# Construction of Bernstein-based words and their patterns

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# Construction of Bernstein-based words and their patterns

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**Abstract**

With inspiration of the definition of Bernstein basis functions and their recurrence relation, in this paper we give construction of new concept so-called Bernstein-based words. By classifying these Bernstein-based words as first and second kind, we investigate their some fundamental properties involving periodicity and symmetricity. Providing schematic algorithms based on tree diagrams, we also illustrate the construction of the Bernstein-based words. Moreover, we give computational implementations of Bernstein-based words in the Wolfram Language. By executing these implementations, we present some tables of Bernstein-based words and their decimal equivalents. In addition, we present black-white and 4-colored patterns arising from the Bernstein-based words with their potential applications. We also give some finite sums and generating functions for the lengths of the Bernstein-based words. We show that these functions are of relationships with the Catalan numbers, the centered $m$-gonal numbers, the Laguerre polynomials, certain finite sums, and hypergeometric functions. We also raise some open questions and provide some comments on our results. Finally, we investigate relations between the slopes of the Bernstein-based words and the Farey fractions.

*Keywords:* Combinatorics on words, Bernstein basis functions, Recurrence

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relation, Tree diagrams, Computational implementation, Special functions, Generating functions, Farey fractions

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## 1. Introduction, definitions and preliminaries

The field of combinatorics on words is a quite new field that has been started to be studied in recent years by the researchers working on multivarious branches of mathematics such as number theory, group theory, theoretical computer science dealing with automata and formal languages. Combinatorics on words concentrates on the study of formal languages, words and strings formed by letters or symbols. In this aspect, the field of combinatorics on words is in essence to differ from combinatorics. The main idea behind the field of combinatorics on words is to make an investigation on words in either algebraic, combinatorial or algorithmic way. With the emergence of the book of Lothaire [13] providing a terminological and well-defined theory on combinatorics on words, this field has started to develop and grow even more. These developments encourage many researchers to define new word classes and still find their interesting and useful applications. Based upon the consequence of these developments, the source of our motivation in this paper is to construct new words, called Bernstein-based words, and present some their fundamental properties.

We first start with reminding terminology regarding the combinatorics on words, which can also be found in the books of Lothaire [13, 14, 15].

Let $\Sigma$ be a nonempty set called the alphabet, each element of which is called a letter. A finite sequence of letters, in the following form:

$$w = (a_1, a_2, \ldots, a_n), \quad \forall a_i \in \Sigma; \quad i = 1, 2, \ldots, n,$$

is called a finite word of length $n$ over the alphabet $\Sigma$. If we use $\Sigma^*$ to denote the set of all finite words over the alphabet $\Sigma$, then $w \in \Sigma^*$. Let $\epsilon$ denote the empty word which is a neutral element for concatenation. Then, $\Sigma^+ = \Sigma^* - \{\epsilon\}$ denotes the set of all finite nonempty words.

Let $w_1 = (a_1, a_2, \ldots, a_n) \in \Sigma^*$ and $w_2 = (b_1, b_2, \ldots, b_m) \in \Sigma^*$. Then, the concatenation $\bowtie$ of two words $w_1$ and $w_2$ is defined by the following function (binary operation):

$$\bowtie \colon \Sigma^* \times \Sigma^* \mapsto \Sigma^*$$

such that

$$w_1 \bowtie w_2 = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)\,. \tag{1.1}$$

It is clear that the concatenation, $\bowtie$, or so-called juxtaposition, of two words, is well-defined, internal and an associative binary operation which is not commutative. Due to this feature, the algebraic structure $(\Sigma^*, \bowtie)$ is a semigroup and called free semigroup over the alphabet $\Sigma$ (*cf.* [8], [13], [14], [15]). Therefore, a word in the form of

$$w = (a_1, a_2, \ldots, a_n)$$

can be expressed as

$$w = a_1 a_2 \ldots a_n. \tag{1.2}$$

We recall the length of the word $w$ which is the number of letters that forms the word $w$, and denoted by $|w|$. Thus, the length of the word $w_1 = (a_1, a_2, \ldots, a_n) \in \Sigma^*$ is given by

$$|w_1| = n$$

(see, for details, [8], [13], [14], [15]).

As for the Bernstein basis functions, $B_k^n(x)$, these functions are given by the following explicit formula involving the classical binomial coefficient:

$$\begin{aligned} &B_k^n(x) = \binom{n}{k} x^k (1-x)^{n-k}, \\ &(k = 0, 1, \ldots, n; \quad n \in \mathbb{N}_0)\,, \end{aligned} \tag{1.3}$$

which have relationships with a large number of concepts including the Catalan numbers, the binomial distribution, the proof of the Weierstrass approximation theorem, the Poisson distribution, Computer Aided Geometric Design (CAGD) involving Bezier curves and surfaces, splines and etc. Moreover, these functions

have found a wide variety of applications to themself in areas of mathematics (especially in generating functions theory, probability theory, approximation theory), engineering (especially in automobile engineering, machine learning, human-computer interaction systems and etc.) and almost all areas in recent years. For details, see [9, 12, 19, 20, 21] and also the cited references therein.

The recurrence relation for the Bernstein basis functions is given by

$$B_k^n(x) = (1-x)\, B_k^{n-1}(x) + x B_{k-1}^{n-1}(x) \tag{1.4}$$

such that $B_0^0(x) = 1$ and $B_k^n(x) = 0$ for $k < 0$ and $k > n$ (*cf.* [12, 19, 20, 21]).

The Bernstein basis functions satisfy the following symmetry identity:

$$B_{n-k}^n(1-x) = B_k^n(x), \tag{1.5}$$

(*cf.* [12, 19, 20, 21]).

As stated in Section 2, the reason why we named our words as *Bernstein-based words* is that they are constructed by the inspiration arising from the combinations of the equations (1.1), (1.2), (1.3), and (1.4).

Before presenting our main results in the next sections, we shall briefly summarize other auxiliary concepts and their definitions needed to obtain the findings of this paper, as follows:

The Catalan numbers are defined by

$$C_m = \frac{1}{m+1}\binom{2m}{m} = \prod_{k=2}^{m} \frac{m+k}{k}; \qquad (m \in \mathbb{N}_0) \tag{1.6}$$

which is also given by the following ordinary generating function:

$$\sum_{m=0}^{\infty} C_m t^m = \frac{1-\sqrt{1-4t}}{2t}, \tag{1.7}$$

where $0 < |t| \leq \frac{1}{4}$ (*cf.* [5, 24]).

The Catalan numbers arise in the solution of many kinds of combinatorial and real-world problems such as the Euler's polygon problem and polygon triangulations, ballot sequences, parenthesizations, and Dyck paths, binary trees, plane trees and various kinds of enumeration problems. For some applications in detail, see the book of Koshy [5] and Stanley [24].

The generalized hypergeometric series ${}_kF_r\left(\alpha_1,...,\alpha_k;\beta_1,...,\beta_r;z\right)$ is defined by

$$ {}_kF_r\left(\alpha_1,...,\alpha_k;\beta_1,...,\beta_r;z\right)=\sum_{n=0}^{\infty}\left(\frac{\prod\limits_{j=1}^{k}\left(\alpha_j\right)_n}{\prod\limits_{j=1}^{r}\left(\beta_j\right)_n}\right)\frac{z^n}{n!}, \tag{1.8} $$

where the above series converges for all $z$ if $k<r+1$, and for $|z|<1$ if $k=r+1$. Assuming that all parameters have general values, real or complex, except for the $\beta_j$; $(j=1,2,...,r)$ none of which is equal to zero or a negative integer such that $(\beta)_v$ denotes the Pochhammer's symbol, defined by

$$ (\beta)_v=\prod_{j=0}^{v-1}(\beta+j), $$

and $(\beta)_0=1$ for $\beta\neq 1$, $v\in\mathbb{N}$, and $\beta\in\mathbb{C}$ (*cf.* [23], [26]).

Considering

$$ \binom{\omega}{m}=\frac{\prod\limits_{j=0}^{m-1}(\omega-j)}{m!} \quad \text{and} \quad \binom{\omega}{0}=1, $$

the second author [23] introduced the sum $B_v(\omega;\lambda,p)$, involving higher powers of inverse binomial coefficients, by the following formula:

$$ B_v(\omega;\lambda,p)=\sum_{m=0}^{\infty}\frac{m^v\lambda^m}{\binom{\omega}{m}^p}, \tag{1.9} $$

whose generating function is given by the following hypergeometric series:

$$ {}_{p+1}F_p\left(1,\ldots,1;-\omega,\ldots,-\omega;(-1)^p\lambda e^z\right)=\sum_{v=0}^{\infty}B_v(\omega;\lambda,p)\frac{z^v}{v!} \tag{1.10} $$

where $v$, $p\in\mathbb{N}_0$, $-\omega\notin\{0,-1,-2,-3,\ldots\}$ and $\lambda\in\mathbb{R}$ (or $\mathbb{C}$) with $|\lambda|<1$ (*cf.* [23]).

In [22], the second author also introduced the combinatorial numbers $y_6(n,k;\lambda,p)$, involving higher powers of inverse binomial coefficients, by the following formula, for $n,m,p\in\mathbb{N}_0$:

$$ y_6(m,n;\lambda,p)=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}^p k^m\lambda^k, \tag{1.11} $$

and constructed the following generating functions for these numbers in terms of the hypergeometric series:

$$\frac{1}{n!}\ {}_pF_{p-1}\left(-n,\ldots,-n;1,\ldots,1;(-1)^p\,\lambda e^z\right)=\sum_{m=0}^{\infty}y_6(m,n;\lambda,p)\frac{z^m}{m!}, \tag{1.12}$$

where $n,p\in\mathbb{N}$ and $\lambda\in\mathbb{R}$ (or $\mathbb{C}$).

Now we briefly summarize our results in the next sections as follows:

In Section 2, we introduce Bernstein-based words and investigate their fundamental properties with examples and tables. We also give schematic algorithms of these words. In Section 3, we provide computational implementations for evaluating the Bernstein-based words in the Wolfram language. In Section 4, we construct some finite sums and generating functions for the lengths of the Bernstein-based words. We also derive some relations and results pertaining to the length of the Bernstein-based words. In the final section, we give relations between the slopes of the Bernstein-based words and the Farey fractions.

## 2. Bernstein-based words

In this section, inspired by the explicit formula (1.3) and the recurrence relation (1.4) of the Bernstein basis functions, we introduce two kinds of Bernstein-based words over the alphabet $\Sigma=\{0,1\}$.

### *2.1. Bernstein words of the first kind*

Here, by the following definition, inspired by the explicit formula (1.3) of the Bernstein basis functions, we first define so-called Bernstein words of the first kind as in the following definition:

**Definition 2.1.** Let $n,k\in\mathbb{N}_0$. Let $x\in\Sigma=\{0,1\}$. Let $\bowtie$ denote a binary operation as the concatenation of two words, based on the definition given in equations (1.1) and (1.2). Then, Bernstein words of the first kind $w_B\left(x;n,k\right)$ over the alphabet $\Sigma=\{0,1\}$ are defined by

$$w_B\left(0;n,k\right)=\underbrace{\bowtie_{i=1}^{k}0\bowtie_{i=1}^{n-k}1}_{\binom{n}{k}-\text{times}} \tag{2.1}$$

and

$$w_B\left(1;n,k\right)=\underbrace{\bowtie_{i=1}^{k}1\bowtie_{i=1}^{n-k}0}_{\binom{n}{k}-\text{times}} \tag{2.2}$$

with $w_B\left(x;0,0\right)=\epsilon$ and $w_B\left(x;n,k\right)=\epsilon$ when $k<0$ or $k>n$.

Using (2.1) and (2.2), some properties of the $w_B\left(x;n,k\right)$ are given as follows:

*Periodicity property:*

It is known that a periodic word can be expressed a positive power of a shorter word (*cf.* [6], [7], and see also cited references therein). The definitions, given by (2.1) and (2.2), mean that we first juxtapose $k$-times 0's (or 1's) with $(n-k)$-times 1's (or 0's). Then, the string obtained from the first process is brought side by side $\binom{n}{k}$ times to obtain the word $w_B\left(x;n,k\right)$. Here, $\binom{n}{k}$ times juxtaposition means that the words $w_B\left(x;n,k\right)$ can be expressed a positive power of a shorter word. That is, the words $w_B\left(x;n,k\right)$ are all periodic.

*Symmetry property with respect to vertical reflection:*

Let $a_1,\ldots,a_n$ be letters of an alphabet $\Sigma$. Then, the reversal of a word $w=a_1a_2\ldots a_n$ is the word reversal$(w)=a_na_{n-1}\ldots a_1$ (*cf.* [15, p. 4]). Consequently, by the aid of (2.1) and (2.2), the words $w_B\left(x;n,k\right)$ satify the following symmetry properties:

$$\text{reversal}\left(w_B\left(0;n,n-k\right)\right)=w_B\left(1;n,k\right)$$

and

$$\text{reversal}\left(w_B\left(1;n,n-k\right)\right)=w_B\left(0;n,k\right).$$

Note that the above symmetry properties are analogues of (1.5).

These symmetry properties also mean that a concatenation of the words

$$w_B\left(0;n,n-k\right)\quad\text{and}\quad w_B\left(1;n,k\right)$$

or

$$w_B\left(1;n,n-k\right)\quad\text{and}\quad w_B\left(0;n,k\right)$$

generates a palindrome word. For some applications of palindrome words, see also [4].

For instance, let us consider the following words, which are reversal of each other:

$$w_B(0;3,2) = 001001001 \quad \text{and} \quad w_B(1;3,1) = 100100100.$$

The spelling or pronunciation of any of the above forwards is the same as the spelling or pronunciation of the other backwards. The concatenation of them is given as

$$w_B(0;3,2) \bowtie w_B(1;3,1) = 001001001100100100.$$

which is a member of palindrome words, spelling the same backward as forward.

*Remark* 2.2. There are many other applications of (2.1) and (2.2). For instance, Ruskey et al. [16] used word analogues associated with (2.1) and (2.2) as factors of gray codes while investigating the binary bubble languages and cool-lex order.

### *2.2. Bernstein words of the second kind*

Here, inspired by the recurrence relation (1.4) of the Bernstein basis functions, secondly we define Bernstein words of the second kind as in the following definition:

**Definition 2.3.** Let $n, k \in \mathbb{N}_0$. Let $x \in \Sigma = \{0,1\}$. Let $\bowtie$ denote a binary operation as the concatenation of two words, based on the definition given in equations (1.1) and (1.2). Then, Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ over the alphabet $\Sigma = \{0,1\}$ are defined by the following recurrence relations:

$$\mathcal{W}_B(0;n,k) = 1 \bowtie \mathcal{W}_B(0;n-1,k) \bowtie 0 \bowtie \mathcal{W}_B(0;n-1,k-1) \tag{2.3}$$

and

$$\mathcal{W}_B(1;n,k) = 0 \bowtie \mathcal{W}_B(1;n-1,k) \bowtie 1 \bowtie \mathcal{W}_B(1;n-1,k-1). \tag{2.4}$$

with $\mathcal{W}_B(x;0,0) = 1$ and $\mathcal{W}_B(x;n,k) = 0$ when $k < 0$ or $k > n$.

For example, substituting $x = 0$, $k = 1$ and $n = 1$ into (2.3), we get

$$\begin{aligned} \mathcal{W}_B(0;1,1) &= 1 \bowtie \mathcal{W}_B(0;1,1) \bowtie 0 \bowtie \mathcal{W}_B(0;0,0) \\ &= 1001. \end{aligned}$$

Substituting $x = 1$, $k = 1$ and $n = 1$ into (2.4), we get

$$\begin{aligned} \mathcal{W}_B(1;1,1) &= 0 \bowtie \mathcal{W}_B(1;0,1) \bowtie 1 \bowtie \mathcal{W}_B(1;0,0) \\ &= 0011. \end{aligned}$$

Using (2.3) and (2.4), some properties of the $\mathcal{W}_B(x;n,k)$ are given as follows:

As can be seen from the two examples above, the Bernstein words of the second kind are *not periodic*.

Observe that unlike the Bernstein words of the first kind, the Bernstein words of the second kind do not satisfy the symmetry property with respect to vertical reflection. However, in this study it is given as an open problem, which subclasses of the set of all Bernstein words of the second kind will satisfy property that of.

**Open Question 1:** *When we consider the set of all Bernstein words of the second kind, which subclasses of this set can be symmetric with respect to vertical reflection or periodic, or none?*

*2.3. Tree diagram for construction of the Bernstein words of the second kind*

To illustrate the construction of the Bernstein words of the second kind in a schematic way, in Figure 1, we give tree diagram which shows the construction of the associated Bernstein words of the second kind by considering the concatenation based on the definition given in equations (1.1) and (1.2). In Figure 1, blue edges (left) of the tree correspond to the concatenation by 1 from left (namely, juxtapose with the prefix 1) and red edges (right) of the tree correspond to the concatenation by 0 from left (namely, juxtapose with the prefix 0). Let the letter 1 be the root of the tree. In order to generate words in any next level of the trees, we concatenate two new words derived from the rule on the edges out of the previous nodes connecting to the corresponding node.

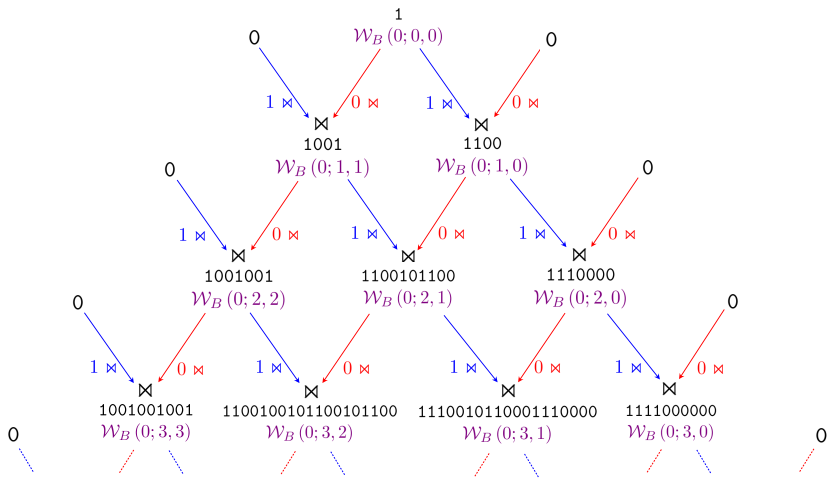


**Figure 1:** Tree diagram which shows the construction of the Bernstein words $\mathcal{W}_B(0;n,k)$ of the second kind.

In Figure 2, red edges (left) of the tree correspond to the concatenation by 0 from left (namely, juxtapose with the prefix 0) and blue edges (right) of the tree correspond to the concatenation by 1 from left (namely, juxtapose with the prefix 1). Similarly, let the letter 1 be the root of the tree. To generate words in any next level of the trees, we concatenate two new words derived from the rule on the edges out of the previous nodes connecting to the corresponding node.

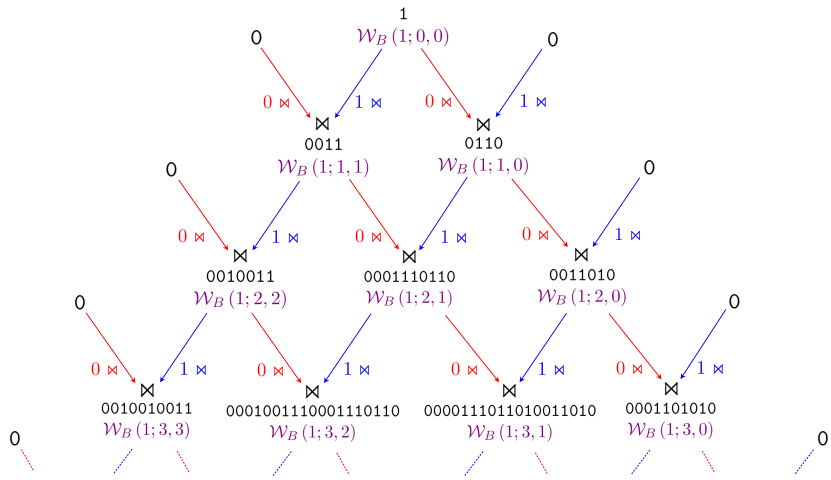


**Figure 2:** Tree diagram which shows the construction of the Bernstein words $\mathcal{W}_B(1;n,k)$ of the second kind.

In Figure 3, we give the lengths of the Bernstein words of the second kind, appeared in the Figure 1 and Figure 2, in the same geometric pattern. The sequences arising from these lengths will be discussed later in Section 4.

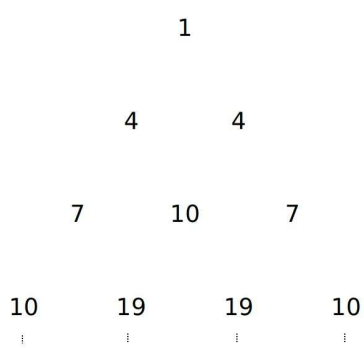


**Figure 3:** Lengths of the Bernstein words of the second kind, appeared in the Figure 1 and Figure 2, in the same geometric pattern.

*Remark* 2.4. The tree diagrams, given in Figure 1 and Figure 2, helps to researchers for making some constructions and algorithmic applications in fields of graph theory, automata theory and cryptology.

## 3. Computational implementations of Bernstein-based words

In this section, we provide a procedure `BernsteinWordsType1` (see: Implementation 1) by implementing (2.1) and (2.2), and also we provide another procedure `BernsteinWordsType2` (see: Implementation 2) by implementing the recurrence relations (2.3) and (2.4) in the Wolfram Language. By executing the procedures `BernsteinWordsType1` and `BernsteinWordsType2` in the Wolfram Mathematica version 12.0, and using the command `TableForm`, we present tables of the Bernstein words of the first kind $w_B(x;n,k)$ and Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ obtained just for a few special cases (among others).

### *3.1. Computational implementations for the Bernstein words of the first kind*

Here, we provide computational implementations for the Bernstein words of the first kind in the Wolfram language.

**Implementation 1:** The following code, involving the procedure `BernsteinWordsType1` written in the Wolfram Language, returns the words $w_B(x;n,k)$ for $x \in \Sigma = \{0,1\}$. Here, `Epsilon` denotes the empty word $\epsilon$.

```
BernsteinWordsType1[x_, n_, k_] /; k < 0 || k > n := "
    \[Epsilon]"
BernsteinWordsType1[x_, 0, 0] := "\[Epsilon]"
BernsteinWordsType1[x_?StringQ, n_, k_] :=First[{
        Factor1CaseZero="" <>Table["0",{j,1,k}];
        Factor2CaseZero="" <>Table["1",{j,1,n-k}];
        Factor1CaseOne="" <>Table["1",{j,1,k}];
        Factor2CaseOne="" <>Table["0",{j,1,n-k}];
```

```
Which[x==”0”,result=Factor1CaseZero<>
    Factor2CaseZero,
x ==”1”,result=Factor1CaseOne<>Factor2CaseOne
    ];
result=””<>Table[””<>result,{j,1,Binomial[n,k
    ]}]}]
```

By using Implementation 1 and the auxiliary commands of Wolfram language, we provide the following code written in Wolfram language:

```
TableForm[Evaluate[Table[BernsteinWordsType1[”0”, n, k
    ], {n, 0, 5}, {k, 0, 2}]], TableHeadings ->{{”n=0”
    , ”n=1”, ”n=2”, ”n=3”, ”n=4”, ”n=5”}, {”k=0”, ”k=1
    ”, ”k=2”}}]
```

which returns Table 1, whose entries are the Bernstein words of the first kind $w_B(x;n,k)$, in the case when $x=0$, $n=\{0,1,2,3,4,5\}$ and $k=\{0,1,2\}$.

| | k=0 | k=1 | k=2 |
|---|---|---|---|
| n=0 | $\epsilon$ | $\epsilon$ | $\epsilon$ |
| n=1 | 1 | 0 | $\epsilon$ |
| n=2 | 11 | 0101 | 00 |
| n=3 | 111 | 011011011 | 001001001 |
| n=4 | 1111 | 0111011101110111 | 001100110011001100110011 |
| n=5 | 11111 | 011110111101111011101111 | 001110011100111001110011100111001110011100111001110011 |

**Table 1:** The Bernstein words of the first kind $w_B(x;n,k)$ in the case when $x=0$, $n=\{0,1,2,3,4,5\}$ and $k=\{0,1,2\}$

In addition, by the following code written in Wolfram language:

```
TableForm[Evaluate[Table[BernsteinWordsType1[”1”, n, k
    ], {n, 0, 5}, {k, 0, 2}]], TableHeadings ->{{”n=0”
    , ”n=1”, ”n=2”, ”n=3”, ”n=4”, ”n=5”}, {”k=0”, ”k=1
    ”, ”k=2”}}]
```

we get Table 2, whose entries are the Bernstein words of the first kind $w_B(x;n,k)$, in the case when $x=1$, $n\in\{0,1,2,3,4,5\}$ and $k\in\{0,1,2\}$.

| | k=0 | k=1 | k=2 |
|---|---|---|---|
| n=0 | $\epsilon$ | $\epsilon$ | $\epsilon$ |
| n=1 | 0 | 1 | $\epsilon$ |
| n=2 | 00 | 1010 | 11 |
| n=3 | 000 | 100100100 | 110110110 |
| n=4 | 0000 | 1000100010001000 | 1100110011001100110011001100 |
| n=5 | 00000 | 10000100001000010000100001000010000 | 1100011000110001100011000110001100011000110001100011000 |

**Table 2:** The Bernstein words of the first kind $w_B(x;n,k)$ in the case when $x=1$, $n \in \{0,1,2,3,4,5\}$ and $k \in \{0,1,2\}$.

Note that the entries $\epsilon$ of Table 1 and Table 2 denote the empty word.

*3.2. Computational implementations for the Bernstein words of the second kind*

Here, we provide computational implementations for the Bernstein words of the second kind in the Wolfram language.

**Implementation 2:** The following code, involving the procedure `BernsteinWordsType2` written in the Wolfram Language, returns the words $\mathcal{W}_B(x;n,k)$ for $x \in \Sigma = \{0,1\}$.

```
BernsteinWordsType2[x_, n_, k_] /; k < 0 || k > n := "
    0"
BernsteinWordsType2[x_, 0, 0 ] := "1"
BernsteinWordsType2[x_?StringQ, n_, k_] :=
Which[x =="0", "1" <> BernsteinWordsType2[x, n-1, k]
    <> "0" <> BernsteinWordsType2[x, n-1, k-1],
x == "1", "0" <> BernsteinWordsType2[x, n-1, k] <> "1"
     <> BernsteinWordsType2[x, n-1, k-1]]
```

By using Implementation 2 and the auxiliary commands of Wolfram language, we also provide the following code written in Wolfram language:

```
TableForm[Evaluate[Table[BernsteinWordsType2["0", n, k
    ], {n, 0, 4}, {k, 0, 2}]], TableHeadings ->{{"n=0"
    , "n=1", "n=2", "n=3", "n=4"}, {"k=0", "k=1", "k=2
    "}}]
```

which returns Table 3, whose entries are the Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ in the case when $x=0$, $n\in\{0,1,2,3,4\}$ and $k\in\{0,1,2\}$.

| | k=0 | k=1 | k=2 |
|---|---|---|---|
| n=0 | 1 | 0 | 0 |
| n=1 | 1100 | 1001 | 0 |
| n=2 | 1110000 | 1100101100 | 1001001 |
| n=3 | 1111000000 | 1110010110001110000 | 1100100101100101100 |
| n=4 | 1111100000000 | 1111001011000111000001111000000 | 1110010010110010110001110010110001110000 |

**Table 3:** The Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ in the case when $x=0$, $n\in\{0,1,2,3,4\}$ and $k\in\{0,1,2\}$

In addition, by the following code written in Wolfram language:

```
TableForm[Evaluate[Table[BernsteinWordsType2["1", n, k
    ], {n, 0, 4}, {k, 0, 2}]], TableHeadings ->{{"n=0"
    , "n=1", "n=2", "n=3", "n=4"}, {"k=0", "k=1", "k=2
    "}}]
```

we get Table 4, whose entries are the Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ in the case when $x=1$, $n\in\{0,1,2,3,4\}$ and $k\in\{0,1,2\}$.

| | k=0 | k=1 | k=2 |
|---|---|---|---|
| n=0 | 1 | 0 | 0 |
| n=1 | 0110 | 0011 | 0 |
| n=2 | 0011010 | 0001110110 | 0010011 |
| n=3 | 0001101010 | 0000111011010011010 | 0001001110001110110 |
| n=4 | 0000110101010 | 0000011101101001101010001101010 | 0000100111000111011010000111011010011010 |

**Table 4:** The Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ in the case when $x=1$, $n\in\{0,1,2,3,4\}$ and $k\in\{0,1,2\}$

In Table 5, we present a decimal equivalents of the Bernstein words of the first kind $w_B(x;n,k)$ for the case when $x=0$, $n\in\{0,1,\ldots,15\}$ and $k=1$.

| | k=1 |
|---|---|
| n=0 | 0 |
| n=1 | 0 |
| n=2 | 5 |
| n=3 | 219 |
| n=4 | 30 583 |
| n=5 | 16 236 015 |
| n=6 | 33 814 345 695 |
| n=7 | 279 258 638 311 359 |
| n=8 | 9 187 201 950 435 737 471 |
| n=9 | 1 206 560 015 662 350 056 947 455 |
| n=10 | 633 205 725 040 689 368 685 058 981 375 |
| n=11 | 1 328 578 641 610 130 862 706 980 579 058 908 159 |
| n=12 | 11 147 649 675 553 647 270 017 976 875 240 829 304 698 879 |
| n=13 | 374 098 741 654 677 608 890 559 610 263 248 398 282 433 696 362 495 |
| n=14 | 50 213 748 704 928 086 076 131 552 136 232 920 089 648 434 055 403 681 079 295 |
| n=15 | 26 959 123 889 762 805 978 944 041 759 736 479 343 619 943 057 007 489 178 619 980 267 519 |

**Table 5:** Integers obtained by converting the Bernstein words of the first kind $w_B(x;n,k)$ to decimal in the case when $x=0$, $n\in\{0,1,\ldots,15\}$ and $k=1$.

In Table 6, we present a decimal equivalents of the Bernstein words of the first kind $w_B(x;n,k)$ for the case when $x=1$, $n=\{0,1,\ldots,15\}$ and $k=1$.

| | k=1 |
|---|---|
| n=0 | 0 |
| n=1 | 1 |
| n=2 | 10 |
| n=3 | 292 |
| n=4 | 34 952 |
| n=5 | 17 318 416 |
| n=6 | 34 905 131 040 |
| n=7 | 283 691 315 109 952 |
| n=8 | 9 259 542 123 273 814 144 |
| n=9 | 1 211 291 623 566 908 292 464 896 |
| n=10 | 634 444 875 187 540 032 811 644 224 000 |
| n=11 | 1 329 877 349 959 700 883 100 633 541 501 780 992 |
| n=12 | 11 153 095 522 976 975 871 517 741 397 407 532 201 281 536 |
| n=13 | 374 190 096 658 744 685 229 727 024 087 488 507 781 403 765 641 216 |
| n=14 | 50 219 879 061 258 806 145 241 078 635 089 742 567 989 253 056 020 871 127 040 |
| n=15 | 26 960 769 444 538 473 610 389 988 414 302 782 003 654 345 788 073 655 783 587 240 230 912 |

**Table 6:** Integers obtained by converting the Bernstein words of the first kind $w_B(x;n,k)$ to decimal in the case when $x=1$, $n\in\{0,1,\ldots,15\}$ and $k=1$.

In Table 7, we present a decimal equivalents of the Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ for the case when $x=0$, $n=\{0,1,\ldots,15\}$ and $k=1$.

| | k=1 |
|---|---|
| n=0 | 0 |
| n=1 | 9 |
| n=2 | 812 |
| n=3 | 470 128 |
| n=4 | 2 036 564 928 |
| n=5 | 68 551 451 877 120 |
| n=6 | 18 208 547 937 292 712 960 |
| n=7 | 38 435 859 475 728 710 580 563 968 |
| n=8 | 646 941 911 943 400 394 188 959 571 230 720 |
| n=9 | 86 971 679 750 389 756 074 485 227 918 487 065 657 344 |
| n=10 | 93 460 617 420 352 574 081 684 338 890 047 069 228 652 262 326 272 |
| n=11 | 803 144 806 349 129 759 355 741 991 213 423 752 868 260 161 293 451 476 860 928 |
| n=12 | 55 202 830 804 936 378 945 685 118 505 712 696 807 874 544 602 082 019 437 294 806 072 557 568 |
| n=13 | 30 351 139 309 558 186 954 230 650 981 997 648 463 698 028 347 652 318 857 320 105 397 792 195 154 374 819 840 |
| n=14 | 133 492 456 046 745 365 861 711 369 659 735 238 250 384 205 909 743 384 530 671 069 002 481 127 412 194 852 729 127 289 487 360 |
| n=15 | 4 696 966 705 074 203 326 538 999 460 271 928 244 237 556 412 863 889 593 327 541 289 556 977 680 853 840 716 057 160 471 539 176 957 687 103 488 |

**Table 7:** Integers obtained by converting the Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ to decimal in the case when $x = 0$, $n = \{0, 1, \ldots, 15\}$ and $k = 1$.

In Table 7, we present a decimal equivalents of the Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ for the case when $x = 1$, $n = \{0, 1, \ldots, 15\}$ and $k = 1$.

| | k=1 |
|---|---|
| n=0 | 0 |
| n=1 | 3 |
| n=2 | 118 |
| n=3 | 30 362 |
| n=4 | 62 182 506 |
| n=5 | 1 018 798 186 922 |
| n=6 | 133 535 915 956 307 626 |
| n=7 | 140 022 556 609 801 225 771 690 |
| n=8 | 1 174 594 338 557 431 440 918 209 129 130 |
| n=9 | 78 825 691 721 420 622 757 904 131 570 377 271 978 |
| n=10 | 42 319 221 003 509 939 675 643 946 324 756 438 191 169 776 298 |
| n=11 | 181 759 670 202 271 492 117 823 597 205 208 297 196 519 585 185 695 181 482 |
| n=12 | 6 245 214 714 004 014 108 473 393 029 547 573 098 615 200 827 168 375 131 936 985 361 066 |
| n=13 | 1 716 671 549 001 294 963 751 412 451 075 916 040 622 186 908 234 260 446 536 258 384 551 127 655 557 802 |
| n=14 | 3 775 000 658 398 322 345 452 452 839 003 268 816 749 619 334 587 136 883 005 228 154 804 677 079 317 460 160 176 892 586 |
| n=15 | 66 410 513 900 336 178 032 575 689 795 589 704 929 428 406 528 004 237 618 310 484 467 320 886 573 094 049 340 385 997 758 146 397 514 410 |

**Table 8:** Integers obtained by converting the Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ to decimal in the case when $x = 1$, $n = \{0, 1, \ldots, 15\}$ and $k = 1$.

### *3.3. Patterns arising from the Bernstein-based words*

Here, by representing each successive letter of the Bernstein-based words as a square block with 1s colored black and 0s colored white. Then, by placing corresponding square blocks side-by-side to be an row of colored squares, we present some patterns of the Bernstein-based words (see Figure 4).

**Figure 4:** The row of square blocks corresponding to the Bernstein word of the second kind $\mathcal{W}_B(1;3,2) = 0001001110001110110$.

By stacking up the row of square block representation of the first few Bernstein-based words, we obtain some patterns which are given in Figure 5-Figure 8.

**Figure 5:** Pattern obtained by the Bernstein words of the first kind $w_B(x;n,k)$ in the case when $x = 0$, $n = \{1, 2, \ldots, 8\}$ and $k = 1$

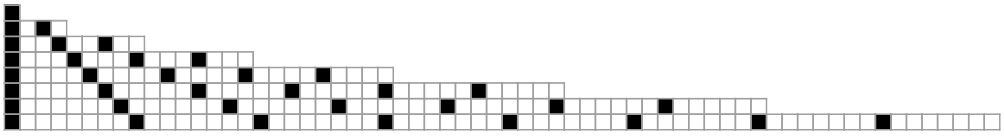

**Figure 6:** Pattern obtained by the Bernstein words of the first kind $w_B(x;n,k)$ in the case when $x = 1$, $n = \{1, 2, \ldots, 8\}$ and $k = 1$

*Remark* 3.1. It is well-known that the logical complement $\neg w$ (namely, so-called ones' complement or the Boolean complement in Boolean algebra) of a binary word $w$ is obtained by changing each 0 in $w$ to 1 and vice versa. Observe that Figure 5 and Figure 6 are logical complement of each other since we draw them by representing zeros in the words with the white square blocks and ones in the words with the black square blocks.

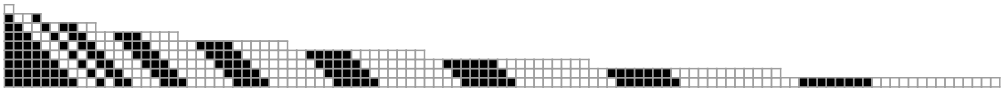

**Figure 7:** Pattern obtained by the Bernstein words of the second kind $\mathcal{W}_B(x; n, k)$ in the case when $x = 0$, $n = \{0, 1, \ldots, 8\}$ and $k = 1$

**Figure 8:** Pattern obtained by the Bernstein words of the second kind $\mathcal{W}_B(x; n, k)$ in the case when $x = 1$, $n = \{0, 1, \ldots, 8\}$ and $k = 1$

*Remark* 3.2. Observe that Figure 7 and Figure 8 are not logical complement of each other as opposed to the Figures arising from the Bernstein words of the first kind.

*Remark* 3.3. The DNA (*deoxyribonucleic acid*) is a nucleic acid that contains the genetic instructions and information used in the development and functioning of all known living organisms. The DNA is a strand composed of four nucleotides or bases called Adenine, Cytosine, Guanine and Thymine, abbreviated by A, C, G and T, respectively (*cf.* [11]). Considering that the obtained words are binary numbers, their 4-ary representations as well as their patterns may find application in pharmaceutical technologies, biotechnology, and DNA sequencing. For example; after associating 4-ary representations of the Bernstein-based words by the following morphism mapping letters 0, 1, 2 and 3 respectively to A, C, G and T:

$$0 \mapsto \mathrm{A}, \quad 1 \mapsto \mathrm{C}, \quad 2 \mapsto \mathrm{G}, \quad 3 \mapsto \mathrm{T},$$

it is also possible to determine of which cell gives the nucleotide base (nucleobase) sequence in the DNA molecule and which biological information this sequence encodes, this type studies also reveals an area of potential applica-

tion of the Bernstein-based words. For nucleotide base (nucleobase) sequences corresponding to the Bernstein-based words, see Table 9-Table 12.

| | k=1 |
|---|---|
| n=1 | A |
| n=2 | CC |
| n=3 | TCGT |
| n=4 | CTCTCTCT |
| n=5 | TTCTGTTCTGTT |
| n=6 | CTTCTTCTTCTTCTTCTT |
| n=7 | TTTCTTGTTTCTTGTTTCTTGTTT |
| n=8 | CTTTCTTTCTTTCTTTCTTTCTTTCTTTCTTT |

**Table 9:** The nucleotide base (nucleobase) sequence corresponding to the Bernstein words of the first kind $w_B(x; n, k)$ in the case when $x = 0$, $n = \{0, 1, \ldots, 8\}$ and $k = 1$

| | k=1 |
|---|---|
| n=1 | C |
| n=2 | GG |
| n=3 | CAGCA |
| n=4 | GAGAGAGA |
| n=5 | CAAGACAAGACAA |
| n=6 | GAAGAAGAAGAAGAAGAA |
| n=7 | CAAAGAACAAAGAACAAAGAACAAA |
| n=8 | GAAAGAAAGAAAGAAAGAAAGAAAGAAAGAAA |

**Table 10:** The nucleotide base (nucleobase) sequence corresponding to the Bernstein words of the first kind $w_B(x; n, k)$ in the case when $x = 1$, $n = \{0, 1, \ldots, 8\}$ and $k = 1$

| | k=1 |
|---|---|
| n=1 | GC |
| n=2 | TAGTA |
| n=3 | CTAGTACTAA |
| n=4 | CTGCCGATGAATTAAA |
| n=5 | TTGCCGATGAATTAAACTTAAAA |
| n=6 | TTTAGTACTAACTGAAATTGAAAATTTAAAAA |
| n=7 | CTTTAGTACTAACTGAAATTGAAAATTTAAAAACTTTAAAAAA |
| n=8 | CTTTGCCGATGAATTAAACTTAAAACTTGAAAAATTTGAAAAAATTTTAAAAAAA |

**Table 11:** The nucleotide base (nucleobase) sequence corresponding to the Bernstein words of the second kind $\mathcal{W}_B(x; n, k)$ in the case when $x = 0$, $n = \{0, 1, \ldots, 8\}$ and $k = 1$

| | k=1 |
|---|---|
| n=1 | T |
| n=2 | CTCG |
| n=3 | CTCGGCGG |
| n=4 | TGTCATCCACGGG |
| n=5 | TGTCATCCACGGGGACGGGG |
| n=6 | CTCGGCGGGATCCCAATCCCCAACGGGGG |
| n=7 | CTCGGCGGGATCCCAATCCCCAACGGGGGGAACGGGGGG |
| n=8 | TGTCATCCACGGGGACGGGGGAATCCCCCAAATCCCCCCAAACGGGGGGG |

**Table 12:** The nucleotide base (nucleobase) sequence corresponding to the Bernstein words of the second kind $\mathcal{W}_B(x;n,k)$ in the case when $x = 1$, $n = \{0, 1, \ldots, 8\}$ and $k = 1$

Moreover, after associating 4-ary representations of the Bernstein-based words by the following morphism mapping letters 0, 1, 2 and 3 respectively to Red, Green, Blue and Yellow colored square blocks:

$$0 \mapsto \text{Red colored square block},$$

$$1 \mapsto \text{Green colored square block},$$

$$2 \mapsto \text{Blue colored square block},$$

$$3 \mapsto \text{Yellow colored square block},$$

we get a row of square blocks for the first few Bernstein-based words and then by stacking up these rows, we also obtain some patterns which are given in Figure 9-Figure 12.

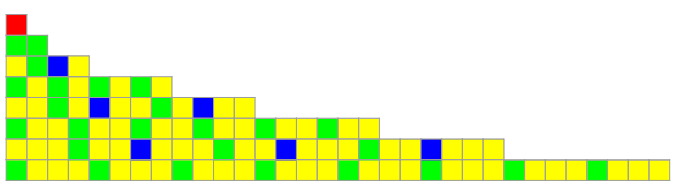

**Figure 9:** Pattern obtained by 4-ary representations of the Bernstein words of the first kind $w_B(x;n,k)$ in the case when $x = 0$, $n = \{1, \ldots, 8\}$ and $k = 1$.

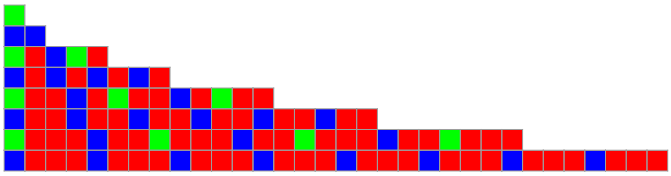

**Figure 10:** Pattern obtained by 4-ary representations of the Bernstein words of the second kind $w_B(x; n, k)$ in the case when $x = 1$, $n = \{1, \ldots, 8\}$ and $k = 1$.

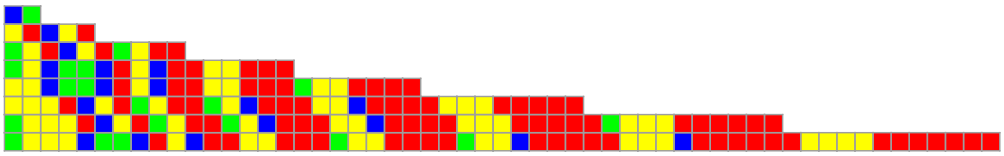

**Figure 11:** Pattern obtained by 4-ary representations of the Bernstein words of the second kind $\mathcal{W}_B(x; n, k)$ in the case when $x = 0$, $n = \{1, \ldots, 8\}$ and $k = 1$.

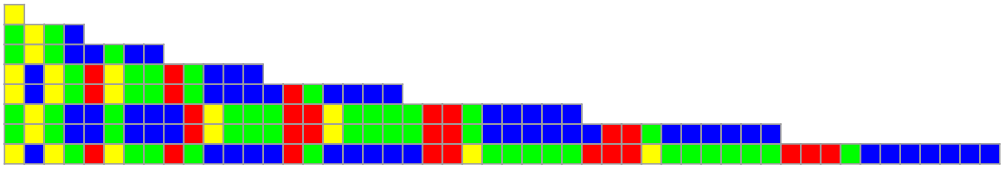

**Figure 12:** Pattern obtained by 4-ary representations of the Bernstein words of the second kind $\mathcal{W}_B(x; n, k)$ in the case when $x = 1$, $n = \{1, \ldots, 8\}$ and $k = 1$.

## 4. Relations arising from finite sums and generating functions for the lengths of the Bernstein-based words

In this section, we give some finite sums and generating functions for the lengths of the Bernstein-based words. Moreover, we give some relations and results derived from the length of the Bernstein words of the first and the second kind.

*4.1. Generating functions for the lengths of the Bernstein words of the first kind*

Here, we give some formulas, finite sums, and generating functions for the lengths $|w_B(x;n,k)|$ of the Bernstein words of the first kind.

The definitions, given by (2.1) and (2.2), mean that we first juxtapose $k$-times 0's or 1's with $(n-k)$-times 0's or 1's. Then, the words obtained from the first process is brought side by side $\binom{n}{k}$ times to obtain the word $w_B(x;n,k)$. Therefore, the length of the word $w_B(x;n,k)$ is equal to the product of $(k+n-k)$ and $\binom{n}{k}$ which yields the assertion of the following theorem:

**Theorem 4.1.** *Let $x \in \Sigma = \{0,1\}$ and $n,k \in \mathbb{N}_0$. Then, the length of the Bernstein words of the first kind $w_B(x;n,k)$ is given by*

$$|w_B(x;n,k)| = n\binom{n}{k}. \tag{4.1}$$

Using (4.1), we get Table 13 and Table 14 involving the lengths of the Bernstein words of the first kind $w_B(x;n,k)$ are provided as tables for the cases of $x \in \Sigma = \{0,1\}$, $n \in \{0,1,2,\ldots,15\}$ and $k \in \{0,1,2,\ldots,10\}$.

| | k=0 | k=1 | k=2 | k=3 | k=4 | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n=0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| n=1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| n=2 | 2 | 4 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| n=3 | 3 | 9 | 9 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| n=4 | 4 | 16 | 24 | 16 | 4 | 0 | 0 | 0 | 0 | 0 | 0 |
| n=5 | 5 | 25 | 50 | 50 | 25 | 5 | 0 | 0 | 0 | 0 | 0 |
| n=6 | 6 | 36 | 90 | 120 | 90 | 36 | 6 | 0 | 0 | 0 | 0 |
| n=7 | 7 | 49 | 147 | 245 | 245 | 147 | 49 | 7 | 0 | 0 | 0 |
| n=8 | 8 | 64 | 224 | 448 | 560 | 448 | 224 | 64 | 8 | 0 | 0 |
| n=9 | 9 | 81 | 324 | 756 | 1134 | 1134 | 756 | 324 | 81 | 9 | 0 |
| n=10 | 10 | 100 | 450 | 1200 | 2100 | 2520 | 2100 | 1200 | 450 | 100 | 10 |
| n=11 | 11 | 121 | 605 | 1815 | 3630 | 5082 | 5082 | 3630 | 1815 | 605 | 121 |
| n=12 | 12 | 144 | 792 | 2640 | 5940 | 9504 | 11 088 | 9504 | 5940 | 2640 | 792 |
| n=13 | 13 | 169 | 1014 | 3718 | 9295 | 16 731 | 22 308 | 22 308 | 16 731 | 9295 | 3718 |
| n=14 | 14 | 196 | 1274 | 5096 | 14 014 | 28 028 | 42 042 | 48 048 | 42 042 | 28 028 | 14 014 |
| n=15 | 15 | 225 | 1575 | 6825 | 20 475 | 45 045 | 75 075 | 96 525 | 96 525 | 75 075 | 45 045 |

**Table 13:** For $x \in \Sigma = \{0,1\}$, $n \in \{0,1,2,\ldots,15\}$ and $k \in \{0,1,2,\ldots,10\}$, the lengths of the words $w_B(x;n,k)$, i.e. $|w_B(x;n,k)|$.

| $k$ | $\left\{\left\|w_B\left(x;n,k\right)\right\|\right\}_{n=k}^{\infty}$ | Corresponding Sequence | Also, see `OEIS` |
|---|---|---|---|
| $k=0$ | $\{0,1,2,3,4,5,6,7,8,9,10,\dots\}$ | $\left\{n\right\}_{n=0}^{\infty}$ | `A001477` |
| $k=1$ | $\{1,4,9,16,25,36,49,64,81,100,\dots\}$ | $\left\{n^2\right\}_{n=1}^{\infty}$ | `A000290` |
| $k=2$ | $\{2,9,24,50,90,147,224,324,450,\dots\}$ | $\left\{\frac{(n-1)n^2}{2}\right\}_{n=2}^{\infty}$ | `A006002` |
| $k=3$ | $\{3,16,50,120,245,448,756,1200,\dots\}$ | $\left\{\frac{(n-2)(n-1)n^2}{6}\right\}_{n=3}^{\infty}$ | `A004320` |
| $k=4$ | $\{4,25,90,245,560,1134,2100,\dots\}$ | $\left\{n\binom{n}{4}\right\}_{n=4}^{\infty}$ | `A027764` |
| $k=5$ | $\{5,36,147,448,1134,2520,5082,\dots\}$ | $\left\{n\binom{n}{5}\right\}_{n=5}^{\infty}$ | `A027765` |

**Table 14:** Table of the lengths of the words $w_B\left(x;n,k\right)$, i.e. $\left|w_B\left(x;n,k\right)\right|$.

In Table 14, the second and third columns respectively shows the first terms of the sequences $\left\{\left|w_B\left(x;n,k\right)\right|\right\}_{n=k}^{\infty}$ for $k\in\{0,\dots,5\}$ and the symbolic notations of the corresponding sequences. As for the last column, it provides the IDs of the corresponding sequences in the Sloane's *On-Line Encyclopedia of Integer Sequences* (OEIS).

Some applications of (4.1) are give as follows:

Substituting $n=2m$ and $k=m$ into (4.1), we get

$$\left|w_B\left(x;2m,m\right)\right|=2m\binom{2m}{m}. \tag{4.2}$$

Combining (1.6) with (4.2) gives a relation, between the length of the words $w_B\left(x;2m,m\right)$ and the Catalan numbers $C_m$, given the following theorem:

**Theorem 4.2.** *Let $x\in\Sigma=\{0,1\}$ and $n\in\mathbb{N}_0$. Then, we have*

$$\left|w_B\left(x;2m,m\right)\right|=2m\left(m+1\right)C_m \tag{4.3}$$

*or, equivalently*

$$\left|w_B\left(x;2m,m\right)\right|=2m\left(m+1\right)\prod_{k=2}^{m}\frac{m+k}{k}. \tag{4.4}$$

The combination of (4.3) with (1.7) also yields the following corollary:

**Corollary 4.3.** *Let $x \in \Sigma = \{0,1\}$ and $0 < |t| \leq \frac{1}{4}$. Then we have*

$$\sum_{m=0}^{\infty} \frac{|w_B(x;2m,m)|}{m(m+1)} t^m = \frac{4}{1+\sqrt{1-4t}}. \tag{4.5}$$

Summing the equation (4.1) over all $0 \leq k \leq n$, we get

$$\sum_{k=0}^{n} |w_B(x;n,k)| = \sum_{k=0}^{n} n \binom{n}{k} \tag{4.6}$$

by which and by the well-known formula of the sum of the binomial coefficients, we have

$$\sum_{k=0}^{n} |w_B(x;n,k)| = \frac{1}{2} \sum_{j=0}^{n} j \binom{n}{j}. \tag{4.7}$$

Combining the above equation with the Eq.(1) of [22, p. 1329], we deduce to the following corollary:

**Corollary 4.4.** *Let $x \in \Sigma = \{0,1\}$ and $n \in \mathbb{N}_0$. Then, we have*

$$\sum_{k=0}^{n} |w_B(x;n,k)| = n2^n. \tag{4.8}$$

Combining (4.8) and (1.11), we obtain a relation, between the numbers $y_6(m,n;\lambda,p)$ and the finite sums of the lengths $|w_B(x;n,k)|$, as in the following corollary:

**Corollary 4.5.** *Let $x \in \Sigma = \{0,1\}$ and $n \in \mathbb{N}_0$. Then, we have*

$$\sum_{k=0}^{n} |w_B(x;n,k)| = nn!y_6(0,n;1,1). \tag{4.9}$$

Using (4.1), we get the ordinary generating functions for the lengths $|w_B(x;n,k)|$, given in the following theorem:

**Theorem 4.6.** *Let $x \in \Sigma = \{0,1\}$ and $n \in \mathbb{N}_0$. Then we have*

$$\sum_{k=0}^{\infty} |w_B(x;n,k)| t^k = (1+t) \frac{d}{dt} \{(1+t)^n\}. \tag{4.10}$$

By combining (4.1) with the following well-known formula of the Laguerre polynomials $L_n(t)$:

$$L_n(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{t^k}{k!},$$

and after some elementary calculations, we get the exponential generating function for the lengths $|w_B(x;n,k)|$, given in the following theorem:

**Theorem 4.7.** *Let $x \in \Sigma = \{0,1\}$ and $n \in \mathbb{N}_0$. Then, we have*

$$nL_n(-t) = \sum_{k=0}^{\infty} |w_B(x;n,k)| \frac{t^k}{k!} \tag{4.11}$$

*Remark* 4.8. By combining (1.8) with (4.11), we also write the exponential generating function for the lengths $|w_B(x;n,k)|$ in terms of the hypergeometric series as follows:

$${}_1F_1(-n;1;-t) = \frac{1}{n} \sum_{k=0}^{\infty} |w_B(x;n,k)| \frac{t^k}{k!}.$$

Summing the reciprocals of the equation (4.1) over all $0 \leq k \leq n$, we get

$$\sum_{k=0}^{n} \frac{1}{|w_B(x;n,k)|} = \sum_{k=0}^{n} \frac{1}{n\binom{n}{k}}. \tag{4.12}$$

Since the following well-known equality holds true (*cf.* [18]; and see also the references cited therein):

$$\sum_{k=0}^{n} \frac{1}{\binom{n}{k}} = \frac{n+1}{2^{n+1}} \sum_{k=0}^{n+1} \frac{2^k}{k}, \tag{4.13}$$

combining (4.12) with the above equation we arrive at the following theorem:

**Theorem 4.9.** *Let $x \in \Sigma = \{0,1\}$ and $n \in \mathbb{N}_0$. Then, we have*

$$\sum_{k=0}^{n} \frac{1}{|w_B(x;n,k)|} = \frac{n+1}{n2^{n+1}} \sum_{k=0}^{n+1} \frac{2^k}{k}. \tag{4.14}$$

By combining (1.8) and (1.10) with (4.1), we get the following theorem, which gives the ordinary generating functions for the reciprocal of the lengths $|w_B(x;n,k)|$:

**Theorem 4.10.** *Let $x \in \Sigma = \{0,1\}$ and $n \in \mathbb{N}_0$. Then we have*

$$\frac{{}_2F_1\left(1,1;-n;-t\right)}{n} = \sum_{k=0}^{\infty} \frac{t^k}{\left|w_B\left(x;n,k\right)\right|}. \tag{4.15}$$

By combining (1.8) and (1.10) with (4.1), we get the following theorem, which gives the exponential generating functions for the reciprocal of the lengths $\left|w_B\left(x;n,k\right)\right|$:

**Theorem 4.11.** *Let $x \in \Sigma = \{0,1\}$ and $n \in \mathbb{N}_0$. Then we have*

$$\frac{{}_1F_1\left(1;-n;-t\right)}{n} = \sum_{k=0}^{\infty} \frac{1}{\left|w_B\left(x;n,k\right)\right|} \frac{t^k}{k!}. \tag{4.16}$$

*Remark* 4.12. By using linear differential equations, the generating function equation (4.15) and (4.16) may be represented by another special functions.

### *4.2. Generating functions for the lengths of the Bernstein words of the second kind*

Here, we provide some tables involving the lengths of the Bernstein words of the second kind. We also give some observations and open questions regarding generating functions for these length.

By Table 15 and Table 16, the lengths of the Bernstein words of the second kind $\mathcal{W}_B\left(x;n,k\right)$ are provided as tables for the cases of $x \in \Sigma = \{0,1\}$, $n \in \{0,1,2,\ldots,15\}$ and $k \in \{0,1,2,\ldots,10\}$.

| | k=0 | k=1 | k=2 | k=3 | k=4 | k=5 | k=6 | k=7 | k=8 | k=9 | k=10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| n=0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| n=1 | 4 | 4 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| n=2 | 7 | 10 | 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| n=3 | 10 | 19 | 19 | 10 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| n=4 | 13 | 31 | 40 | 31 | 13 | 1 | 1 | 1 | 1 | 1 | 1 |
| n=5 | 16 | 46 | 73 | 73 | 46 | 16 | 1 | 1 | 1 | 1 | 1 |
| n=6 | 19 | 64 | 121 | 148 | 121 | 64 | 19 | 1 | 1 | 1 | 1 |
| n=7 | 22 | 85 | 187 | 271 | 271 | 187 | 85 | 22 | 1 | 1 | 1 |
| n=8 | 25 | 109 | 274 | 460 | 544 | 460 | 274 | 109 | 25 | 1 | 1 |
| n=9 | 28 | 136 | 385 | 736 | 1006 | 1006 | 736 | 385 | 136 | 28 | 1 |
| n=10 | 31 | 166 | 523 | 1123 | 1744 | 2014 | 1744 | 1123 | 523 | 166 | 31 |
| n=11 | 34 | 199 | 691 | 1648 | 2869 | 3760 | 3760 | 2869 | 1648 | 691 | 199 |
| n=12 | 37 | 235 | 892 | 2341 | 4519 | 6631 | 7522 | 6631 | 4519 | 2341 | 892 |
| n=13 | 40 | 274 | 1129 | 3235 | 6862 | 11 152 | 14 155 | 14 155 | 11 152 | 6862 | 3235 |
| n=14 | 43 | 316 | 1405 | 4366 | 10 099 | 18 016 | 25 309 | 28 312 | 25 309 | 18 016 | 10 099 |
| n=15 | 46 | 361 | 1723 | 5773 | 14 467 | 28 117 | 43 327 | 53 623 | 53 623 | 43 327 | 28 117 |

**Table 15:** For $x \in \Sigma = \{0,1\}$, $n \in \{0,1,2,\ldots,15\}$ and $k \in \{0,1,2,\ldots,10\}$, the lengths of the words $\mathcal{W}_B\left(x;n,k\right)$, i.e. $\left|\mathcal{W}_B\left(x;n,k\right)\right|$.

By comparing Table 15 with the trees in Figure 1 and Figure 2, we come up with some novel sequences of words with their lengths derived from the tree diagrams. Some of these given in the rows of Table 16. For example, the sequence in the first row of Table 16 is obtained from the lengths of the words encountered on the nodes while traveling the first left branches of the trees in Figure 1 and Figure 2.

| $k$ | $\left\{\left\|\mathcal{W}_B\left(x;n,k\right)\right\|\right\}_{n=k}^{\infty}$ | Corresponding Sequence | Also, see `OEIS` |
|---|---|---|---|
| $k=0$ | $\{1,4,7,10,13,16,19,22,25,28,31,\dots\}$ | $\left\{3n+1\right\}_{n=0}^{\infty}$ | `A016777` |
| $k=1$ | $\{4,10,19,31,46,64,85,109,136,166,\dots\}$ | $\left\{\frac{3n(n-1)}{2}+1\right\}_{n=0}^{\infty}$ | `A005448` |
| $k=2$ | $\{7,19,40,73,121,187,274,385,523,\dots\}$ | `New Sequence` | `Does Not Exist` |
| $k=3$ | $\{10,31,73,148,271,460,736,1123,\dots\}$ | `New Sequence` | `Does Not Exist` |
| $k=4$ | $\{13,46,121,271,544,1006,1744,\dots\}$ | `New Sequence` | `Does Not Exist` |
| $k=5$ | $\{16,64,187,460,1006,2014,3760\dots\}$ | `New Sequence` | `Does Not Exist` |

**Table 16:** Table of the lengths of the words $\mathcal{W}_B\left(x;n,k\right)$, i.e. $\left|\mathcal{W}_B\left(x;n,k\right)\right|$.

In Table 16, the second and third columns respectively shows the first terms of the sequences $\left\{\left|\mathcal{W}_B\left(x;n,k\right)\right|\right\}_{n=k}^{\infty}$ for $k\in\{0,\dots,5\}$ and the symbolic notations of the corresponding sequences (if exist). As for the last column, it provides the IDs (if exist) of the corresponding sequences in the Sloane's *On-Line Encyclopedia of Integer Sequences* (OEIS).

*Remark* 4.13. Observe from the second column of the Table 15 that the length of the words $\mathcal{W}_B\left(x;n,1\right)$ gives the following sequence, for $n\in\mathbb{N}_0$:

$$\{1,4,10,19,31,46,64,85,109,136,166,199,235,274,316,361,\dots\}$$

which is overlapping with the $(n+1)$*-th centered* 3*-gonal numbers* or so-called *centered triangular numbers*, originate from a centered polygonal number con-

sisting of a central dot with three dots around it, and then additional dots in the gaps between adjacent dots, see Figure 13 (*cf.* [3], [25, OEIS: A005448]).

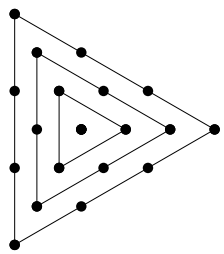

**Figure 13:** The geometric origin of the centered triangular numbers (*cf.* [3, p. 48]).

The $n$-th centered $m$-gonal number $CS_m(n)$ is given by the following explicit formulas:

$$\begin{aligned} CS_m(n) &= 1 + m\binom{n}{2} \\ &= \frac{mn^2 - mn + 2}{2} \end{aligned} \tag{4.17}$$

which have the following generating function:

$$\frac{t\left(1 + (m-2)t + t^2\right)}{(1-t)^3} = \sum_{n=1}^{\infty} CS_3(n)\, t^n$$

where $|t| < 1$ (*cf.* [3, p. 51]; see also [25, sequence A005448 in the OEIS]).

Thus, from the Remark 4.13, we deduce that we have the following relation, for $n \in \mathbb{N}_0$:

$$CS_3(n+1) = |\mathcal{W}_B(x; n, 1)|, \tag{4.18}$$

and we thus conclude that the generating function for $|\mathcal{W}_B(x; n, 1)|$ is given by

$$\frac{t^2 + t + 1}{(1-t)^3} = \sum_{n=0}^{\infty} |\mathcal{W}_B(x; n, 1)|\, t^n; \qquad |t| < 1 \tag{4.19}$$

which is also related to the sequence A005448 in the OEIS [25], and [3, p. 51].

At this stage, the following questions come to mind for the case $k \neq 1$:

What is the explicit formula for the lengths $|\mathcal{W}_B(x;n,k)|$? How can we construct ordinary or exponential function for the lengths $|\mathcal{W}_B(x;n,k)|$? Namely, we have the following open questions:

**Open Question 2:** *Is there an explicit formula for the following generating functions:*

$$\mathcal{G}_3(t;k) = \sum_{n=0}^{\infty} |\mathcal{W}_B(x;n,k)| \frac{t^n}{n!} = ? \tag{4.20}$$

and

$$\mathcal{G}_4(t;n) = \sum_{k=0}^{\infty} |\mathcal{W}_B(x;n,k)| \frac{t^k}{k!} = ? \tag{4.21}$$

## 5. Slopes of the Bernstein-based words and their relations with Farey fractions

In this section, we give relations between slopes of the Bernstein-based words and the Farey fractions. It is known form the book of Lothaire [14], the slope of a nonempty $w$ is defined by

$$\text{slope}(w) = \frac{\text{height}(w)}{|w|} \tag{5.1}$$

where $\text{height}(w)$ denotes the height of the word $w$ which corresponds to the number of letters equal to 1 in the word $w$ (*cf.* [14, pp. 42-45]; see also [2]). For instance, the height of the word $w_B(1;3,2) = 110110110$ is 6, and the length of the word $w_B(1;3,2)$ is 9. Therefore, the slope of the word $w_B(1;3,2)$ is $\frac{6}{9}$, namely $\frac{1}{3}$. The word $w_B(1;3,2)$ can be drawn on a grid by representing a 0 (resp. a 1) as horizontal (resp. a diagonal) unit segment. This gives a polygonal line from the origin to the point $(|w|, \text{slope}(w))$, and the line from the origin to this point has the slope $\text{slope}(w)$. See Figure 14.

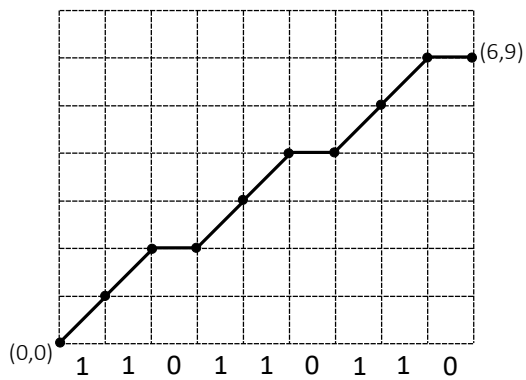


**Figure 14:** Slope diagram the word $w_B(1;3,2) = 110110110$.

In order to present some relations between the slopes of the Bernstein-based words and the Farey fractions, we briefly recall the definition of the set of consecutive Farey fractions of order $n$, denoted by $F_n$, as follows:

$F_n$ is a set of reduced fractions in the closed interval $[0,1]$ with denominators $\leq n$.

The first ten set of consecutive Farey fractions are given as follows:

$$F_1 : \left\{\frac{0}{1}, \frac{1}{1}\right\}$$
$$F_2 : \left\{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\right\}$$
$$F_3 : \left\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\right\}$$
$$F_4 : \left\{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\right\}$$
$$F_5 : \left\{\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}\right\}$$
$$F_6 : \left\{\frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}\right\}$$
$$F_7 : \left\{\frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}\right\}$$
$$F_8 : \left\{\frac{0}{1}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{1}{1}\right\}$$
$$F_9 : \left\{\frac{0}{1}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{1}{1}\right\}$$
$$F_{10} : \left\{\frac{0}{1}, \frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{3}{10}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{7}{10}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}, \frac{1}{1}\right\}$$

and so on.

Some properties of Farey fractions are given as follows:

It is easy to see that $F_n \subset F_{n+1}$ with $n \in \mathbb{N}$.

Let $\frac{a}{b} < \frac{c}{d}$ be consecutive Farey fractions. Then, their mediant $\frac{a+b}{c+d}$ which satisfies

$$\frac{a}{b} < \frac{a+b}{c+d} < \frac{c}{d}.$$

If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive Farey fractions, then the following equality holds true:

$$ad - bc = -1$$

(*cf.* [1, p. 98]).

Fractions that appear as neighbors in a set of consecutive Farey fractions have closely associated with the concept of the continued fraction expansions and every fraction has two continued fraction expansions. For further properties on the Farey fractions and continued fractions, the interested reader may refer to the book of Apostol [1].

By choosing the penultimate element of each set of consecutive Farey fractions $\{F_1, F_2, \ldots, F_{n-1}, F_n\}$, we obtain the following new set of Farey fractions:

$$\mathcal{F}_{0,n} := \left\{ \frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \ldots, \frac{n}{n+1}, \frac{n+1}{n+2}, \ldots \right\} \tag{5.2}$$

so that $n \in \mathbb{N}_0$.

On the other hand, by choosing the second element of each set of consecutive Farey fractions $\{F_1, F_2, \ldots, F_{n-1}, F_n\}$, we obtain the following another new set of Farey fractions:

$$\mathcal{F}_{1,n} := \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots \right\} \tag{5.3}$$

so that $n \in \mathbb{N}_0$.

By implementing the equation (5.1) in the Wolfram Language, we write the following procedure, `CalculateWordSlope`:

```
CalculateWordSlope[w_?StringQ]
:=StringCount[w, "1"] / StringLength[w]
```

for calculating the slope of the word $w$.

By executing the procedure `CalculateWordSlope` with the input words $w_B(x; n, 1)$ for $x \in \Sigma = \{0, 1\}$, we obtain the same list respectively given in (5.2) and (5.3). Therefore, we arrive at the assertion of the following theorem:

**Theorem 5.1.** *Let $n \in \mathbb{N}_0$. Then, we have*

$$\mathcal{F}_{0,n} = \left\{ \operatorname{slope}\left(w_B\left(0; n, 1\right)\right) \right\}_{n=1}^{\infty}. \tag{5.4}$$

**Theorem 5.2.** *Let $n \in \mathbb{N}_0$. Then, we have*

$$\mathcal{F}_{1,n} = \left\{ \operatorname{slope}\left(w_B\left(1; n, 1\right)\right) \right\}_{n=1}^{\infty}. \tag{5.5}$$

| $k$ | $\left\{ \operatorname{slope}\left(w_B\left(0; n, k\right)\right) \right\}_{n=k}^{\infty}$ |
|---|---|
| $k = 2$ | $\{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{9}{11}, \frac{5}{6}, \frac{11}{13}, \frac{6}{7}, \frac{13}{15}, \ldots\}$ |
| $k = 3$ | $\{\frac{0}{1}, \frac{1}{4}, \frac{2}{5}, \frac{1}{2}, \frac{4}{7}, \frac{5}{8}, \frac{2}{3}, \frac{7}{10}, \frac{8}{11}, \frac{3}{4}, \frac{10}{13}, \frac{11}{14}, \frac{4}{5}, \frac{13}{16}, \ldots\}$ |
| $k = 4$ | $\{\frac{0}{1}, \frac{1}{5}, \frac{1}{3}, \frac{3}{7}, \frac{1}{2}, \frac{5}{9}, \frac{3}{5}, \frac{7}{11}, \frac{2}{3}, \frac{9}{13}, \frac{5}{7}, \frac{11}{15}, \frac{3}{4}, \frac{13}{17}, \ldots\}$ |
| $k = 5$ | $\{\frac{0}{1}, \frac{1}{6}, \frac{2}{7}, \frac{3}{8}, \frac{4}{9}, \frac{1}{2}, \frac{6}{11}, \frac{7}{12}, \frac{8}{13}, \frac{9}{14}, \frac{2}{3}, \frac{11}{16}, \frac{12}{17}, \frac{13}{18}, \ldots\}$ |

**Table 17:** Table of the slope of the words $w_B(0; n, k)$, i.e. $\operatorname{slope}(w_B(0; n, k))$, for the cases when $k \in \{2, 3, 4, 5\}$.

| $k$ | $\left\{ \text{slope}\left(w_B\left(1;n,k\right)\right) \right\}_{n=k}^{\infty}$ |
|---|---|
| $k=2$ | $\{\frac{1}{1},\frac{2}{3},\frac{1}{2},\frac{2}{5},\frac{1}{3},\frac{2}{7},\frac{1}{4},\frac{2}{9},\frac{1}{5},\frac{2}{11},\frac{1}{6},\frac{2}{13},\frac{1}{7},\frac{2}{15},\frac{1}{8},\ldots\}$ |
| $k=3$ | $\{\frac{1}{1},\frac{3}{4},\frac{3}{5},\frac{1}{2},\frac{3}{7},\frac{3}{8},\frac{1}{3},\frac{3}{10},\frac{3}{11},\frac{1}{4},\frac{3}{13},\frac{3}{14},\frac{1}{5},\frac{3}{16},\ldots\}$ |
| $k=4$ | $\{\frac{1}{1},\frac{4}{5},\frac{2}{3},\frac{4}{7},\frac{1}{2},\frac{4}{9},\frac{2}{5},\frac{4}{11},\frac{1}{3},\frac{4}{13},\frac{2}{7},\frac{4}{15},\frac{1}{4},\frac{4}{17},\frac{2}{9},\ldots\}$ |
| $k=5$ | $\{\frac{1}{1},\frac{5}{6},\frac{5}{7},\frac{5}{8},\frac{5}{9},\frac{1}{2},\frac{5}{11},\frac{5}{12},\frac{5}{13},\frac{5}{14},\frac{1}{3},\frac{5}{16},\frac{5}{17},\frac{5}{18},\ldots\}$ |

**Table 18:** Table of the slope of the words $w_B\left(1;n,k\right)$, i.e. $\text{slope}\left(w_B\left(1;n,k\right)\right)$, for the cases when $k \in \{2,3,4,5\}$.

*Remark* 5.3. Observe from Table 17 and Table 18 that each sequence in the tables is a sequence of Farey fractions. However, in case $k > 1$, it is an open problem how the sequence of the slopes of the words to be chosen and from which set of Farey fractions similar to the above methods.

*Remark* 5.4. Let $\frac{a}{b}$ be a rational number whose numerator and denominator are co-primes, i.e. $(a,b)=1$. Then, the Ford circle $C\left(a,b\right)$ belonging to the fraction $\frac{a}{b}$ is defined as the circle in the complex plane with radius $\frac{1}{2b^2}$ and center at the point $\frac{a}{b}+\frac{i}{2b^2}$ so that $i^2=-1$ (*cf.* [1, p. 99]).

At this stage, the following another question comes to mind:

If so, what are the relations between the Ford circles and the geometry arising from the sets $\mathcal{F}_{0,n}$ and $\mathcal{F}_{1,n}$, respectively?

*Remark* 5.5. Observe that the sets $\mathcal{F}_{0,n}$ and $\mathcal{F}_{1,n}$ forms a convergent subsequences derived from the sequence of consecutive Farey fractions although each of $F_1, F_2, \ldots, F_{n-1}, F_n, \ldots$ is not convergent.

Since every convergent sequence is a Cauchy sequence, we also conclude that each of the sets $\mathcal{F}_{0,n}$ and $\mathcal{F}_{1,n}$ forms a Cauchy sequence.

| $k$ | $\left\{ \text{slope}\left(\mathcal{W}_B\left(x;n,k\right)\right) \right\}_{n=0}^{\infty}$ | Corresponding Sequence |
|---|---|---|
| $k=0$ | $\left\{\frac{1}{1},\frac{2}{4},\frac{3}{7},\frac{4}{10},\frac{5}{13},\frac{6}{16},\frac{7}{19},\frac{8}{22},\frac{9}{25},\frac{10}{28},\frac{11}{31},\dots\right\}$ | $\left\{\frac{n+1}{3n+1}\right\}_{n=0}^{\infty}$ |
| $k=1$ | $\left\{\frac{0}{1},\frac{2}{4},\frac{5}{10},\frac{9}{19},\frac{14}{31},\frac{20}{46},\frac{27}{64},\frac{35}{85},\frac{44}{109},\frac{54}{136},\frac{65}{166},\dots\right\}$ | $\left\{\frac{n(n+3)}{3n^2+3n+1}\right\}_{n=0}^{\infty}$ |
| $k=2$ | $\left\{\frac{0}{1},\frac{0}{1},\frac{3}{7},\frac{9}{19},\frac{19}{40},\frac{34}{73},\frac{55}{121},\frac{83}{187},\frac{119}{274},\frac{164}{385},\frac{219}{523},\dots\right\}$ | `New Sequence` |

**Table 19:** Table of the slopes of the words $\mathcal{W}_B\left(x;n,k\right)$, i.e. $\text{slope}\left(\mathcal{W}_B\left(x;n,k\right)\right)$.

In Table 19, the second and third columns respectively shows the first terms of the sequences $\left\{ \text{slope}\left(\mathcal{W}_B\left(x;n,k\right)\right) \right\}_{n=0}^{\infty}$ for $k \in \{0,1,2\}$ and the symbolic notations of the corresponding sequences (if exist).

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