)}{t^\lambda}, \quad \text{where} \ f^* \in H^\mu(0),
\]
and
\[
\lambda = \alpha + i\beta, \quad 0 \leq \alpha < 1.
\]
In addition, denote by
\[
H^\nu_\alpha(0) = \cup_{0 \leq \alpha < 1} H^\nu_\alpha(0).
\]

Let $f_m(\tau) = \tau^m f(\tau)$. If $f_m \in \hat{H}^\mu(\infty), f_m \in \hat{H}^\mu_0(\infty)$ or $f_m \in \hat{H}^\mu_\nu(\infty)$, denote by $f \in \hat{H}^\nu_\alpha(\infty), f \in \hat{H}^\nu_{m,0}(\infty)$ or $f \in \hat{H}^\nu_{m,\nu}(\infty)$, respectively.

3 Sectional holomorphic functions jumping on the Archimedean spiral

To state the Riemann boundary value problem on the Archimedean spiral, we introduce the sectional holomorphic function on the complex plane cut along the Archimedean spiral. We also need to generalize the concepts of the principal part at both 0 and $\infty$.

If $F$ is holomorphic in the complex plane cut by the Archimedean spiral, denote by $F \in A(C \setminus L)$.

Definition 31 Let $f$ be defined on $L$ and locally integrable. If
\[
C(f)(z) = \frac{1}{2\pi i} \int_{L} \frac{f(\tau)}{\tau - z}, \quad z \notin L
\]
is integrable, it is called the Cauchy-type integral with kernel density $f$ on $L$. 

\[
\text{(1)}
\]
The integrability of $f$ if different from the case of \[11\] since the Archimedean spiral rotates around $\infty$. For example, since
\[
\int_{L_1 \to \infty} \frac{1}{|t|^\nu} |dt| = \int_1^\infty \frac{1}{|\theta|^\nu} (1 + i\theta) d\theta \leq 2 \int_1^\infty \frac{1}{|\theta|^{\nu-1}} d\theta,
\]
\(\frac{1}{|t|^\nu}\) is integrable on $L_1L_\infty$ if and only if $\nu > 2$. While since
\[
\int_{L_0 \to 1} \frac{1}{|t|^\alpha} |dt| = \int_1^1 \frac{1}{|\theta|^\alpha} (1 + i\theta) d\theta \leq 2 \int_0^1 \frac{1}{|\theta|^\alpha} d\theta,
\]
\(\frac{1}{|t|^\alpha}\) is integrable on $L_0L_1$ when $\alpha < 1$. Thus for $f \in O^\alpha(0) \cap O^\nu(\infty) (\alpha < 1, \nu > 1)$, the Cauchy integral (1) is integrable.

**Lemma 32** Let $f \in O^\alpha(0) \cap O^\nu(\infty) (\alpha > 0, \nu > 1)$ and locally integrable on $L$. Then
\[C(f) \in A(C \setminus L).\]

The proof of Lemma 32 is similar to the case of the positive real axis.

As the same as \[11\], we introduce the concept of generalized principal part and order of singularity.

**Definition 33** Let $F \in A(C \setminus L)$ If there exists an entire function $E(z)$ such that
\[
\lim_{z \to \infty} (F(z) - E(z)) = 0,
\]
then $E(z)$ is said to be the generalized principal part of $F(z)$ at $\infty$, denote by $G.P(F, \infty)$.

Let $F \in A(C \setminus L)$ If there exists an entire function $E(z)$ such that
\[
\lim_{z \to 0} (F(z) - E(z^{-1})) = 0, \tag{2}
\]
then $E(z^{-1})$ is said to be the generalized principal part of $F(z)$ at 0, denote by $G.P(F, 0)$.

**Remark 1** If $F$ is analytic near $\infty$, it has the Laurent series
\[
F(z) = \sum_{k=0}^{+\infty} a_k z^k + \sum_{k=1}^{+\infty} a_{-k} z^{-k}
\]
near $\infty$, which the principal part is $P.P(F, \infty)(z) = \sum_{k=0}^{+\infty} a_k z^k$. It is easy to check that $G.P(F, \infty) = P.P(F, \infty)$. In fact, by (2), we have
\[
\lim_{z \in C \setminus L, z \to \infty} (G.P(F, \infty)(z) - P.P(F, \infty)(z)) = 0.
\]
Since both $G.P(F, \infty)$ and $P.P(F, \infty)$ are entire functions, we obtain that
\[G.P(F, \infty)(z) - P.P(F, \infty)(z) \equiv 0.\]
Similarly, if \( F \) is analytic near 0, we also have \( \text{P.P}(F,0) = \text{G.P}(F,0) \).

**Remark 2** In this paper, in general, \( F(z) \) may not be analytic near both 0 or \( \infty \). So it has no principal part in the classical sense near 0 or \( \infty \). For example, let \( F(z) = \frac{\ln(-z)}{z^m}, \ m = 1, 2, \cdots \), where the logarithmic function \( \ln z \) is chosen as the main branch on the complex plane cut by \( L \). Then \( \text{G.P}(F,\infty)(z) = 0 \).

**Definition 34** Let \( F \in A(\mathbb{C} \setminus L) \). If
\[
0 < \beta_m = \limsup_{z \to \infty} |z^{-m}F(z)| < +\infty,
\]
\( F \) is said to be of order \( m \) at \( \infty \). Denote by \( \text{ord}(F, \infty) = m \). And if
\[
0 < \alpha_m = \limsup_{z \to 0} |z^mF(z)| < +\infty,
\]
\( F \) is said to be of order \( m \) at 0. Denote by \( \text{ord}(F, 0) = m \).

**Remark 3** Obviously, if \( \text{G.P}(z^{-m}F, \infty) = \beta_m \neq 0, \text{ord}(F, \infty) = m \). And if \( \text{G.P}(z^mF, 0) = \alpha_m \neq 0, \text{ord}(F, 0) = m \). When \( \text{G.P}(F, \infty) \) is a polynomial of degree \( m \), we have \( \text{ord}(F, \infty) = m \); when \( \text{G.P}(F, 0) \) is a polynomial of degree \( m \), we have \( \text{ord}(F, 0) = m \). If \( \text{ord}(F, \infty) \leq m \), then \( \text{G.P}(z^{-m-1}F, \infty) = 0 \). Similarly, if \( \text{ord}(F, 0) \leq m \), then \( \text{G.P}(z^{m+1}F, 0) = 0 \).

**Theorem 35** Let \( f \in O^{\alpha}(0) \cap \bar{H}_\mu^{\nu}(\infty) \cap H_\nu^{*}(\infty) (\alpha < 1, \nu > 1) \) and locally integrable. We obtain
\[ \text{G.P}(C(f), \infty) = 0. \]

**Proof** Let \( \Delta \) be large enough that there exists \( f^*(t) \in H_\nu^{*}(\infty) \) such that, for \( t = \theta e^{i\rho} \in \overline{L_\Delta L_\infty} \),
\[ f^*(t) = \frac{f^*(t)}{t^\nu} \text{ and } \left| \frac{1 + i\theta}{\theta} \right| \leq 2. \]

We divide the Cauchy integral into two parts as follows:
\[
C(f)(z) = \frac{1}{2\pi i} \int_{L_\infty}^{L_\Delta} \frac{f(\tau)}{\tau - z} d\tau + \frac{1}{2\pi i} \int_{L_\Delta}^{\infty} \frac{f(\tau)}{\tau - z} d\tau, \ z \notin L. \tag{3}
\]
It is obvious that
\[ \frac{1}{2\pi i} \int_{L_\infty}^{L_\Delta} \frac{f(\tau)}{\tau - z} d\tau \to 0, \ z \to \infty. \tag{4} \]

For \( z \notin L \), chose respectively \( z_1, z_2 \in L \) such that
\[ |z - z_1| = \min\{|\rho e^{i\rho} - z|, 0 \leq \rho \leq |z|\} \tag{5} \]
and
\[ |z - z_2| = \min\{|\rho e^{i\rho} - z|, \rho \geq |z|\}. \tag{6} \]
Assume \( z_1 = \rho_1 e^{i\rho_1} \) and \( z_2 = \rho_2 e^{i\rho_2} \). By Lemma A1 we have
\[ \rho_2 < \rho_1 + 4\pi. \tag{7} \]
Now consider the second part of (3). Denote by
\[
\frac{1}{2\pi i} \int_{L_\Delta}^\infty \frac{f(\tau)}{\tau - z} \, d\tau - \frac{1}{2\pi i} \int_{L_{\rho_1}}^\infty \frac{f(\tau)}{\tau - z} \, d\tau
\]
\[
+ \frac{1}{2\pi i} \int_{L_{\rho_2+1}}^\infty \frac{f(\tau)}{\tau - z} \, d\tau + \frac{1}{2\pi i} \int_{L_{\rho_2+1}}^\infty \frac{f(\tau)}{\tau - z} \, d\tau
\]
\[= \delta_1 + \delta_2 + \delta_3.\]  
(8)

Let \( \tau = \theta e^{i \theta} \). By (5) we obtain that
\[2|\tau - z| \geq |\tau - z| + |z_1 - z| \geq |z_1 - \tau| \geq |z_1| - |\tau| = \rho_1 - \theta.\]

Then we obtain
\[|\delta_1| \leq \frac{1}{2\pi} \int_{L_\Delta}^{\rho_1-1} \frac{|f(\tau)|}{\tau - z} \, d\tau\]
\[\leq \frac{1}{2\pi} \int_{\Delta}^{\nu-1} \frac{\nu}{\theta^{\nu-1}} \frac{1}{\rho_1 + \theta} \, d\theta\]
\[\leq 2M \frac{1}{\nu-1} \frac{1}{\rho_1 - \theta} \, d\theta.\]

When \( \nu \geq 2 \), obviously
\[\int_{\Delta}^{\nu-1} \frac{1}{\rho_1 + \theta} \, d\theta \to 0, \rho_1 \to \infty.\]  
(9)

When \( 1 < \nu < 2 \), it is easy to check that \( \frac{1}{\theta^{\nu-1}} \) decreases on \((\Delta, \rho_1 - 1)\) and increases on\((\rho_1 - 1, \rho_1 - 1)\). We divide the integral into two parts
\[\int_{\Delta}^{\nu-1} \frac{1}{\theta^{\nu-1}(\rho_1 - \theta)} \, d\theta = \int_{\Delta}^{\nu-1} \frac{1}{\theta^{\nu-1}(\rho_1 - \theta)} \, d\theta + \int_{\nu-1}^{\nu-1} \frac{1}{\theta^{\nu-1}(\rho_1 - \theta)} \, d\theta.\]

It is easy to prove
\[\int_{\Delta}^{\nu-1} \frac{1}{\theta^{\nu-1}(\rho_1 - \theta)} \, d\theta \leq \nu \frac{1}{\rho_1(\nu - 1)} (\nu - 1)^{2-\nu} \Delta^{2-\nu} \to 0, \rho_1 \to \infty\]
and
\[\int_{\nu-1}^{\nu-1} \frac{1}{\theta^{\nu-1}(\rho_1 - \theta)} \, d\theta \leq \nu - 1)^{\nu-1} \frac{1}{\rho_1(\nu - 1)} \ln(\nu \rho_1) \to 0, \rho_1 \to \infty,\]

So we obtain
\[\lim_{z \to \infty} \delta_1 = \lim_{\rho_1 \to \infty} \delta_1 = 0.\]  
(10)
Choose $z_L \in L$ such that $|z - z_L|$ be the distance between $z$ and $L$. It is obvious that $z_L = z_1$ or $z_2$. Assume that $z_L = z_1$. Denote by

$$
|\delta_z| \leq \left| \frac{1}{2\pi} \int_{L_{\rho_1}}^{L_{\rho_2+1}} \frac{f^*(\tau) - f^*(z_L)}{\tau - z} \, d\tau \right| = \left| \frac{1}{2\pi} \int_{L_{\rho_1}}^{L_{\rho_2+1}} \frac{f^*(\tau)}{\tau - z} \, d\tau \right| + \left| \frac{1}{2\pi} \int_{L_{\rho_1}}^{L_{\rho_2+1}} \frac{f(z_L)}{\tau - z} \, d\tau \right|
$$

By (7), for $\rho_1 > 4\pi + 3$, we have $\rho_2 + 1 \leq 2(\rho_1 - 1)$. By $|\tau| > \rho_1 - 1$, $f^* \in H^\mu(\infty)$ and

$$
2|\tau - z| \geq |\tau - z_L| \geq ||\tau| - |z_L|| = |\theta - \rho_1|,
$$

we have

$$
\varepsilon_1 \leq \frac{1}{2\pi} \int_{L_{\rho_1}}^{L_{\rho_2+1}} \frac{|f^*(\tau) - f^*(z_L)|}{|\tau^\nu| |\tau - z|} \, |d\tau| \leq \frac{1}{4\pi} \int_{L_{\rho_1}}^{L_{\rho_2+1}} \frac{M|\tau - z_L|^{\mu-1}}{(\rho_1 - 1)^{\nu}} \, |d\tau|
$$

$$
\leq \frac{1}{2\pi} \int_{L_{\rho_1}}^{L_{\rho_2+1}} \frac{M|\tau - z_L|^{\mu-1}}{(\rho_1 - 1)^{\nu}} \, |d\tau|
$$

and

$$
\leq \frac{2M}{\pi} \int_{L_{\rho_1}}^{L_{\rho_2+1}} \frac{(\rho_1 - 1)^{\nu}}{(\rho_2 + 1)^{\nu}|\rho_1 - \theta|^{1-\nu}} \, d\theta
$$

$$
\leq \frac{4M}{\pi(\rho_1 - 1)^{\nu-1}} \int_{L_{\rho_1}}^{L_{\rho_2+1}} |\rho_1 - \theta|^{-\nu} d\theta
$$

$$
\leq \frac{4M}{\pi(\rho_1 - 1)^{\nu-1}} \mu
$$

So we obtain

$$
\lim_{\rho_1 \to \infty} \varepsilon_1 = \lim_{\rho_1 \to \infty} \varepsilon_1 = 0.
$$

Now we deal with the second part of (11). Recall that

$$
\frac{|z^\nu \tau^\nu|}{|\tau^\nu| |\tau - z|} \leq \frac{1}{2(\rho_2 + 1)^{\nu-1}} |\tau - z_L| \leq \frac{2\nu}{\nu(\rho_1 - 1)^{\nu}}.
$$

We obtain

$$
\varepsilon_2 \leq \frac{|f^*(z_L)|^{2\nu+1}}{\pi\nu|z^\nu_L|} (4\pi + 2) \leq \frac{M}{(\rho_1 - 1)^{\nu}}.
$$
which gives
\[ \lim_{z \to \infty} \varepsilon_2 = \lim_{\rho_1 \to \infty} \varepsilon_2 = 0. \] (13)
We take the main branch of \( \log(\tau - z) \) on some simply connected domain containing the arc \( L_{\rho_1-1}L_{\rho_2+1} \). Notice that for large enough \( \rho_1 \), the increment of the argument of \( \tau - z \) from \( (\rho_1 - 1)e^{i(\rho_1-1)} \) to \( (\rho_2 + 1)e^{i(\rho_2+1)} \) do not exceed 5\( \pi \) since \( \rho_2 < \rho_1 + 4\pi \). Then we have
\[
\varepsilon_3 = \left| \frac{f(z)}{2\pi} \right| \log((\rho_2 + 1)e^{i(\rho_2+1)} - z) - \log((\rho_1 - 1)e^{i(\rho_1-1)} - z)] \\
\leq \frac{M}{2\pi\rho_1^3} \left| \log |(\rho_2 + 1)e^{i(\rho_2+1)} - z| \right| + \left| \log |(\rho_1 - 1)e^{i(\rho_1-1)} - z| \right| + 5\pi. \] (14)
Again remember that
\[
2(\rho_1 + 2\pi) \geq (\rho_1 - 1) + |z| \geq |(\rho_1 - 1)e^{i(\rho_1-1)} - z| \\
\geq \frac{1}{2} \left| (\rho_1 - 1)e^{i(\rho_1-1)} - z + |\rho_1 e^{i\rho_1} - z| \right| \\
\geq \frac{1}{2} (\rho_1 - 1)e^{i(\rho_1-1)} - \rho_1 e^{i\rho_1} \geq \frac{1}{2} \] (15)
and
\[
3(\rho_1 + 2\pi) \geq (\rho_2 + 1) + |z| \geq |(\rho_2 + 1)e^{i(\rho_2+1)} - z| \\
\geq \frac{1}{2} \left| (\rho_2 + 1)e^{i(\rho_2+1)} - z + |\rho_2 e^{i\rho_2} - z| \right| \\
\geq \frac{1}{2} (\rho_2 + 1)e^{i(\rho_2+1)} - \rho_2 e^{i\rho_2} \geq \frac{1}{2}. \] (16)
Then by (14), (15) and (16) we obtain
\[
\lim_{z \to \infty} \varepsilon_3 = 0. \] (17)
Now by (12), (13) and (17) we obtain
\[
\lim_{z \to \infty} \delta_2 \to 0. \] (18)
In the following we prove \( \delta_3 \to 0, z \to \infty \). Recall that for \( \tau = \theta e^{i\theta} \in L_{\rho_2+1}L_{\infty} \) we have \(|\tau - z| \geq \frac{1}{2}|(\tau - z_1)| \geq \frac{1}{2}|\tau - z_2| \geq \frac{1}{2}(\theta - \rho_2) \). Then by \( f \in O^{\nu'}(\infty) \) we obtain
\[
\delta_3 \leq \frac{M}{\pi} \int_{\rho_2+1}^{\infty} \frac{|1 + i\theta|}{\theta^{\nu}(\theta - \rho_2)} d\theta \leq \frac{2M}{\pi} \int_{\rho_2+1}^{\infty} \frac{1}{\theta^{\nu-1}(\theta - \rho_2)} d\theta. \]
Since \( \nu > 1 \) we take \( p, q > 1 \) such that \( (\nu - 1)p > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). Then by the Hölder inequality, we obtain
\[
\int_{\rho_2+1}^{\infty} \frac{1}{\theta^{\nu-1}(\theta - \rho_2)} d\theta \leq \left( \int_{\rho_2+1}^{\infty} \frac{1}{\theta^{(\nu - 1)p}} d\theta \right)^{\frac{1}{p}} \left( \int_{\rho_2+1}^{\infty} \frac{1}{(\theta - \rho_2)^{q}} d\theta \right)^{\frac{1}{q}} \]
So we get
\[ \lim_{z \to \infty} \delta_3 = \lim_{\rho_2 \to \infty} \delta_3 \to 0 \] (19)

since
\[ \int_{\rho_2+1}^{\infty} \frac{1}{(\theta - \rho_2)^q} d\theta = \int_1^{\infty} \frac{1}{s^q} ds \]
is bounded and
\[ \int_{\rho_2+1}^{\infty} \frac{1}{s^{(\nu-1)p}} d\theta \leq \frac{1}{(\nu-1)p-1} (\rho_2 + 1)^{1-(\nu-1)p}. \]

Now by (3), (4), (8), (10), (18) and (19) we obtain
\[ \lim_{z \to \infty} C(f) = 0, \]
which means G.P(C(f), \infty) = 0.

Remark 4 Theorem 35 says that the Cauchy-type integral vanishes at infinity. The conclusion also holds for the Cauchy principal value integral, i.e.,
\[ \frac{1}{2\pi i} \int_L \frac{f(\tau)}{\tau - t} d\tau \to 0, \quad t \to \infty. \]

Moreover, if we track the proof of Theorem 35, we have more precise results. That is, there exists \( M, \eta > 0 \) such that
\[ |C(f)(z)| \leq M \frac{1}{|z|^\eta}, \quad z \to \infty. \] (20)

Theorem 36 Let \( f \in H^\mu(0) \cap O^\nu(\infty)(\nu > 1), f(0) = 0, \) and locally integrable, then
\[ \text{G.P}(C(f), 0) = \frac{1}{2\pi i} \int_L \frac{f(\tau)}{\tau} d\tau, \]
where \( C(f) \) is the Cauchy-type integral on \( L. \)

Proof Given \( b > 0. \) It is easy to check that
\[ \lim_{z \to 0} (C_{L_b \to \infty}(f))(z) = \frac{1}{2\pi i} \int_{L_b}^{\infty} \frac{f(\tau)}{\tau} d\tau. \]
So it only need to prove
\[ \lim_{z \to 0} (C_{L_b \to \infty}(f))(z) = \frac{1}{2\pi i} \int_0^{L_b} \frac{f(\tau)}{\tau} d\tau. \]
Take \(0 < r < b\), such that \(f \in H^\mu(\overline{L_0L_r})\). For \(z\) near \(0\), let \(z_L \in L\) be the closest point to \(z\). Let \(z_L = \alpha e^{i\alpha}\). Then we have
\[
\left| \frac{1}{2\pi i} \int_{L_0}^{L_r} \frac{f(\tau)}{\tau - z} d\tau - \frac{1}{2\pi i} \int_{L_0}^{L_r} \frac{f(\tau)}{\tau} d\tau \right| 
\leq \left| \frac{1}{2\pi} \int_{L_0}^{L_r} \frac{f(\tau) - f(z_L)}{\tau - z} d\tau \right| + \left| \int_{L_0}^{L_r} \frac{1}{\tau} d\tau \right| 
+ \left| \int_{L_0}^{L_r} \frac{f(\tau)}{\tau - z} d\tau - \int_{L_0}^{L_r} \frac{f(\tau)}{\tau} d\tau \right| 
\triangleq \frac{1}{2\pi} (\rho_1 + \rho_2 + \rho_3 + \rho_4). 
\] (21)

First we have
\[
\rho_1 \leq \int_{L_0}^{L_r} \frac{2|f(\tau) - f(z_L)|}{|\tau - z_L|} d\tau \leq \int_{L_0}^{L_r} 2M|\tau - z_L|^{\mu - 1} d\tau 
= \int_0^r 2M|\theta e^{i\theta} - \alpha e^{i\alpha}|^{\mu - 1} |1 + i\theta| d\theta 
\leq 4M \int_0^r |\theta - \alpha|^{\mu - 1} d\theta \leq \frac{8Mr^\mu}{\mu}, 
\] (22)

since
\[
\int_0^r |\theta - \alpha|^{\mu - 1} d\theta \leq \int_0^\alpha (\alpha - \theta)^{\mu - 1} d\theta + \int_\alpha^r (\theta - \alpha)^{\mu - 1} d\theta = \frac{2r^\mu}{\mu}. 
\]

Notice that \(f(0) = 0\), we get
\[
\rho_3 \leq \int_{L_0}^{L_r} \frac{|f(\tau) - f(0)|}{|\tau - 0|} d\tau \leq M \int_0^r \theta^{\mu - 1} |1 + i\theta| d\theta \leq \frac{2Mr^\mu}{\mu}. 
\] (23)

Recall that \(|z_L| \leq |z - z_L| + |z| \leq 2|z|\). We have
\[
\rho_2 \leq |f(z_L)||\log(L_r - z) - \log(-z)| 
\leq M|z_L|^{\mu} (|\log |L_r - z|| + |\log |z|| + 4\pi), 
\] (24)

where we choose the main branch of \(\log(\tau - z)\) on \(C\) cut by \(L\).

Again noice that
\[
C_{L_rL_0}(f)(z) = \frac{1}{2\pi i} \int_{L_r}^{L_0} \frac{f(\tau)}{\tau - z} d\tau \in A(C \backslash L_rL_0) 
\]
which means
\[
\lim_{z \to 0} \rho_4 = 0. 
\] (25)

Since we can choose \(r\) arbitrarily small, by (21)-(25), we complete the proof of Theorem 36.
Corollary 37 (Finite Expansion of Generalized Principal Parts of Cauchy-Type Integrals at Infinity) Let \( f \in O^\alpha (0) \cap H^\mu \rightarrow (\infty) \cap O^{\nu+\lambda}(\infty)(\alpha < 1, \nu > 1) \), where \( \lambda \) is a positive integer, and locally integrable, then
\[
G.P(z^\lambda C(f), \infty)(z) = -\sum_{k=0}^{\lambda-1} \frac{z^{\lambda-1-k}}{2\pi i} \int_L f(\tau) \tau^k d\tau.
\]

Proof Let \( f_k(t) = t^k f(t) (k = 0, 1, \cdots, \lambda) \). Since \( f \in O^\alpha (0) \cap H^\mu \rightarrow (\infty) \cap O^{\nu+\lambda}(\infty), f_k \in O^\alpha (0) \cap O^{\nu+\lambda-k}(\infty) \). So \( \frac{1}{2\pi i} \int_L \frac{f(\tau)}{\tau - z} \tau^k d\tau \) is well defined.

\[
z^\lambda (C(f))(z) = \frac{1}{2\pi i} \int_L \frac{f(\tau)(z^\lambda - \tau^\lambda)}{\tau - z} d\tau + \frac{1}{2\pi i} \int_L \frac{f(\tau)\tau^\lambda}{\tau - z} d\tau
\]
\[
= -\sum_{k=0}^{\lambda} \frac{z^k}{2\pi i} \int_L f(\tau) \tau^{\lambda-1-k} d\tau + \frac{1}{2\pi} \int_L \frac{f(\tau)\tau^\lambda}{\tau - z} d\tau.
\]

By Theorem 35 we have
\[
\lim_{z \in C, \lambda \to \infty} \frac{1}{2\pi i} \int_L \frac{f(\tau)\tau^\lambda}{\tau - z} d\tau = 0,
\]
which finishes the proof.

Lemma 38 Let \( f \in H^* (0) \cap O^\nu(\infty)(\alpha < 1, \nu > 1) \) and integrable in any finite interval \([\delta, \Delta] \), then
\[
G.P(z C(f), 0) = 0.
\]

In general, if
\[
f(t) = \frac{f^*(\tau)}{\tau^\lambda}, \quad \text{where } f^* \in H^\mu (L_0 L_\Delta), \quad 0 \leq \alpha = \text{Re}(\lambda) < 1,
\]
we have
\[
C(f)(z) = \begin{cases} \frac{f(0)}{2\pi i} \log(-z) + \Phi(z), & \lambda = 0, \\ \frac{e^{\lambda \pi} f(0)}{2\sin(\lambda \pi)} z^{\lambda} + \Psi(z), & \lambda \neq 0, \end{cases}
\]
where both \( \Phi \) and \( \Psi \) are respectively holomorphic in some cut neighborhood of \( 0 \), and \( \Phi(z) \) tends to a definite limit as \( z \to 0 \), and \( \Psi(z) = O(|z|^{-\alpha}) \).

Proof We can prove Lemma 38 as the same as Lemma 3.3 in [11] since it only involves in the neighbourhood of origin. In fact their forms are all the same as the theorems in [9, 10].
Let $F \in A(C \setminus L)$. For $t \in L \setminus L_0$, if $F$ is continuous up to both sides of $L$, denoted by $F^\pm(t)$ the boundary values respectively.

To analyze the boundary value of the Cauchy-type integral, we introduce the Cauchy principal value integral. Assume that $f \in O^\alpha(0) \cap H^\nu(\infty) \cap O^\nu(\infty)(\alpha < 1, \nu > 1)$. For $t = \alpha e^{\iota \theta}(\alpha > 0)$, take $b > |t| + 1$. Since

$$
\int_{L_b}^\infty \frac{f(\tau)}{\tau - z} \, d\tau
$$

is analytic at $z = t$,

$$
\lim_{\delta \to 0} \frac{1}{2\pi i} \int_0^\infty \frac{f(\tau)}{\tau - t} \, d\tau + \int_{L_{a+b}}^L \frac{f(\tau)}{\tau - t} \, d\tau + \frac{1}{2\pi i} \int_{L_b}^\infty \frac{f(\tau)}{\tau - t} \, d\tau
$$

tends to a definite limit, which is called the Cauchy principal value integral with kernel density $f$ on $L$.

**Remark 5** The Cauchy principal value integral can also be divided into two parts, that is 

$$
C(f)(t) = (C_{L_0 L_\infty}(f))(t) + (C_{L_0 L_b}(f))(t), \quad t \in L \setminus \{L_0, L_b\}.
$$

**Theorem 39** (Boundary Values of Cauchy-Type Integrals) Let $f \in H^\nu(L) \cap O^\alpha(0) \cap O^\nu(\infty)(\alpha < 1, \nu > 1)$. Then the boundary values of the Cauchy-type integral exist and the following Plemelj formula hold

$$
\begin{align*}
C(f)^+(t) &= \frac{1}{2} f(t) + \frac{1}{2\pi i} \int_{L_0}^{L_\infty} \frac{f(\tau)}{\tau - t} \, d\tau, \\
C(f)^-(t) &= -\frac{1}{2} f(t) + \frac{1}{2\pi i} \int_{L_0}^{L_\infty} \frac{f(\tau)}{\tau - t} \, d\tau,
\end{align*}
$$

for $t \in L \setminus L_0$.

**Proof** It is obvious that

$$
(C(f))^\pm(t) = (C_{L_0 L_\infty}(f))^\pm(t) + (C_{L_0 L_b}(f))^\pm(t), \quad t \in L \setminus \{L_0, L_b\}. \tag{27}
$$

Recall that

$$
(C_{L_0 L_\infty}(f))^+(t) = (C_{L_0 L_\infty}(f))^-(t), \tag{28}
$$

and

$$
\begin{align*}
(C_{L_0 L_b}(f))^+(t) &= \frac{1}{2} f(t) + \frac{1}{2\pi i} \int_{L_0}^{L_b} \frac{f(\tau)}{\tau - t} \, d\tau, \\
(C_{L_0 L_b}(f))^-(t) &= -\frac{1}{2} f(t) + \frac{1}{2\pi i} \int_{L_0}^{L_b} \frac{f(\tau)}{\tau - t} \, d\tau. \tag{29}
\end{align*}
$$

By (27), (28) and (29), we obtain (26).

We end this section by a corollary.

**Corollary 310** If $f \in H^\nu(0) \cap H^\nu(L) \cap O^\nu(\infty)(\alpha < 1, \nu > 1)$, then the Cauchy-type integral given by (1) is a sectionally holomorphic function with jumping curve $L$. 

4 Behavior of of the Cauchy principal value integral

Theorem 41 (Privalov) Let \( f \in O^\alpha(0) \cap O^\nu(\infty)(\alpha < 1, \nu > 1) \), and \( f \in H^\mu(L) \). Then we have

\[
C[f], C[f]^\pm \in \begin{cases} 
H^\mu(L), & 0 < \mu < 1, \\
H^\mu_c(L)(0 < \epsilon < 1), & \mu = 1.
\end{cases}
\]  

(30)

Proof For any connected sub-arc \( L_\delta L_\Delta \) of \( L \setminus L_0 \), rewrite \( C[f] \) as

\[
C[f](t) = \frac{1}{2\pi i} \int_{L_0}^{L_\Delta} \frac{f(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{L_{\nu,\Delta}}^{L_{\nu,\epsilon}} \frac{f(\tau)}{\tau - t} d\tau + \frac{1}{2\pi i} \int_{L_{\tau,\Delta}}^{L_{\tau,\epsilon}} \frac{f(\tau)}{\tau - t} d\tau
\]

\[
\triangleq F_1(t) + F_2(t) + F_3(t), t \in L_\delta L_\Delta.
\]  

(31)

Since \( F_1 \) and \( F_3 \) are analytic functions on \( L_\delta L_\Delta \), so \( F_1, F_3 \in H^1(L_\delta L_\Delta) \). By [9,10] we have

\[
F_2(t) \in \begin{cases} 
H^\mu(L_\delta L_\Delta), & 0 < \mu < 1, \\
H^\mu(L_\delta L_\Delta)(0 < \epsilon < 1), & \mu = 1.
\end{cases}
\]  

(32)

Then (30) follows.

Theorem 42 Let \( f \in H^\alpha_*(0) \cap O^\nu(\infty)(\alpha < 1, \nu > 1) \), and locally integrable. Then \( C[f] \in H^*(0) \). Moreover, if

\[
f(\tau) = \frac{f^*(\tau)}{\tau^\lambda},
\]

where \( f^* \in H^\mu(L_\Delta \overline{L_\delta}) \), \( 0 \leq \alpha = \text{Re}(\lambda) < 1 \), and \( \tau^\lambda \) is given by the boundary value of \( z^\lambda \) as \( z \to \tau^+ \). Then we have

\[
\langle C(f) \rangle(t) = \begin{cases} 
-\frac{f(0)}{2\pi} \log(-t) + \phi(t), & \lambda = 0, \\
\frac{f(0)}{2\pi} \frac{\phi(0)}{\tau^\lambda} + \frac{\phi(0)}{\tau^\lambda}, & \lambda \neq 0,
\end{cases}
\]  

(33)

near \( t = 0 \), where \( \phi \in H(L_0 L_\eta)(\eta < \delta) \).

Proof By [9], we have

\[
\left( C_{L_\delta L_\alpha}[f] \right)(t) = \begin{cases} 
-\frac{f(0)}{2\pi} \log(-t) + \phi_0(t), & \lambda = 0, \\
\frac{f(0)}{2\pi} \frac{\phi_0(0)}{\tau^\lambda} + \frac{\phi_0(t)}{\tau^\lambda}, & \lambda \neq 0,
\end{cases}
\]  

(34)

near \( t = 0 \), where \( \phi \in H(L_0 L_\eta)(\eta < \delta) \).

Again by (34) and \( \left( C_{L_\delta L_\infty}[f] \right)(t) \in H^1(0) \), Theorem 42 follows.

Theorem 43 Let \( f \in O^\nu(0) \cap O^\nu(\infty) \cap H^\mu_*(\infty)(\alpha < 1, \nu > 1, \nu - 1 > 2\mu) \), and locally integrable. Then \( C[f] \in H^0(\infty) \).
Proof
Since \( f \in O^\nu(\infty) \), there exist \( \exists \Delta > 1 \) and \( f^*(t) \in \hat{H}^\nu(\infty) \) such that \( f(t) = \frac{f^*(t)}{t} \) for \( t \in \hat{L}_\Delta \hat{L}_\infty = L \cap \{ t \in \mathbb{C} : |t| \geq \Delta \} \). For \( t_1 = \rho_1 e^{i\phi_1}, t_2 = \rho_2 e^{i\phi_2} \in \hat{L}_\Delta \hat{L}_\infty \), assume \( \rho_2 > \rho_1 \geq 2\Delta \).

Take \( h = \frac{\tau}{\nu} \). Let \( |t_2 - t_1| \leq h \). Then by Lemma A2 in Appendix, we have

\[
|\tau - t_2| \geq \frac{1}{2} |t_1 - t_2|
\]  

(35)

for \( \tau \in \hat{L}_{\frac{\rho_1}{2}, \frac{\rho_1}{2} + \rho_2} \).

If \( h \leq |t_1 - t_2| \) and \( |t_2| \leq 2 |t_1| \), by (20) we obtain

\[
|C[f](t_1) - C[f](t_2)| \leq M\left( \frac{1}{|t_1|^\eta} + \frac{1}{|t_2|^\eta} \right) \leq 2M \frac{1}{|t_1|^\eta}
\]

\[
\leq 2M \frac{|t_1 - t_2|^\eta}{|t_1 - t_2|^\eta|t_1|^\frac{\eta}{2}}
\]

\[
\leq 2M \frac{|t_1 - t_2|^\eta}{h^\frac{\eta}{2}|t_1|^\frac{\eta}{2}|t_2|^\frac{\eta}{2}}
\]

\[
\leq M' \frac{|t_1 - t_2|^\eta}{|t_1|^\frac{\eta}{2} |t_2|^\frac{\eta}{2}} = M' \frac{1}{t_1} - \frac{1}{t_2} |^\eta. \quad (36)
\]

If \( 2|t_1| \leq |t_2| \), by (20) we obtain

\[
|C[f](t_1) - C[f](t_2)| \leq M\left( \frac{1}{|t_1|^\eta} + \frac{1}{|t_2|^\eta} \right) \leq M \frac{2|t_2|^\eta}{|t_1|^\eta |t_2|^\eta}
\]

\[
\leq 2^{\eta + 1} M \frac{|\rho_2 - \rho_1|^\eta}{|t_1|^\eta |t_2|^\eta}
\]

\[
\leq M' \frac{1}{t_1} - \frac{1}{t_2} |^\eta. \quad (37)
\]

So we assume that \( \rho_2 \leq 2\rho_1 \), or both \( |t_2 - t_1| \leq h \) and \( |t_2| \leq 2 |t_1| \) hold. Now we rewrite \( C[f] \) as follows,

\[
C[f](t) = \frac{1}{2\pi i} \left( \int_{L_0^{\frac{\rho_1}{2}}} \frac{f(\tau)}{\tau - t} d\tau + \int_{L_{\frac{\rho_1}{2}}}^{L_{2\rho_1}} \frac{f(\tau)}{\tau - t} d\tau + \int_{L_{2\rho_1}}^{L_{\infty}} \frac{f(\tau)}{\tau - t} d\tau \right)
\]

\[
\triangleq \frac{1}{2\pi i} \left( A_1(t) + A_2(t) + A_3(t) \right) \quad (38)
\]

We first consider \( A_1(t) \). For \( \tau \in \hat{L}_0 \hat{L}_{\frac{\rho_1}{2}} \), we have \( |\tau - t_1| > |t_1| - |t| > \frac{1}{2} \rho_1 \), \( |\tau - t_2| > |t_2| - |\tau| > \frac{1}{2} \rho_2 \), and

\[
|\frac{1}{t_1} - \frac{1}{t_2}| = \left| \frac{t_1 - t_2}{t_1 t_2} \right| < \frac{2}{|t_1|} < \frac{1}{\Delta} < 1.
\]
Thus we obtain

\[ |A_1(t_1) - A_1(t_2)| = \left| \int_{L_0}^{L_{\rho_1}} \frac{(t_1 - t_2)f(\tau)}{(\tau - t_1)(\tau - t_2)} \, d\tau \right| \]
\[ \leq \frac{4|t_1 - t_2|}{|t_1||t_2|} \left| \int_{L_0}^{L_{\rho_1}} |f(\tau)||d\tau| \right| \]
\[ \leq 4 \frac{1}{t_1 - t_2} \left( \int_0^\Delta |f(\theta e^{i\theta})|(1 + \Delta) \, d\theta \right) \]
\[ + \int_\Delta ^{2\rho_1} |f(\theta e^{i\theta})||(1 + i\theta)| \, d\theta \]
\[ \leq 4 \frac{1}{t_1 - t_2} \left( M_1 + \int_\Delta ^{2\rho_1} 2M^* \frac{1}{\theta^{\nu-1}} \, d\theta \right) \]
\[ \leq 4M_1 \left( \frac{1}{t_1 - t_2} \right)^{\frac{1}{2}} + 4 \left( \frac{1}{t_1 - t_2} \right)^{\frac{1}{2}} \int_\Delta ^{2\rho_1} 2M^* \frac{1}{\theta^{\nu-1}} \, d\theta. \] (39)

But

\[ \int_\Delta ^{2\rho_1} \frac{1}{\theta^{\nu-1}} \, d\theta = \begin{cases} \frac{1}{2-\nu} \left( \frac{1}{2\rho_1} \right)^{2-\nu} - \Delta^{2-\nu} < \frac{1}{2-\nu} \left( \frac{1}{2\rho_1} \right)^{2-\nu}, & \nu < 2, \\ \log \left( \frac{1}{2\rho_1} \right) - \log \Delta < \rho_1^2, & \nu = 2, \\ \frac{1}{\nu-2} \left( \frac{1}{2\rho_1} \right)^{2-\nu} - \Delta^{2-\nu} < \frac{1}{\nu-2} \, \rho_1, & \nu > 2. \end{cases} \] (40)

Notice that \( \frac{|t_1 - t_2|}{|t_1|} = \frac{2}{\rho_1} \). Take \( s = \min \{ \nu - 1, \frac{1}{2} \} \). By (39) and (40) we obtain

\[ |A_1(t_1) - A_1(t_2)| \leq M \left| \frac{1}{t_1 - t_2} \right|^s \] (41)

Recall that \( |\theta - \rho_1| > \frac{1}{2} \theta, |\theta - \rho_2| > \frac{1}{2} \theta \) for \( \theta \in L_{2\rho_2}L_\infty \). Then we have

\[ |A_3(t_1) - A_3(t_2)| = \left| \int_{L_{2\rho_2}}^{L_\infty} \frac{(t_1 - t_2)f(\tau)}{(\tau - t_1)(\tau - t_2)} \, d\tau \right| \]
\[ \leq |t_1 - t_2| \int_{2\rho_2}^{\infty} \frac{|f(\theta e^{i\theta})|2\theta}{|\theta - \rho_1||\theta - \rho_2|} \, d\theta \]
\[ \leq 8M^* |t_1 - t_2| \int_{2\rho_2}^{\infty} \frac{1}{\theta^{\nu+1}} \, d\theta \]
\[ \leq 8M^* |t_1 - t_2| \frac{1}{\nu(2\rho_2)^\nu} = \frac{8M^* |t_1 - t_2|}{\nu 2^\nu 12^{\nu-2}} |t_2|^{\nu-2}. \] (42)

When \( \nu \geq 2 \), we have

\[ \frac{|t_1 - t_2|}{|t_2|^{\nu-2}} < \frac{|t_1 - t_2|}{|t_2|^2} < \frac{|t_1 - t_2|}{|t_1 t_2|}. \] (43)
When $\nu < 2$, take $s = \nu - 1$, $p = \frac{1}{s}$ and $q = \frac{1}{1-s}$. By Young inequality, we obtain
\[
\frac{|t_1 - t_2|}{|t_2|^\nu} < \frac{|t_1 - t_2|}{t_1 t_2} + \frac{|t_1|^s |t_1 - t_2|^{1-s}}{|t_2|}
\]
\[
< \frac{|t_1 - t_2|}{t_1 t_2} + \frac{|\tau_1|^{s} + |\tau_1 - \tau_2|^{1-s}}{q}
\]
\[
= \frac{|t_1 - t_2|}{t_1 t_2} + \frac{|t_1|^s}{|t_2|} + \frac{|t_1 - t_2|^{1-s}}{q} \leq 2 \frac{|t_1 - t_2|}{t_1 t_2} \quad (44)
\]
since $|t_1| < |t_2|$ and $|t_1 - t_2| < 2|t_2|$. Now by (42), (43) and (44), we obtain
\[
|A_3(t_1) - A_3(t_2)| < M \frac{1}{|t_1|} - \frac{1}{|t_2|^s}. \quad (45)
\]

Now we rewrite $A_2(t)$ as
\[
A_2(t) = \int_{L_{\frac{1}{2}, \rho_1}} f(t) \frac{f(t)}{\tau - t} \, d\tau + \int_{L_{\frac{1}{2}, \rho_1}} \frac{1}{\tau - t} \, d\tau
\]
\[
\leq B_1(t) + B_2(t). \quad (46)
\]
Notice that
\[
B_2(t) = f(t)(\log \frac{L_{2\rho_2} - t}{L_{2\rho_2} - t} + i\pi)
\]
and
\[
|\log \frac{L_{2\rho_2} - t}{L_{2\rho_2} - t} + i\pi| < 5|t_2|^\nu - 2\mu
\]
when $\rho_1$ large enough. And since $\log \frac{L_{2\rho_2} - z}{L_{2\rho_2} - z}$ is analytic on $L_{t_1, t_2}$ we obtain
\[
|\log \frac{L_{2\rho_2} - t}{L_{2\rho_2} - t} | \leq 3 \frac{|t_1 - t_2|}{|t_1|}
\]
when $\rho_1$ large enough. So we obtain
\[
|B_2(t_1) - B_2(t_2)| \leq |(f(t_1) - f(t_2)) \log \frac{L_{2\rho_2} - t}{L_{2\rho_2} - t}|
\]
\[
+ |f(t_2)(\log \frac{L_{2\rho_2} - t_1}{L_{2\rho_2} - t_1} - \log \frac{L_{2\rho_2} - t_2}{L_{2\rho_2} - t_2})|
\]
\[
\leq M \frac{|t_1 - t_2|^\mu}{|t_2|^\nu} + M \frac{|t_1 - t_2|^{1-\mu}}{|t_2|^{\nu-1}} + M \frac{|t_1 - t_2|^{1-\mu}}{|t_1 t_2|^\nu} + M \frac{|t_1 - t_2|^\mu}{|t_1 t_2|^\nu}
\]
\[
= 5M \frac{|t_1 - t_2|^\mu}{|t_2|^\nu} + 3M \frac{|t_1 - t_2|^{1-\mu}}{|t_1 t_2|^\nu} \quad (47)
\]
when \( \rho_1 \) is large enough.

Notice that
\[
\frac{f(\tau) - f(t_1)}{\tau - t_1} - \frac{f(\tau) - f(t_2)}{\tau - t_2} = \frac{(f(\tau) - f(t_1))(t_1 - t_2) + (f(t_1) - f(t_2))(t_1 - \tau)}{(\tau - t_1)(\tau - t_2)} = \frac{(f(\tau) - f(t_2))(t_1 - t_2) + (f(t_1) - f(t_2))(t_2 - \tau)}{(\tau - t_1)(\tau - t_2)}.
\]

We obtain
\[
|B_1(t_1) - B_1(t_2)| \leq \int_{L_{\frac{1}{2}\rho_1}} \left| \frac{(f(\tau) - f(t_1))(t_1 - t_2) + (f(t_1) - f(t_2))(t_1 - \tau)}{(\tau - t_1)(\tau - t_2)} \right| d\tau
\]
\[
+ |f(t_1) - f(t_2)| \int_{L_{\frac{1}{2}\rho_1}} \frac{1}{\tau - t_2} d\tau
\]
\[
+ \int_{L_{\frac{1}{2}\rho_1}} \frac{(f(\tau) - f(t_1))(t_1 - t_2)}{(\tau - t_1)(\tau - t_2)} d\tau
\]
\[
+ |f(t_1) - f(t_2)| \int_{L_{\frac{1}{2}\rho_1}} \frac{1}{\tau - t_1} d\tau
\]
\[
\triangleq C_1 + C_2 + C_3 + C_4. \tag{48}
\]

By \( \nu - 2\mu > \nu - 1 - \mu > 0 \), we obtain
\[
C_2 \leq M \frac{|t_1 - t_2|^{\mu}}{|t_2|} \log \rho_2 = M \frac{|t_1 - t_2|^{\mu}}{|t_2|} \log \rho_2 < M \frac{|t_1 - t_2|^{\mu}}{|t_1 t_2|^{\mu}}. \tag{49}
\]

and
\[
C_4 \leq M \frac{|t_1 - t_2|^{\mu}}{|t_1 t_2|^{\mu}}. \tag{50}
\]

Let \( |t_2 - t_1| < h \) and \( 2\rho_1 > \rho_2 \). By \( (35) \), we obtain
\[
C_1 \leq |t_1 - t_2|^{\frac{\mu}{2}} \int_{L_{\frac{1}{2}\rho_1}} \frac{M_1 |t_1 - t_2|^{1 - \frac{\mu}{2}}}{\max \{|\tau|, |t_1|\} |\tau - t_1||\tau - t_2|} |d\tau|
\]
\[
\leq \frac{2^{\nu+1} M_1 |t_1 - t_2|^{\mu}}{\rho_1^{\nu-1}} \int_{\frac{\pi}{4}}^{\frac{\pi + \phi_2}{4}} \frac{1}{|\rho_1 - \theta|^{1 - \mu}(\rho_2 - \theta)^{2 - \frac{\mu}{2}}} d\theta \tag{51}
\]
take $p = \frac{1}{1-\frac{4}{n}}$, $q = \frac{2}{n}$, by H"{o}lder inequality
\begin{align*}
C_1 &\leq \frac{2^{n+1} M_1 |t_1 - t_2|^{\frac{q}{n}}}{\rho_1^{1-\frac{1}{n}}} \left( \int \frac{1}{|\theta_1 - (1-\mu)p\theta|^{\frac{1}{n}}} \frac{1}{(\rho_2 - \theta)^{\frac{1}{n}}} d\theta \right)^{\frac{n}{n+1}} \\
&\quad \cdot \left( \int \frac{\rho_1^{\frac{4}{n}}}{\rho_2 - \theta} d\theta \right)^{\frac{1}{n}} \\
&\leq M_3 |t_1 - t_2|^{\frac{q}{n}} \rho_1^{\frac{4}{n}} \rho_1 \mu_1 \leq M_3 \left| \frac{1}{t_1} - \frac{1}{t_2} \right| \mu_1 \quad \text{(52)}
\end{align*}

Substitute (35) by Lemma A3 in Appendix. By the same as the argument for $C_1$, we obtain
\begin{align*}
C_3 &\leq M_1 \left| \frac{1}{t_1} - \frac{1}{t_2} \right|^{\mu_1}, \mu_1 \in (0, 1). \quad \text{(53)}
\end{align*}

Take $\epsilon = \min\{\frac{q}{n}, \mu_1\}$. By (48)-(52) and (53) we obtain
\begin{align*}
|B_1(t_1) - B_1(t_2)| < M_1 \left| \frac{1}{t_1} - \frac{1}{t_2} \right| \epsilon. \quad \text{(54)}
\end{align*}

Again by (46), (47) and (54), we obtain
\begin{align*}
|A_2(t_1) - A_2(t_2)| \leq M_1 \left| \frac{1}{t_1} - \frac{1}{t_2} \right| \epsilon. \quad \text{(55)}
\end{align*}

At last, by Theorem 35, (36)-(38), (41), (45) and (55) we obtain $C[f] \in \hat{H}_0(\infty)$.

\section{5 Riemann boundary value problem}

Consider the Riemann boundary value problem on $L$.

\textbf{Problem A} (Riemann boundary value problem) Find a sectional holomorphic function $\Phi(z)$ with the jumping curve $L$, satisfying the boundary value condition and the growth condition at infinity
\begin{align*}
\begin{cases}
\Phi^+(t) = G(t)\Phi^-(t) + g(t), \quad t \in L \setminus L_0, \\
G.P(z^{-(m+1)}\Phi, \infty) = 0,
\end{cases}
\end{align*}

where $m$ is an integer, $G$ and $g$ are given functions on $L \setminus L_0$. This problem is denoted by $R_m$.

For the solvability of $R_m$, $G$ and $g$ need to satisfy some requirements. We state it step by step in the rest of this section.

The simplest $R_m$ problem is the Liouville problem.
**Problem B** (The Liouville Problem) Find a sectionally holomorphic function $\Phi(z)$ satisfying
\[
\begin{align*}
\Phi^+(t) &= \Phi^-(t), \quad t \in L \setminus L_0, \\
G.P(z^{-(m+1)}\Phi, \infty) &= 0,
\end{align*}
\] (57)

**Lemma 51** When $m \geq 0$, the solution of Liouville problem is an arbitrary polynomial of degree not exceeding $m$. When $m < 0$, there exists only trivial solution $\Phi(z) = 0$.

**Problem C** (The Jump Problem $R_m$) Find a sectional holomorphic function $\Phi(z)$ satisfying
\[
\begin{align*}
\Phi^+(t) &= \Phi^-(t) + g(t), \quad t \in L \setminus L_0, \\
G.P(z^{-(m+1)}\Phi, \infty) &= 0,
\end{align*}
\] (58)
where
\[
g \in H^+(0) \cap H_c(L \setminus L_0) \cap \mathcal{O}^{\nu+m_0}(\infty) \cap \mathcal{H}^{\mu}_{\nu,m_0}(\infty) \ (\nu > 1, 0 < \mu < 1),
\]
where $m_0 = \max\{0, -m - 1\}$.

When $m = -1$, we get the $R_{-1}$ problem.

**Problem D** (The Jump Problem $R_{-1}$) Find a sectional holomorphic function $\Phi(z)$ satisfying
\[
\begin{align*}
\Phi^+(t) &= \Phi^-(t) + g(t), \quad t \in L \setminus L_0, \\
G.P(\Phi, \infty) &= 0,
\end{align*}
\] (59)
where $g$ is the same as Problem C.

By Theorem 39 and Theorem 41 we obtain the solution of Problem D as follows.

**Lemma 52** The unique solution of the $R_{-1}$ problem is
\[
\Phi(z) = (C(g))(z) = \frac{1}{2\pi i} \int_L \frac{g(\tau)}{\tau - z} d\tau, \quad z \in \mathbb{C} \setminus L
\] (60)

**Theorem 53** When $m \geq 0$, the solution of the $R_m$ problem is
\[
\Phi(z) = (C(g))(z) = \frac{1}{2\pi i} \int_L \frac{g(\tau)}{\tau - z} d\tau + P_m(z), \quad z \in \mathbb{C} \setminus L,
\] (61)
where $P_m(z)$ is arbitrary polynomial of degree not greater than $m$. When $m = -1$, the unique solution of $R_m$ problem is (60). When $m < -1$, it is (uniquely) solvable if and only if
\[
\frac{1}{2\pi i} \int_L g(\tau)\tau^k d\tau = 0, \quad k = 0, 1, 2, \ldots, -m - 2.
\] (62)
And the solution is just (60) if solvable.
Proof When $m \geq 0$, by Lemma 5.2, $C(g)$ is a special solution of the $R_m$ problem. Therefore, $\Phi$ are the general solutions of the $R_m$ problem since $\Delta = \Phi - C(g)$ are the general solutions of the Liouville problem:

$$\begin{align*}
\Delta^+(t) &= \Delta^-(t), \ t \in L \setminus L_0, \\
G.P(z^{-(m+1)} \Delta, \infty) &= 0.
\end{align*}$$

(63)

When $m < 0$, it is obvious that $C(g)$ satisfies the boundary value condition in this problem. Thus, by Lemma 5.1, it is the unique solution of $R_m$ problem if and only if the following condition is fulfilled

$$G.P(z^{-(m+1)}C(g), \infty) = 0.$$ 

By

$$g \in H^\ast(0) \cap H_c(L \setminus L_0) \cap O^{\nu+m}(\infty) \cap \hat{H}_c^{\mu}(\infty) \ (\nu > 1, 0 < \mu < 1),$$

and Corollary 3.7, we obtain (62).

Remark 6 When $m < 0$, by Corollary 3.7 and (62), the solution of (5.5) can be rewritten as

$$\Phi(z) = \frac{z^{m+1}}{2\pi i} \int_L \frac{\tau^{-(m+1)} g(\tau)}{\tau - z} d\tau, \ z \in \mathbb{C} \setminus L.$$  

(64)

If $\nu - 1 > 2\mu$, by Theorem 4.3, $\Phi^\pm \in \hat{H}^{-\mu}(\infty).$

Problem E (The Jump Problem $Q_m$) Find a sectional holomorphic function $\Phi(z)$ with $L$ as its jump curve such that

$$\begin{align*}
\Phi^+(t) &= \Phi^-(t) + g(t), \ t \in L \setminus L_0, \\
G.P(z^{-m} \Phi, \infty) &= 1,
\end{align*}$$

(65)

where $g \in H^\ast(0) \cap H_c(L \setminus L_0).$ When $m \geq 0$, $g \in \hat{H}_c^{\mu}(\infty) \cap O^{\nu}(\infty).$ When $m < 0$, $g \in \hat{H}_c^{\mu,-m}(\infty) \cap O^{\nu-m}(\infty).$

Lemma 54 When $m \geq 0$, the solution of Problem E is

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{g(\tau)}{\tau - z} d\tau + P_m(z), \ z \in \mathbb{C} \setminus L,$$

(66)

where $P_m$ is an arbitrary polynomial of degree $m$ and the leading coefficient is 1. When $m < 0$, the solution of Problem E is

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{g(\tau)}{\tau - z} d\tau, \ z \in \mathbb{C} \setminus L,$$

(67)

if and only if the following conditions are fulfilled

$$\begin{align*}
\frac{1}{2\pi i} \int_L g(\tau) \tau^k d\tau &= 0, \ k = 0, 1, 2, \cdots, -m - 2, \\
\frac{1}{2\pi i} \int_L g(\tau) \tau^{-(m-1)} d\tau &= -1.
\end{align*}$$

(68)

when $m = -1$, $\frac{1}{2\pi i} \int_L g(\tau) \tau^k d\tau = 0, \ k = 0, 1, 2, \cdots, -m - 2$ does not occur.
Proof By Theorem 5.3 and Corollary 3.7, Lemma 5.4 can be proved.

Problem F (Homogeneous Boundary Value Problems) Find a sectional holomorphic function \( \Phi(z) \) satisfying
\[
\begin{align*}
\Phi^+(t) &= G(t)\Phi^-(t), \quad t \in L \setminus L_0, \\
G.P(z^{-m-1}\Phi, \infty) &= 0,
\end{align*}
\]
where \( G \in H(L) \) and \( G(t) \neq 0, \ t \in L \). In addition, \( G \) satisfies the infinity growth condition
\[
G(\infty) = 1, \ \log G \in O^\nu(\infty) \cap \mathcal{H}^\mu(\infty) \ (\nu > 1, 0 < \mu < 1),
\]
where \( \log G \) is the principle branch such that \( \log G(\infty) = 0 \).

Assume
\[
\frac{\log G(0)}{2\pi i} = \alpha + i\beta.
\]
Then, see [9,10], the integer
\[
\kappa = -[\alpha]
\]
is called the index of homogeneous boundary value problem (69), where \([\alpha]\) is the largest integer not exceeding \( \alpha \).

Problem G (Canonical Problem) Find a sectional holomorphic function with \( L \) as its jump curve such that
\[
\begin{align*}
\Phi^+(t) &= G(t)\Phi^-(t), \quad t \in L \setminus L_0, \\
G.P(z^\kappa \Phi, \infty) &= 1.
\end{align*}
\]
Denote by
\[
\Gamma(z) = \frac{1}{2\pi i} \int_L \frac{\log G(\tau)}{\tau - z} \, d\tau, \ z \in \mathbb{C} \setminus L.
\]
By \( \log G \in O^\nu(\infty) \cap \mathcal{H}^\mu(\infty) \ (\nu > 1, 0 < \mu < 1) \), Theorem 3.5 and Lemma 3.8, we obtain
\[
\Gamma(\infty) = 0, \ \Gamma(z) = -(\alpha + i\beta) \log(-z) + \Delta(z),
\]
where \( \Delta \) is holomorphic in the neighborhood cut by \( L \) near 0, and \( \lim_{z \to 0} \Delta(z) \) exists.

Let
\[
X(z) = z^{-\kappa} e^{\Gamma(z)}, \ z \in \mathbb{C} \setminus L,
\]
by (72),
\[
\begin{align*}
G.P(z^\kappa X, \infty) &= 1, \ G.P(zX, 0) &= 0.
\end{align*}
\]
then, by \( \log G \in H(L) \cap O^\nu(\infty) \cap \mathcal{H}^\mu(\infty) \ (\nu > 1, 0 < \mu < 1) \) and Theorem 3.9, we have
\[
X^+(t) = G(t)X^-(t), \ t \in L \setminus L_0.
\]
so, by (74) and (75), \( X \) is the solution to the canonical problem (70).
The uniqueness of the solution is shown below. If $\Phi$ is the solution of the canonical problem (70), let
\[ Q(z) = \frac{\Phi(z)}{X(z)}, \quad z \in \mathbb{C} \setminus L. \]  
(76)

Then by (70) and (74), $Q$ satisfies
\[
\begin{cases}
Q^+(t) = Q^-(t), & t \in L \setminus L_0, \\
G.P(Q, \infty) = 1.
\end{cases}
\]

By Lemma 5.4, $Q = 1$. That is, $\Phi = X$.

Summarizing the above discussion, we have the following theorem.

**Lemma 55** If $G \in H(L)$, $G(t) \neq 0$, $t \in L$ and $G(\infty) = 1$, $\log G \in O^\nu(\infty) \cap H^\mu_0(\infty)$ ($\nu > 1, 0 < \mu < 1$), the canonical problem (70) has a unique solution $X$, which is given by (73).

**Remark 7** $X$ is called the canonical solution of the homogeneous problem (69).

If $\Phi$ is the solution of the homogeneous problem (69), then $Q$ in (76) is the solution of the following Liouville problem
\[
\begin{cases}
Q^+(t) = Q^-(t), & t \in L \setminus L_0, \\
G.P(z^{-(m+1+\kappa)}Q, \infty) = 0.
\end{cases}
\]

By Lemma 5.1, one gets
\[ \Phi(z) = X(z)P_{\kappa+m}(z), \quad z \in \mathbb{C} \setminus L, \]
where $\kappa = -[\alpha]$, $P_{\kappa+m}(z)$ is arbitrary polynomial of degree not greater than $m + \kappa$, if $\kappa + m < 0$, $P_{\kappa+m} = 0$. In addition, it is easy to prove that $\Phi$ is the solution of the homogeneous problem (69).

**Lemma 56** When $\kappa \geq -m$, $\Phi$ is the solution of the homogeneous problem (69); When $\kappa < -m$, the unique solution of problem is $\Phi = 0$.

Now solve the problem (56). Let $X$ be given by (73). Denote by
\[ F(z) = \frac{\Phi(z)}{X(z)}, \quad z \in \mathbb{C} \setminus L. \]

If $\Phi$ is the solution of problem (56), then $F$ is a solution of the following jump problem:
\[
\begin{cases}
F^+(t) = F^-(t) + \frac{Q(t)}{X(t)}, & t \in L \setminus L_0, \\
G.P(z^{-(m+1+\kappa)}F, \infty) = 0.
\end{cases}
\]
(77)
By the conditions of the \( R_m \) Problem (58) and (69), and the additional condition \( \nu - 1 > 2\mu \), it can be proved that the following formula holds

\[
\frac{g}{X^+} \in H^\nu_\ast(0) \cap H_c(L \setminus L_0) \cap O^{\nu + m_\kappa}(\infty) \cap \bar{H}_{\nu_0, m_\kappa}(\infty),
\]

(78)

where \( m_\kappa = \max \{0, - (m + 1 + \kappa)\} \), \( \rho < 1, 0 < \mu' < 1, \nu' > 1 \).

We first prove \( \frac{g}{X^+} \in H^\nu_\ast(0) \). By Theorem 4.2, there exists \( X^* \in H(0) \) such that

\[
1 \frac{X^+(t)}{t^{\alpha - \alpha - i\beta}} = X^*(t).
\]

By \( g \in H^\nu_\ast(0) \), there exist \( \gamma \in (\alpha - [\alpha], 1) \) and \( g^* \in H(0) \) such that

\[
g(t) = g^*(t) t^\gamma.
\]

(79)

Thus (see [9, p. 222]), there exists \( \rho \in (\gamma + [\alpha] - \alpha, 1) \) such that

\[
\frac{g}{X^+} \in H^\nu_\ast(0).
\]

(80)

Next, we prove \( \frac{g}{X^+} \in H_c(L \setminus L_0) \cap O^{\nu + m_0}(\infty) \cap \bar{H}_{\nu, m_0}(\infty) \). By Theorem 4.1, Theorem 4.3 and \( \log G \in H(L) \cap \bar{H}_\nu(\infty) \) \( (\nu - 1 > 2\mu) \),

\[
\Gamma^+ \in H_c(L \setminus L_0) \cap \bar{H}(\infty),
\]

then

\[
X^+ \in H_c(L \setminus L_0), \quad e^{\Gamma^+} \in \bar{H}(\infty),
\]

(81)

by \( g \in H_c(L \setminus L_0) \) and (81),

\[
\frac{g}{X^+} \in H_c(L \setminus L_0).
\]

(82)

Let \( f(t) = \frac{g(t)}{X^+(t)}, f_{m_\kappa}(t) = t^{m_\kappa} f(t), b y \ X^+(t) = t^{\kappa} e^{\Gamma^+} \) and \( m_0 = m_\kappa + \kappa \),

\[
f_{m_\kappa}(t) = \frac{t^{m_\kappa} g(t)}{t^{-\kappa} e^{\Gamma^+}} = \frac{t^{m_0} g(t)}{e^{\Gamma^+}},
\]

by \( g \in O^{\nu + m_0}(\infty) \),

\[
f_{m_\kappa}(t) \in O^\nu(\infty), \quad \text{that is, } f(t) \in O^{\nu + \kappa}(\infty),
\]

(83)

by \( g \in \bar{H}^{\nu}_{\nu_0, m_0}(\infty) \) \( (0 < \mu < 1, \nu - 1 > 2\mu) \),

\[
g_{m_0}(t) = t^{m_0} g(t) \in \bar{H}^{\nu}_{\nu}(\infty),
\]

(84)
by (81) and (84), there exists $\Delta > 0$, for $|t_2| > |t_1| > \Delta(t_1, t_2 \in L)$, such that

\[
\frac{g_{m_0}(t_1)}{e^{\Gamma^+(t_1)}} \frac{g_{m_0}(t_2)}{e^{\Gamma^+(t_2)}} \leq \frac{g_{m_0}(t_1)}{e^{\Gamma^+(t_1)}} - \frac{g_{m_0}(t_2)}{e^{\Gamma^+(t_2)}} + \frac{g_{m_0}(t_2)}{e^{\Gamma^+(t_2)}},
\]

\[
\leq \frac{1}{e^{\Gamma^+(t_1)}} |M_1(t_1-t_2)|^\mu + \frac{1}{|t_2|^\mu} |\frac{1}{t_2}|^{1 - \frac{1}{\mu_1}} \leq \frac{M_1}{|t_2|^{\mu + \min\{\mu, \mu_1\}}}.
\]

Let $\mu' = \min\{\mu, \mu_1\}$, $\nu' = \nu - 2\mu + \min\{\mu, \mu_1\}$, by $\nu - 2\mu > 1$ and (85), we have $\mu' \in (0, 1), \nu' > 1$, then

\[
\frac{g}{X^+} \in H_{\nu', \mu^0}(\infty).
\]

From (80), (82), (83) and (86), one gets (78).

By Theorem 5.3, when $m + \kappa \geq 0$, the solution to the jumping problem (77) is

\[
F(z) = \frac{1}{2\pi i} \int_L \frac{g(\tau)}{X^+(\tau)(\tau - z)} d\tau + P_{m+\kappa}(z), \quad z \in \mathbb{C} \setminus L,
\]

where $P_{m+\kappa}$ is an arbitrary polynomial of degree not greater than $m + \kappa$. When $m + \kappa < 0$, the unique solution of problem $R_{m+\kappa}$ is $C(g/X^+)$, or $P_{m+\kappa} = 0$ in (87), if and only if the following conditions are fulfilled:

\[
\frac{1}{2\pi i} \int_L \frac{g(\tau)}{X^+(\tau)} z^k d\tau = 0, \quad k = 0, 1, \cdots, -m - \kappa - 2.
\]

When $k = m + \kappa = -1$, the condition (88) does not appear. So, if $\Phi$ is the solution to problem (56), then

\[
\Phi(z) = \frac{X(z)}{2\pi i} \int_L \frac{g(\tau)}{X^+(\tau)(\tau - z)} d\tau + X(z)P_{m+\kappa}(z), \quad z \in \mathbb{C} \setminus L.
\]

On the other hand, we want to verify that (89) is the solution to problem (56), obviously, we only need to verify the condition $G.P(z\Phi, 0) = 0$. Let

\[
\Phi_0(z) = \frac{1}{2\pi i} \int_L \frac{g(\tau)}{X^+(\tau)(\tau - z)} d\tau, \quad z \in \mathbb{C} \setminus L.
\]

By (79), there exist $f^* \in H(0)$ such that

\[
g(t) = \frac{f^*(t)}{X^+(t)} = \frac{f^*(t)}{t^\lambda}, \quad \lambda = \gamma + [\alpha] - \alpha - i\beta.
\]

By Lemma 3.8 and (90)

\[
\Phi_0(z) = \frac{e^{i\lambda\pi}}{2i\sin \lambda\pi} \frac{f^*(0)}{z^\lambda} + \Psi(z), \quad \text{where } \Psi = O(|z|^{-[\alpha] + \alpha}) \text{ near } z = 0.
\]
At the same time, by (72),
\[(\frac{X(z)}{(-z)^{\alpha_1 - \alpha_2}}) = O(1) \text{ near } z = 0.\] (92)

By (91), (92) and \(\gamma\) in (79),
\[\Phi(z) = O(|z|^{-\gamma}) \text{ near } z = 0, \text{ thus } G.P(z\Phi, 0) = 0.\] (93)

Summarizing the above discussion, we have the following theorem.

**Theorem 57** Under the conditions of the \(R_m\) Problem (58) and the homogeneous problem (69), and the additional condition \(\nu - 1 > 2\mu\), the solution of problem (56) is (89), where \(P_{m+\kappa}(z)\) is arbitrary polynomial of degree not greater than \(m + \kappa\). When \(m + \kappa = -1\), the solution of problem (56) is (89) and \(P_{m+\kappa}(z) = 0\). When \(m + \kappa < -1\), the unique solution of problem (56) is (89) and \(P_{m+\kappa}(z) = 0\) if and only if the solvability condition (88) is satisfied.

**References**

A

**Lemma A1** For \( z \notin L \), chose respectively \( z_1, z_2 \in L \) such that

\[
|z - z_1| = \min\{|\rho e^{i\rho} - z|, 0 \leq \rho \leq |z|\}
\]

and

\[
|z - z_2| = \min\{|\rho e^{i\rho} - z|, \rho \geq |z|\}.
\]

Then

\[
|z| < |z_1| + 4\pi.
\]

**Proof** Assume \( z_1 = \rho_1 e^{i\rho_1} \) and \( z_2 = \rho_2 e^{i\rho_2} \). Let \( \arg z \) be the principle branch, i.e., \( 0 \leq \arg z < 2\pi \). Then

\[
|\rho e^{i\rho} - z| \geq \rho - |z| \geq \left( \left\lfloor \frac{|z|}{2\pi} \right\rfloor + 1 \right) 2\pi + \arg z - |z|
\]

for \( \rho \geq \left( \left\lfloor \frac{|z|}{2\pi} \right\rfloor + 1 \right) 2\pi + \arg z \). Thus

\[
|z_2| \leq \left( \left\lfloor \frac{|z|}{2\pi} \right\rfloor + 1 \right) 2\pi + \arg z. \tag{94}
\]

Similarly

\[
|\rho e^{i\rho} - z| \geq |z| - \rho \geq |z| - \left( \left\lfloor \frac{|z|}{2\pi} \right\rfloor - 1 \right) 2\pi + \arg z
\]

for \( \rho \leq \left( \left\lfloor \frac{|z|}{2\pi} \right\rfloor - 1 \right) 2\pi + \arg z \). Thus again we obtain

\[
|z_1| \geq \left( \left\lfloor \frac{|z|}{2\pi} \right\rfloor - 1 \right) 2\pi + \arg z. \tag{95}
\]

By (94) and (95) we complete the proof of lemma.

**Lemma A2** Let \( t_1, t_2 \in L \) and \( |t_2 - t_1| \leq h = \frac{\pi}{3} \), then

\[
|\tau - t_2| \geq \frac{1}{2}|t_2 - t_1|
\]

for \( \tau \in L_{\frac{1}{2}}^{1/2} L_{\frac{1}{2}(\rho_1 + \rho_2)} \).

**Proof** Assume that \( |\tau - t_2| < 1 \) because \( 1 > \frac{\pi}{3} \). Denote by \( \tau = \theta e^{i\theta} \) and \( T = L_{\frac{1}{2}(\rho_1 + \rho_2)} \). Then we have

\[
|T - t_1|^2 = \rho_1^2 + \frac{(\rho_2 + \rho_1)^2}{4} - \rho_1 (\rho_2 + \rho_1) \cos \left( \frac{\rho_2 - \rho_1}{2} \right)
\]

and

\[
|T - t_2|^2 = \rho_2^2 + \frac{(\rho_2 + \rho_1)^2}{4} - \rho_2 (\rho_2 + \rho_1) \cos \left( \frac{\rho_2 - \rho_1}{2} \right).
\]
It is easy to check
\[ |T - t_2| - |T - t_1| \geq 0. \tag{96} \]

Denote by
\[ f(\theta) = |\tau - t_2|^2 = \rho_2^2 + \theta^2 - 2\theta \rho_2 \cos(\rho_2 - \theta), \theta \in (\rho_2 - \frac{\pi}{3}, \rho_2). \]
\( f(\theta) \) is decreasing since \( f'(\theta) < 0 \) for \( \theta \in (\rho_2 - \frac{\pi}{3}, \rho_2) \). Thus we obtain
\[ |\tau - t_2|^2 = f(\theta) \geq f\left(\frac{1}{2}(\rho_2 + \rho_1)\right) = |T - t_2|^2, \tau \in \mathbb{L}_{\frac{1}{2}\rho_1 L_{\frac{1}{2}(\rho_1 + \rho_2)}}^2. \tag{97} \]

Now by (96) and (97) we obtain
\[ \frac{|t_2 - t_1|}{|T - t_2|} \leq \frac{|t_2 - t_1|}{|T - t_1|} \leq \frac{|t_2 - T| + |T - t_1|}{|T - T|} \leq 2. \]

**Lemma A3** Let \( \rho_1 > \pi \) and \( |t_2 - t_1| \leq h = \frac{\pi}{3} \). Then
\[ |\tau - t_1| \geq \frac{1}{3}|t_2 - t_1| \]
for any \( \tau \in \mathbb{L}_{\frac{1}{2}(\rho_1 + \rho_2) L_{2\rho_2}}^2 \).

**Proof** By the same as the argument of (97), we have
\[ |\tau - t_1| \geq |T - t_1| \tag{98} \]
for the corresponding \( \tau \in \mathbb{L}_{\frac{1}{2}(\rho_1 + \rho_2) L_{2\rho_2}}^2 \).

Denote by \( T = L_{\frac{1}{2}(\rho_1 + \rho_2)} \), then
\[ |T - t_2|^2 = \frac{(\rho_1 + \rho_2)^2}{4} + \rho_2^2 - (\rho_1 + \rho_2)\rho_2 \cos\frac{\rho_2 - \rho_1}{2} \]
\[ = \rho_2^2(1 - \cos\frac{\rho_2 - \rho_1}{2}) + \frac{(\rho_1 + \rho_2)^2}{4} - \rho_1\rho_2 \cos\frac{\rho_2 - \rho_1}{2}. \]

Similarly,
\[ |T - t_1|^2 = \rho_1^2(1 - \cos\frac{\rho_2 - \rho_1}{2}) + \frac{(\rho_1 + \rho_2)^2}{4} - \rho_1\rho_2 \cos\frac{\rho_2 - \rho_1}{2}. \]

Because of \( \frac{(\rho_1 + \rho_2)^2}{4} - \rho_1\rho_2 \cos\frac{\rho_2 - \rho_1}{2} > 0 \) and \( \rho_2 \leq 2\rho_1 \), then we obtain
\[ \frac{|T - t_2|^2}{|T - t_1|^2} \leq \frac{\rho_2^2}{\rho_1^2} \leq 4, \]
thus we obtain
\[ |T - t_2| \leq 2|T - t_1|. \tag{99} \]

Now by (98) and (99) we obtain
\[ \frac{|t_2 - t_1|}{|\tau - t_1|} \leq \frac{|t_2 - t_1|}{|T - t_1|} \leq \frac{|t_2 - T| + |T - t_1|}{|t_1 - T|} \leq 3. \]