Sharp threshold of global existence and mass concentration for the Schrödinger-Hartree equation with anisotropic harmonic confinement

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Abstract
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Keywords: Schrödinger-Hartree equation; Harmonic confinement; Sharp threshold; Mass concentration; Minimal mass blow-up solutions

Mathematics Subject Classification (2020) : 35A01; 35B44; 35E15; 35Q55.

1. Introduction
In this paper, we consider the initial-value problem of the following Schrödinger-Hartree equation in the presence of anisotropic partial/whole harmonic confinement

\[
\begin{aligned}
\begin{cases}
i \varphi_t + \Delta \varphi - \sum_{i=1}^{k} \nu_i^2 x_i^2 \varphi + \lambda (I_\alpha \ast |\varphi|^p)|\varphi|^{p-2} \varphi = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^N, \\
\varphi(0, x) = \varphi_0, \quad x \in \mathbb{R}^N,
\end{cases}
\end{aligned}
\]

(1.1)

where $\varphi : [0, T) \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function, $0 < T \leq \infty$ and $\varphi_0$ is a given function in $\mathbb{R}^N$, $1 \leq k \leq N$, $\nu_i \neq 0$ and $\nu_i \in \mathbb{R}$ ($1 \leq i \leq k$), $\lambda > 0$, $2 \leq p < \frac{N+\alpha}{N-2}$, $I_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined by

\[
I_\alpha(x) = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{N}{2}\right)\pi^{\frac{N}{2}}} 2^{\alpha} |x|^{N-\alpha}
\]
with $0 < \alpha < N$ and $\Gamma$ is the Gamma function.

Nonlinear Schrödinger equations of Hartree-type have a broad physical background. They often appear as models of quantum semiconductor devices [1]. When $k = N$, Eq.(1.1), known as Schrödinger-Hartree equation with complete harmonic confinement, can be used to characterize Bose-Einstein condensation (BEC) in a gas with very weak two-body interactions, which was found in $^{23}$Na or $^{87}$Rb atomic systems [2]. When $1 \leq k < N$, Eq.(1.1) is called nonlinear Hartree equation with partial confinement, arising also as a typical model to describe the BEC [3]. When removing the harmonic confinement in Eq.(1.1), for $N = 3$, $p = 2$ and $\alpha = 2$, Eq.(1.1) is used to describe electrons trapped in their own holes, which is similar to the Hartree-Fock theory of single component plasma to some extent [4].

When $k = N$, Eq.(1.1) with complete harmonic confinement has been well-studied. In the special case $k = N$, $\nu_1 = \nu_2 = \cdots = \nu_N$ and $p = 2$, Huang et al. [5] applied the Hamiltonian invariants and the Gagliardo-Nirenberg inequality of convolution type and scaling technique to investigate the sharp threshold of global existence and showed the stability of standing waves in the mass-critical case $\alpha = N - 2$. Wang [6] proved the existence of blow-up solutions and studied the strong instability of standing waves by variational methods in the mass-supercritical case $2 < N - \alpha < \min(4, N)$. It’s worth mentioning that, Feng [7] derived the sharp threshold for global existence and finite time blow-up on mass for $\nu_1 = \nu_2 = \cdots = \nu_N$ and $p = 1 + \frac{2 + \alpha}{N} \geq 2$ in Eq.(1.1), by using the variational characterization of the ground state solution to a nonlinear Schrödinger-Hartree equation without potential (see Eq.(2.6)). Moreover, in the general $L^2$-supercritical case $1 + \frac{2 + \alpha}{N} \leq p < \frac{N + \alpha}{N - 2}$ with $0 < \alpha < N$, Feng [7] obtained blow-up versus global well-posedness criteria by constructing some cross-invariant manifolds and variational problems and studied the stability and instability of standing waves. If the nonlinearity $\left( I_\alpha * |\varphi|^p \right) |\varphi|^{p-2} \varphi$ is replaced by $|\varphi|^{p-1} \varphi$, there exists extensive literatures on the Cauchy problem of nonlinear Schrödinger equation with complete harmonic potential, see e.g., [8-10]. In particular, Shu and Zhang [9] and Zhang [10] derived the sharp criterion of global existence to Eq.(1.1) by constructing different cross-constrained variational problems and so-called invariant sets.

When $1 \leq k < N$, the main difference between nonlinear Schrödinger-type equation with partial harmonic confinement and complete confinement is that the embedding from natural energy space $\Sigma$ (see Section 2) to $L^p(\mathbb{R}^N)(p \in [2, \frac{2N}{N-2}])$ is lack of compactness, resulting the main difficulty on the study of corresponding Cauchy problem. Due to the fact, the existence of stable standing waves, global and blow-up dynamics, and sharp criterion of global existence to the nonlinear Schrödinger-type equation with partial confinement have attracted considerable interest. There exists several studies in these directions to Eq.(1.1) with power type nonlinearity $|x|^{-b} |\varphi|^{p-1} \varphi$ ($b \geq 0$), see [11-16] for example and the references therein. More precisely, for the case $b = 0$, Ardila and Carles [12] studied the criteria of blow-up and scattering below the ground state in the focusing $L^2$-supercritical case. The papers [13, 14] studied the sharp threshold for finite time blow-up and global existence in the mass-critical case by utilizing the variational characteristic of the ground state to a classical nonlinear elliptic equation without harmonic confinement and Hamilton conservation. It is worth noting that, by exploiting the refined compactness lemma proposed by Hmidi and Keraani [17] and the variational characterization of the ground state, Pan and Zhang [14] investigated the mass concentration properties and limiting profile of the blow-up solutions possessing small supercritical mass in the $L^2$-critical case in dimension $N = 2$. More recently, when $k = 1$, that’s, the
harmonic potential is confined in one direction, Wang and Zhang [15] derived the sharp condition for global existence and blowup to the solutions by making using of the irregular variance identity and constructing cross-constrained variational problems and invariant manifolds of the evolution flow. Liu, He and Feng [16] studied the existence and stability of normalized standing waves for Eq.(1.1) with anisotropic partial confinement and inhomogeneous nonlinearity \( |x|^{-b} |\varphi|^{p-1} \varphi \) \((b > 0)\). As far as we know, there is no paper concerning the sharp threshold of global existence and mass concentration phenomenon to the blow-up solutions of nonlinear Schrödinger-type equation with Hartree nonlinearity \((I_\alpha * |\varphi|^p)|\varphi|^{p-2} \varphi\) and partial confinement, which are greatly pursued in physics. This is the main motivation for us to study these problems for Eq.(1.1).

In the absence of harmonic confinement in Eq.(1.1), the corresponding equation is also known as Choquard equation, whose Cauchy problem has also been extensively studied, see for instance [8, 18-22]. In particular, by constructing invariant sets and using variational methods, Chen and Guo [18] obtained the existence of blow-up solutions for some suitable initial data and showed strong instability of standing waves in the case \( N = 3 \) and \( 2 < N - \alpha < 3 \). Miao et al. [19] studied the mass concentration properties of blow-up solutions as well as the dynamics of blow-up solutions with minimal mass for Eq.(1.1) in the \( L^2\)-critical case with \( \alpha = 2 \) and \( N = 4 \). When \( p = 1 \frac{2+\alpha}{N} \) \((N = 3, 4)\), Genev and Venkov [20] gave a sharp sufficient condition of global existence to Eq.(1.1). Furthermore, they proved the existence of blow-up solutions and considered the blow-up dynamics to the solutions in the \( L^2\)-critical setting, i.e., \( p = 1 + \frac{2+\alpha}{N} \) with \( \alpha = 2 \). Notice that Feng and Yuan [21] not only considered the local and global well-posedness and finite time blow-up to the corresponding initial-value problem (1.1) in the general case \( 2 \leq p < \frac{N+\alpha}{N-2} \) with \( \max\{0, N - 4\} < \alpha < N \), but also took into account the concentration phenomenon of blow-up solutions and the blow-up dynamics of blow-up solutions possessing minimal mass in the case \( p = 1 + \frac{2+\alpha}{N} \geq 2 \), by establishing a new refined compactness lemma with respect to the nonlocal nonlinearity \((I_\alpha * |\varphi|^p)|\varphi|^{p-2} \varphi\).

To the best of our knowledge, there are few papers dealing with the global well-posedness and blow-up dynamics to the Cauchy problem (1.1) in the presence of anisotropic partial/whole harmonic confinement. Inspired by the literatures aforementioned, the purposes of this present article are devoted to investigate the sharp criterion of global existence and mass-concentration phenomenon of blow-up solutions as well as the dynamical properties to minimal mass blow-up solutions of Eq.(1.1) with anisotropic partial/whole confinement. To achieve these aims, the main difficulties that we will encounter come from the presence of anisotropic harmonic confinement \( \sum_{j=1}^{k} \omega_j^2 x_j^2 \) and the nonlocal nonlinearity \((I_\alpha * |\varphi|^p)|\varphi|^{p-2} \varphi\), resulting in the loss of scale invariance and pseudo-conformal transformation. Motivated by [7, 13, 14, 23], we utilize the ground state to the nonlinear Schrödinger-Hartree equation (2.6), which is without any confined potential, to overcome the lack of the above two symmetries. Firstly, we get a sharp threshold for global existence and finite time blow-up at the ground state mass in the \( L^2\)-critical case, which extend the global existence and blow-up results of Feng [7] to the case with anisotropic partial/complete confinement. Then, in the \( L^2\)-critical and \( L^2\)-supercritical cases, by constructing some new cross-invariant manifolds of the evolution flow and some variational problems associated to Eq.(1.1), we derive blow-up versus global well-posedness criteria for Eq.(1.1). In the present case, the constructed cross-invariant sets and variational problems are in light of Shu and Zhang [9], which differ from those of Feng [7], and some new criterions of global existence are derived. Finally, based on
the ideas of [14, 17, 21], we research the mass concentration phenomenon of blow-up solutions and the dynamics of the $L^2$-minimal blow-up solutions, including the precise mass-concentration and blow-up rate of the minimal mass blow-up solutions. The main ingredients of the proofs are the variational characterization of the ground state to Eq.(2.6), a refined compactness lemma established by Feng and Yuan [21] and scaling techniques. Our conclusions extend and compensate for some previous results of [7] and [21].

The rest of this paper is organized as follows. In section 2, some notations and preliminaries are given. Section 3 considers the sharp threshold for global existence and finite time blow-up of Eq.(1.1) in both the $L^2$-critical and $L^2$-supercritical cases. The last section focuses on the mass concentration phenomenon of blow-up solutions and the dynamics of the $L^2$-minimal blow-up solutions.

2. Notations and Preliminaries

Throughout this paper, we use $\int \cdot \, dx$ to represent $\int_{\mathbb{R}^N} \cdot \, dx$ and denote $\| \varphi \|_p = \| \varphi \|_{L^p(\mathbb{R}^N)} = (\int |\varphi|^p \, dx)^{\frac{1}{p}}$, and use $C$ to stand for positive constants, which may vary from line to line. Without loss of generality, we assume $\lambda = 1$ in this and subsequent sections.

For Eq.(1.1), we equip the natural energy space $\Sigma = \{ \varphi \in H^1(\mathbb{R}^N), \int \sum_{i=1}^k \nu_i^2 x_i^2 |\varphi|^2 \, dx < \infty, \nu_i \in \mathbb{R} \setminus \{0\} \}$ with the inner product

$$\langle \phi, \varphi \rangle_\Sigma = \text{Re} \int \left( \phi \bar{\varphi} + \nabla \phi \cdot \nabla \bar{\varphi} + \sum_{i=1}^k \nu_i^2 x_i^2 \varphi \bar{\varphi} \right) \, dx, \quad \forall \phi, \varphi \in \Sigma,$$

and the corresponding norm is given by

$$\| \varphi \|_\Sigma^2 = \| \varphi \|_2^2 + \| \nabla \varphi \|_2^2 + \int \sum_{i=1}^k \nu_i^2 x_i^2 |\varphi|^2 \, dx, \quad \forall \varphi \in \Sigma.$$

The energy functional associated to Eq.(1.1) is defined as

$$E(\varphi(t)) = \frac{1}{2} \int \left( |\nabla \varphi(t)|^2 + \sum_{i=1}^k \nu_i^2 x_i^2 |\varphi(t)|^2 - \frac{1}{p} (I_\alpha * |\varphi|^p) |\varphi|^p \right) \, dx, \quad \varphi \in \Sigma. \quad (2.1)$$

Let us now state the local well-posedness of Eq.(1.1) in energy space $\Sigma$ according to [7, 21].

**Proposition 2.1.** Let $\varphi_0 \in \Sigma$ and $2 \leq p < \frac{N+\alpha}{N-2}$. Then there exists $T = T(\| \varphi_0 \|_\Sigma)$ such that Eq.(1.1) admits a unique solution $\varphi(t, x) \in C([0, T), \Sigma)$. Let $[0, T)$ be the maximal time interval such that the solution $\varphi(t, x)$ is well-defined. If $T < \infty$, then $\lim_{t \to T} \| \varphi(t) \|_\Sigma = \infty$ (blow-up). Furthermore, $\varphi(t, x)$ depends continuously on initial data $\varphi_0$ and for any $t \in [0, T)$, the following conservation laws of mass and energy hold,

$$\int |\varphi(t)|^2 \, dx = \int |\varphi_0|^2 \, dx, \quad (2.2)$$

$$E(\varphi(t)) = E(\varphi_0). \quad (2.3)$$

Then we introduce some vital lemmas.
Lemma 2.2. ([24]) Let $0 < \lambda < N$ and $s, r > 1$ be constants such that
\[
\frac{1}{r} + \frac{1}{s} + \frac{\lambda}{N} = 2.
\]
Assume that $g \in L^r(\mathbb{R}^N)$ and $h \in L^s(\mathbb{R}^N)$. Then
\[
\left| \int \int g(x)|x - y|^{-\lambda} h(y) dx dy \right| \leq C(N, s, \lambda) \|g\|_r \|h\|_s.
\] (2.4)

By inequality (2.4), we can obtain the following generalized Gagliardo-Nirenberg inequality
\[
\int (I_\alpha * |\varphi|^p)|\varphi|^p dx \leq C_{\alpha,p} \left( \int |\nabla \varphi|^2 dx \right)^{\frac{Np-N-\alpha}{2}} \left( \int |\varphi|^2 dx \right)^{\frac{N+\alpha-Np+2p}{2}}.
\] (2.5)

Following Weinstein [25], Feng and Yuan [21] derived the best constant in the inequality (2.5) by discussing the existence of the minimizer of the functional
\[
J_{\alpha,p}(\varphi) = \frac{\left( \int |\nabla \varphi|^2 dx \right)^{\frac{Np-N-\alpha}{2}} \left( \int |\varphi|^2 dx \right)^{\frac{N+\alpha-Np+2p}{2}}}{\int (I_\alpha * |\varphi|^p)|\varphi|^p dx}.
\]

Lemma 2.3. ([21]) It follows that the best constant in the generalized Gagliardo-Nirenberg inequality (2.5) is
\[
C_{\alpha,p} = \frac{2p}{2p - Np + N + \alpha} \left( 2p - Np + N + \alpha \right)^{\frac{Np-N-\alpha}{2}} \|Q(x)\|^{2-2p},
\]
where $Q(x)$ is the ground state of the elliptic equation
\[
-\Delta \varphi + \varphi - (I_\alpha * |\varphi|^p)|\varphi|^{p-2}\varphi = 0.
\] (2.6)

In particular, in the $L^2$-critical case, $C_{\alpha,p} = \|Q(x)\|^{2-2p}$.

It’s known that ground state is of great importance in studying global existence and blow-up dynamics to the initial-value problem of nonlinear Schrödinger equation, in the following lemma we recall some existence results and properties of the ground state solution to Eq.(2.6).

Lemma 2.4. ([26]) Let $\alpha \in (0, N)$ and $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$. It follows that Eq.(2.6) admits a ground state solution $Q(x)$ in $H^1(\mathbb{R}^N)$. Every ground state $Q(x)$ of Eq.(2.6) is in $L^1 \cap C^\infty$, and there exist $x_0 \in \mathbb{R}^N$ and a monotone real function $\tau \in C^\infty(0, \infty)$ such that for every $x \in \mathbb{R}^N$, $Q(x) = \tau(|x - x_0|)$. Moreover, the following pohožaev identity holds.
\[
\frac{N-2}{2} \int |\nabla Q(x)|^2 dx + \frac{N}{2} \int |Q(x)|^2 dx = \frac{N + \alpha}{2p} \int (I_\alpha * |Q(x)|^p)|Q(x)|^p dx,
\] (2.7)
\[
\int |\nabla Q(x)|^2 dx + \int |Q(x)|^2 dx = \int (I_\alpha * |Q(x)|^p)|Q(x)|^p dx.
\] (2.8)

From (2.7) and (2.8), one has
\[
p \int |\nabla Q(x)|^2 dx = \int (I_\alpha * |Q(x)|^p)|Q(x)|^p dx.
\] (2.9)

In order to study the blow-up phenomenon of Eq.(1.1), we also need the following lemma obtained in Weinstein [25].
Lemma 2.5. ([25]) Let \( \varphi \in H^1(\mathbb{R}^N) \), then we have that

\[
\int |\varphi|^2 \, dx \leq \frac{2}{N} \left( \int |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}} \left( \int |\varphi|^2 \, dx \right)^{\frac{1}{2}}.
\]

Following the idea of Glassey [27] (see also Feng [7]), we will adopt the convexity method to study the existence of blow-up solutions. More precisely, we need to consider the variance

\[
V(t) = \int |x|^2 |\varphi(t,x)|^2 \, dx,
\]

and show that there exists time \( T > 0 \) such that \( V(T) = 0 \). With some formal computations (which can be rigorously proved as in [8]), we have the following virial identities.

Proposition 2.6. Let \( 2 \leq p < \frac{N+\alpha}{N-2} \) and assume that \( \varphi(t,x) \) is a solution of problem (1.1) in \( C([0,T]; \Sigma) \) with \( \varphi_0 \in H^1(\mathbb{R}^N) \) and \( |x| \varphi_0 \in L^2(\mathbb{R}^N) \). Then the function \( t \to | \cdot | \varphi(t, \cdot) \) belongs to \( C([0,T), L^2) \). Furthermore, the function \( V(t) = \int |x|^2 |\varphi(t,x)|^2 \, dx \) belongs to \( C^2(0,T) \), then we obtain that

\[
V'(t) = 4\text{Im} \int x \nabla \varphi \bar{\varphi} \, dx,
\]

and

\[
V''(t) = 8 \int |\nabla \varphi|^2 \, dx - 8 \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 \, dx - \frac{4Np - 4N - 4\alpha}{p} \int (I_a * |\varphi|^p)|\varphi|^p \, dx
\]

\[
= 8(Np - N - \alpha)E(\varphi) + (8 + 4N + 4\alpha - 4Np) \int |\nabla \varphi|^2 \, dx
\]

\[
+ (4N + 4\alpha - 4Np - 8) \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 \, dx,
\]

for all \( t \in [0,T) \). In particular, when \( p = 1 + \frac{2+\alpha}{N} \), we have

\[
V''(t) = 16E(\varphi_0) - 16 \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 \, dx.
\]

(2.10)

Using Lemma 2.5 and Proposition 2.6, we can easily get the following sufficient conditions on the existence of blow-up solutions.

Corollary 2.7. Assume that \( \max\{0, N - 4\} < \alpha < N \) and \( \max\{1 + \frac{2+\alpha}{N}, 2\} \leq p < \frac{N+\alpha}{N-2} \). Let \( \varphi_0 \in H^1(\mathbb{R}^N) \) and \( |x| \varphi_0 \in L^2(\mathbb{R}^N) \), and satisfy one of the following conditions:

Case (1): \( E(\varphi_0) < 0 \);
Case (2): \( E(\varphi_0) = 0 \) and \( \text{Im} \int x \nabla \varphi_0 \bar{\varphi}_0 \, dx < 0 \);
Case (3): \( E(\varphi_0) > 0 \) and \( \text{Im} \int x \nabla \varphi_0 \bar{\varphi}_0 \, dx + (2V(0)E(\varphi_0))^\frac{1}{2} \leq 0 \).

Then the corresponding solution \( \varphi(t,x) \) of Eq.(1.1) blows up in finite time.

3. Sharp threshold for global existence and blow-up

3.1. The \( L^2 \)-critical case

The aim of this subsection is mainly to consider the global existence and blowup of the solutions to Eq.(1.1) in the \( L^2 \)-critical case, i.e. \( p = 1 + \frac{2+\alpha}{N} \). The ground state mass \( \|Q(x)\|_2 \) gives a sufficient condition on the global existence of the solution to Eq.(1.1).
Theorem 3.1. Let $p = 1 + \frac{2+\alpha}{N} \geq 2$ and $Q(x)$ be the positive radially symmetric ground state solution of Eq.(2.6). If $\varphi_0 \in \Sigma$ and $\varphi_0$ satisfies

$$\|\varphi_0\|_2 < \|Q(x)\|_2,$$  \hfill (3.1)

then the Cauchy problem (1.1) has a global solution $\varphi(t,x)$ in $C([0,\infty), \Sigma)$. Furthermore, we have for any $0 \leq t < \infty$,

$$\int \left( \left| \nabla \varphi \right|^2 + \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 \right) dx < \frac{2E(\varphi_0)}{1 - \|Q(x)\|_2^{2-2p} \left( \int |\varphi_0|^2 dx \right)^{p-1}} + 2E(\varphi_0).$$  \hfill (3.2)

Proof. Let $\varphi(t,x)$ be the corresponding solution of Eq.(1.1) in $C([0,T), \Sigma)$ with initial value $\varphi_0 \in \Sigma$. By (2.3), (2.1), Lemma 2.3 and (2.2), we obtain

$$E(\varphi_0) = E(\varphi) = \frac{1}{2} \int \left( \left| \nabla \varphi \right|^2 + \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 - \frac{1}{p} (I_{\alpha} + |\varphi|^p) |\varphi|^p \right) dx$$

$$\geq \frac{1}{2} \int \left( \left| \nabla \varphi \right|^2 + \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 \right) dx - \frac{1}{2p} \|Q(x)\|_2^{2-2p} \int |\nabla \varphi|^2 dx \left( \int |\varphi|^2 dx \right)^{p-1}$$

$$= \frac{1}{2} \int \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx + \frac{1}{2} \left( 1 - \|Q(x)\|_2^{2-2p} \left( \int |\varphi|^2 dx \right)^{p-1} \right) |\nabla \varphi|^2 dx$$

$$= \frac{1}{2} \int \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx + \frac{1}{2} \left( 1 - \|Q(x)\|_2^{2-2p} \left( \int |\varphi_0|^2 dx \right)^{p-1} \right) |\nabla \varphi|^2 dx.$$ \hfill (3.3)

From (3.3) and (3.1), we have for all $t \in [0,T)$, where $T$ is arbitrary and $T < \infty$, there exists $C$ such that

$$\int |\nabla \varphi|^2 dx + \int \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx \leq C.$$

Then according to Proposition 2.1, $\varphi(t,x)$ exists globally in time. Moreover, we have

$$\int |\nabla \varphi|^2 dx < \frac{2E(\varphi_0)}{1 - \|Q(x)\|_2^{2-2p} \left( \int |\varphi_0|^2 dx \right)^{p-1}},$$ \hfill (3.4)

and

$$\int \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx < 2E(\varphi_0).$$ \hfill (3.5)

It follows from (3.4) and (3.5) that (3.2) holds true. \hfill \Box

Remark 3.2. (i) When $k = N$ and $\nu_1 = \nu_2 = \cdots = \nu_k = 1$ in Eq.(1.1), Feng [7] proved that the solution $\varphi(t,x)$ of Eq.(1.1) exists globally (see Theorem 3.2 in [7]). Theorem 3.1 can be viewed as the complement of the corresponding result of [7] for Eq.(1.1) with whole harmonic confinement.

(ii) We give an explicit bound to the global solution of Eq.(1.1) in $\Sigma$ (see (3.2)).

By using the variational characterization of the ground state solution to Eq.(2.6), some scaling arguments and energy conservation, we can get the existence result of blow-up solutions to Eq.(1.1).

Theorem 3.3. Let $Q(x)$ be the positive radially symmetric solution of Eq.(2.6), $p = 1 + \frac{2+\alpha}{N} \geq 2$. Then for any $\varepsilon > 0$, there exist $\varphi_0 \in \Sigma$ and $\int |x|^2 |\varphi_0|^2 dx < \infty$ such that

$$\|\varphi_0\|_2^2 = \|Q(x)\|_2^2 + \varepsilon,$$
and the solution \( \varphi(t, x) \) of the Cauchy problem (1.1) blows up in finite time.

**Proof.** For any \( a > 1, b > 0 \), we take \( Q_{a,b}(x) = ab^{\frac{N}{2}}Q(bx) \). Based on some scaling arguments, one has that

\[
\int |Q_{a,b}(x)|^2 dx = a^2 \int |Q(x)|^2 dx, \quad \tag{3.6}
\]
\[
\int |\nabla Q_{a,b}(x)|^2 dx = a^2 b^2 \int |\nabla Q(x)|^2 dx, \quad \tag{3.7}
\]
\[
\int \sum_{i=1}^k \nu_i^2 x_i^2 |Q_{a,b}(x)|^2 dx = a^2 b^{-2} \int \sum_{i=1}^k \nu_i^2 x_i^2 |Q(x)|^2 dx, \quad \tag{3.8}
\]
\[
\int (I_a + |Q_{a,b}(x)|^p) |Q_{a,b}(x)|^p dx = a^{2+\frac{2(2+n)}{N}} b^2 \int (I_a + |Q(x)|^p) |Q(x)|^p dx. \quad \tag{3.9}
\]

Now we set

\[
a = \frac{\int |Q(x)|^2 dx + \varepsilon}{\int |Q(x)|^2 dx} > 1,
\]
\[
b > \left[ \frac{\int \sum_{i=1}^k \nu_i^2 x_i^2 |Q(x)|^2 dx}{\left( a^{\frac{2(2+n)}{N}} - 1 \right) \int |\nabla Q(x)|^2 dx} \right]^{\frac{1}{2}}, \quad \text{and} \quad \varphi_0(x) = ab^{\frac{N}{2}}Q(bx),
\]

then we have \( \varphi_0(x) \in \Sigma \) and \( \int |x|^2 |\varphi_0|^2 dx < \infty \). In fact, since \( Q_{a,b}(x) = ab^{\frac{N}{2}}Q(bx) \in H^1(\mathbb{R}^N) \), by utilizing the exponential decay of ground state solution \( Q(x) \) (see [26]):

\[
Q(|x|), \nabla Q(|x|) = O(|x|^{-\frac{N+1}{2}} e^{-|x|}), \quad \text{as} \quad |x| \to \infty,
\]

we conclude that \( Q_{a,b}(x) \in L^2(\mathbb{R}^N) \) and \( \varphi_0 = ab^{\frac{N}{2}}Q(bx) \in H^1(\mathbb{R}^N) \) and \( \int |x|^2 |\varphi_0|^2 dx < \infty \). Thus, we also deduce that \( \varphi_0 \in \Sigma \). Moreover, it follows from (3.6) that

\[
\int |\varphi_0|^2 dx = \int |Q(x)|^2 dx + \varepsilon.
\]

From (2.1), (2.3), (2.9) and (3.7)-(3.9), we get

\[
E(\varphi) = E(\varphi_0) = \frac{1}{2} \int \left( |\nabla \varphi_0|^2 + \sum_{i=1}^k \nu_i^2 x_i^2 |\varphi_0|^2 - \frac{1}{p} (I_a + |\varphi_0|^p) |\varphi_0|^p \right) dx
\]
\[
= \frac{1}{2} \left( 1 - a^{\frac{2(2+n)}{N}} \right) a^2 b^2 \int |\nabla Q(x)|^2 dx + \frac{a^2}{2b^2} \int \sum_{i=1}^k \nu_i^2 x_i^2 |Q(x)|^2 dx
\]
\[
= \frac{1}{2} a^2 b^2 \left( 1 - a^{\frac{2(2+n)}{N}} \right) \int |\nabla Q(x)|^2 dx + \frac{1}{b^4} \int \sum_{i=1}^k \nu_i^2 x_i^2 |Q(x)|^2 dx
\]
\[
< 0.
\]

Thus, it follows from Corollary 2.7 that the solution \( \varphi(t, x) \) of Eq.(1.1) blows up in finite time. \( \square \)

**Remark 3.4.** (i) When \( k = N \) and \( \nu_1 = \nu_2 = \cdots = \nu_k = 1 \) in Eq.(1.1), Feng [7] proved the existence of blow-up solutions (see Theorem 3.2 in [7]). When considering Eq.(1.1) in the presence of anisotropic partial/complete harmonic confinement, we derive the corresponding blow-up result by scaling approach, which differs from the method of Feng [7].

(ii) Theorems 3.1 and 3.3 declare that \( \|Q(x)\|_2 \) provides a sharp threshold for global existence and blow-up to Eq.(1.1) in terms of the initial data, which is called minimal mass for the blow-up solutions.
3.2. The $L^2$-supercritical case

For $\varphi \in \Sigma$ and $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$, define the following functionals:

\[
I(\varphi) = \frac{1}{2} \int (|\nabla \varphi|^2 + |\varphi|^2 + \sum_{i=1}^k \nu_i^2 x_i^2 |\varphi|^2) dx - \frac{1}{2p} \int (I_\alpha * |\varphi|^p)|\varphi|^p dx; \quad (3.10)
\]

\[
J(\varphi) = \int (|\nabla \varphi|^2 + |\varphi|^2 - (I_\alpha * |\varphi|^p)|\varphi|^p) dx; \quad (3.11)
\]

\[
S(\varphi) = \int (|\nabla \varphi|^2 - \frac{Np-N-\alpha}{2p}(I_\alpha * |\varphi|^p)|\varphi|^p) dx. \quad (3.12)
\]

Then we define the set

\[
M = \{ \varphi \in \Sigma \mid \{0\}, J(\varphi) < 0, S(\varphi) = 0 \},
\]

and consider the following two constrained minimization problems:

\[
d_1 = \inf_{\{\varphi \in \Sigma \mid \{0\}, J(\varphi) = 0\}} I(\varphi), \quad (3.13)
\]

\[
d_2 = \inf_{\varphi \in M} I(\varphi). \quad (3.14)
\]

**Proposition 3.5.** If $1 + \frac{2+\alpha}{N} \leq p < \frac{N+\alpha}{N-2}$, then $d_2 > 0$.

**Proof.** Firstly, we prove $M \neq \emptyset$. According to Lemma 2.3, there exists $\varphi \in \Sigma \setminus \{0\}$ such that $\varphi$ is a solution of Eq.(2.6). By multiplying both sides of Eq.(2.6) by $\varphi$ and integrating over $\mathbb{R}^N$, we get

\[
\int |\nabla \varphi|^2 dx + \int |\varphi|^2 dx = \int (I_\alpha * |\varphi|^p)|\varphi|^p dx. \quad (3.15)
\]

It follows from (3.15) that $J(\varphi) = 0$. Moreover, by taking the inner product of Eq.(2.6) with $x \cdot \nabla \varphi$, we have the following Pohozăev identity

\[
-\frac{N-2}{2} \int |\nabla \varphi|^2 dx - \frac{N}{2} \int |\varphi|^2 + \frac{N+\alpha}{2p} \int (I_\alpha * |\varphi|^p)|\varphi|^p dx = 0. \quad (3.16)
\]

Then multiplying both sides of (3.15) by $\frac{N}{2}$, we have

\[
\frac{N}{2} \int (|\nabla \varphi|^2 + |\varphi|^2 - (I_\alpha * |\varphi|^p)|\varphi|^p) dx = 0. \quad (3.17)
\]

From (3.16) and (3.17), one has that

\[
\int |\nabla \varphi|^2 dx + \left( \frac{N+\alpha-Np}{2p} \right) \int (I_\alpha * |\varphi|^p)|\varphi|^p dx = 0,
\]

which implies $S(\varphi) = 0$. Thus, there exists $\varphi \in \Sigma \setminus \{0\}$ such that $S(\varphi) = 0$ and $J(\varphi) = 0$.

Set

\[
u(x) = \mu^{\frac{2+\alpha}{p-1}} \varphi(\mu x), \quad \mu > 0.
\]

By some simply computations, we obtain

\[
J(\nu(x)) = \mu^{\frac{2p+\alpha-Np+N}{p-1}} \left( \int |\nabla \varphi|^2 dx - \int (I_\alpha * |\varphi|^p)|\varphi|^p dx \right) + \mu^{\frac{2+\alpha}{p-1}} \int |\varphi|^2 dx,
\]

and

\[
S(\nu(x)) = \mu^{\frac{2p+\alpha-Np+N}{p-1}} \left( \int |\nabla \varphi|^2 dx - \frac{Np-N-\alpha}{2p} \int (I_\alpha * |\varphi|^p)|\varphi|^p dx \right).
\]
Note that \( S(\varphi) = 0 \). Thus \( S(u(x)) = 0 \) for every \( \mu > 0 \). Moreover,

\[
J(u(x)) = (\frac{2+\alpha}{p-1} - \mu \frac{2p+\alpha-Np+N}{p-1}) \int |\varphi|^2 dx
\]

\[
= \frac{2+\alpha}{p-1} - N(1 - \mu^2) \int |\varphi|^2 dx.
\]

Thus, there exists \( \mu > 1 \) such that \( J(u(x)) < 0 \). Therefore, when \( \mu > 1 \), we have \( S(u(x)) = 0 \) and \( J(u(x)) < 0 \) which implies \( \mathcal{M} \neq \emptyset \).

Next, we prove \( d_2 > 0 \). Let \( \varphi \in \mathcal{M}, \) from \( J(\varphi) < 0 \), we get \( \varphi \neq 0 \). Since \( S(\varphi) = 0 \), we have

\[
I(\varphi) = \left( \frac{1}{2} - \frac{1}{Np - N - \alpha} \right) \int |\nabla \varphi|^2 dx + \frac{1}{2} \int \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx + \frac{1}{2} \int |\varphi|^2 dx. \tag{3.18}
\]

It follows from \( 1 + \frac{2+\alpha}{N} \leq p < \frac{N+\alpha}{N-2} \), (3.18) and \( \varphi \neq 0 \) that \( I(\varphi) > 0 \) for all \( \varphi \in \mathcal{M} \). Thus, by (3.14), we obtain \( d_2 \geq 0 \). In the following, we will divide the proof into two cases: the \( L^2 \)-supercritical case and the \( L^2 \)-critical case.

We first consider the \( L^2 \)-supercritical case \( 2 \leq 1 + \frac{2+\alpha}{N} < p < \frac{N+\alpha}{N-2} \). In this case, it follows from (2.4) that

\[
\int (I_{\alpha} * |\varphi|^p)|\varphi|^p dx \leq C \left( \int |\varphi|^{\frac{2Np}{N-\alpha}} dx \right)^{\frac{N+\alpha}{N}}
\]

\[
\leq C \left( \int (|\nabla \varphi|^2 + |\varphi|^2) dx \right)^p,
\]

which together with \( J(\varphi) < 0 \) implies

\[
\int (|\nabla \varphi|^2 + |\varphi|^2) dx < \int (I_{\alpha} * |\varphi|^p)|\varphi|^p dx
\]

\[
\leq C \left( \int (|\nabla \varphi|^2 + |\varphi|^2) dx \right)^p.
\]

Thus, one has that

\[
\int (|\nabla \varphi|^2 + |\varphi|^2) dx \geq C > 0. \tag{3.19}
\]

Since \( p > 1 + \frac{2+\alpha}{N} \), we deduce from (3.18) and (3.19) that

\[
I(\varphi) \geq C > 0, \quad \text{for all } \varphi \in \mathcal{M},
\]

which implies \( d_2 > 0 \) for \( 1 + \frac{2+\alpha}{N} < p < \frac{N+\alpha}{N-2} \).

Now we deal with the \( L^2 \)-critical case \( p = 1 + \frac{2+\alpha}{N} \). Suppose by contradiction that \( d_2 = 0 \), then we derive from (3.14) that there exists a sequence \( \{\varphi_n\} \subset \mathcal{M} \) such that \( S(\varphi_n) = 0 \), \( J(\varphi_n) < 0 \) and \( I(\varphi_n) \to 0 \) as \( n \to \infty \). Since \( p = 1 + \frac{2+\alpha}{N} \), one can derive from (3.18) that

\[
\int |\varphi_n|^2 dx \to 0, \quad \int \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi_n|^2 dx \to 0, \quad \text{as } n \to \infty. \tag{3.20}
\]

On the other hand, it follows from \( J(\varphi_n) < 0 \) and (2.5) that

\[
\int (|\nabla \varphi_n|^2 + |\varphi_n|^2) dx < \int (I_{\alpha} * |\varphi_n|^p)|\varphi_n|^p dx \leq C \int |\nabla \varphi_n|^2 dx \left( \int |\varphi_n|^2 dx \right)^{p-1}. \tag{3.21}
\]
It is obvious that (3.22) contradicts (3.21). Thus, Proposition 3.7. Define introduce some new cross-constrained invariant sets as follows.

From (3.10) and (3.11), we obtain

\[ d \geq \frac{1}{2} \int (|\nabla \varphi|^2 + |\varphi|^2) dx + \frac{1}{2} \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx. \tag{3.24} \]

Therefore, \( d \geq 0 \). This, together with Proposition 3.5, implies that the proposition holds true. \( \square \)

To study the sharp threshold of global existence for Eq. (1.1) in the \( L^2 \)-supercritical case, we introduce some new cross-constrained invariant sets as follows.

**Proposition 3.7.** Define

\( \mathbb{K} = \{ \varphi \in \Sigma, I(\varphi) < d, S(\varphi) < 0, J(\varphi) < 0 \} \),

\( \mathbb{K}_+ = \{ \varphi \in \Sigma, I(\varphi) < d, S(\varphi) > 0, J(\varphi) < 0 \} \),

\( \mathbb{R}_- = \{ \varphi \in \Sigma, I(\varphi) < d, J(\varphi) < 0 \} \),

\( \mathbb{R}_+ = \{ \varphi \in \Sigma, I(\varphi) < d, J(\varphi) > 0 \} \).

Then \( \mathbb{K}, \mathbb{K}_+, \mathbb{R}_-, \mathbb{R}_+ \) are invariant sets of Eq. (1.1), that is, if \( \varphi_0 \in \mathbb{K}, \mathbb{K}_+, \mathbb{R}_- \) or \( \mathbb{R}_+ \) then the solution \( \varphi(t, x) \) of the Eq. (1.1) also satisfies \( \varphi(t, x) \in \mathbb{K}, \mathbb{K}_+, \mathbb{R}_- \) or \( \mathbb{R}_+ \) for any \( t \in [0, T] \).

**Proof.** We first prove that \( \mathbb{K} \) is an invariant set of Eq. (1.1). Let \( \varphi_0 \in \Sigma \) and \( \varphi(t, x) \) be the corresponding solution of Eq. (1.1). From (2.2) and (2.3), one has that

\[ I(\varphi) = I(\varphi_0), \text{ for } t \in [0, T]. \tag{3.24} \]

Thus \( I(\varphi_0) \leq d \) implies that \( I(\varphi) \leq d \) for any \( t \in [0, T] \).

Now show \( J(\varphi) < 0 \) for \( t \in [0, T] \). If otherwise, by the continuity of \( J(\varphi) \) on \( t \), there exists \( t_0 \in [0, T] \) such that \( J(\varphi(t_0, \cdot)) = 0 \). By (3.24), we have \( \varphi(t_0, \cdot) \neq 0 \). It is clear that (3.13) and (3.23) implies \( I(\varphi(t_0, \cdot)) \geq d \). This is contradictory to \( I(\varphi(t, \cdot)) < d \) for all \( t \in [0, T] \). Thus \( J(\varphi(t, \cdot)) \leq 0 \) for all \( t \in [0, T] \).

Then show \( S(\varphi(t, \cdot)) < 0 \) for all \( t \in [0, T] \). On the contrary, from the continuity, there exists \( t' \in [0, T] \) such that \( S(\varphi(t', \cdot)) = 0 \). Because we have shown \( S(\varphi(t', \cdot)) = 0 \) and \( J(\varphi(t', \cdot)) < 0 \), it follows that \( \varphi(t', \cdot) \in \mathbb{M} \). Thus, (3.14) and (3.23) imply \( I(\varphi(t', \cdot)) \geq d \). This contradict to \( I(\varphi(t, \cdot)) < d \) for all \( t \in [0, T] \). Therefore \( S(\varphi(t, \cdot)) < 0 \) for all \( t \in [0, T] \). From the above we have proved \( \varphi(t, x) \in \mathbb{K} \) for any \( t \in [0, T] \).

Similar to the proof above, we can also prove that \( \mathbb{K}_+, \mathbb{R}_-, \mathbb{R}_+ \) are invariant manifolds. \( \square \)

In the below, we will use the cross-constrained variational approach to investigate the sharp condition of global existence for Eq. (1.1).
Theorem 3.8. If \( \varphi_0 \in K_+ \cup \mathbb{R}_+ \), then the solution \( \varphi(t, x) \) of the Cauchy problem (1.1) globally exists.

Proof. For \( \varphi_0 \in K_+ \), we have \( \varphi(t, x) \in K_+ \) for \( t \in [0, T) \) by Proposition 3.7. For \( t \in [0, T) \), one has \( I(\varphi) < d \) and \( S(\varphi) > 0 \). It follows from (3.8) and (3.10) that

\[
\int \left( \frac{1}{2} - \frac{1}{Np - N - \alpha} \right) |\nabla \varphi|^2 + \frac{1}{2} |\varphi|^2 + \frac{1}{2} \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 \right) dx < d. \tag{3.25}
\]

Firstly, we deal with the \( L^2 \)-critical case \( p = 1 + \frac{2 + \alpha}{N} \). In this case, we infer from (3.25) that

\[
\frac{1}{2} \int \left( |\varphi|^2 + \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 \right) dx < d. \tag{3.26}
\]

Denote \( \varphi^\omega(x) = \omega^{\frac{\alpha + N}{2p}} \varphi(\omega x) \), then one has

\[
S(\varphi^\omega(x)) = \omega^{\frac{2\alpha + 4}{N + 2 + \alpha}} \int |\nabla \varphi(x)|^2 dx + \frac{N + \alpha - Np}{2p} \int (I_\alpha * |\varphi|^p)|\varphi|^p dx.
\]

It follows from \( S(\varphi) > 0 \) that there exists \( 0 < \omega_1 < 1 \) such that \( S(\varphi^\omega_1(x)) = 0 \). Combining (3.10) with (3.12), we deduce that

\[
I(\varphi^\omega_1(x)) = \frac{1}{2} \int \left( |\varphi^\omega_1(x)|^2 + \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi^\omega_1(x)|^2 \right) dx
\]

\[
= \frac{1}{2} \int \left( \omega_1^{-\frac{2N}{N + 2 + \alpha}} |\varphi(x)|^2 + \omega_1^{-\frac{4N + 4 + 2\alpha}{N + 2 + \alpha}} \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi(x)|^2 \right) dx,
\]

which together with (3.25) yields

\[
I(\varphi^\omega_1(x)) < \omega_1^{-\frac{4N + 4 + 2\alpha}{N + 2 + \alpha}} d. \tag{3.27}
\]

Now we see \( J(\varphi^\omega_1) \), which only has two possibilities. One is \( J(\varphi^\omega_1) < 0 \). In this case, noting that \( S(\varphi^\omega_1) = 0 \), we infer from (3.14) and (3.23) that

\[
I(\varphi^\omega_1) \geq d_2 \geq d > I(\varphi).
\]

Thus,

\[
I(\varphi) - I(\varphi^\omega_1) < 0.
\]

That is,

\[
\frac{1}{2} (1 - \omega_1^{-\frac{4 + 2\alpha}{N + 2 + \alpha}}) \int |\nabla \varphi|^2 dx + \frac{1}{2} (1 - \omega_1^{-\frac{4N + 4 + 2\alpha}{N + 2 + \alpha}}) \int \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx + \frac{1}{2} (1 - \omega_1^{-\frac{2N}{N + 2 + \alpha}}) \int |\varphi|^2 dx < 0.
\]

It follows that

\[
\int |\nabla \varphi|^2 dx < C \int \left( \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 + |\varphi|^2 \right) dx.
\]

By (3.26), we obtain

\[
\int |\nabla \varphi|^2 dx < C.
\]
For $J(\varphi^\omega)$, the other possible case is $J(\varphi^\omega) \geq 0$. In the present case, we deduce from the inequality (3.27) that

$$I(\varphi^\omega(x)) - \frac{1}{2p} J(\varphi^\omega(x)) < \omega_1^{-\frac{4N+4+2\alpha}{N+2+\alpha}} d.$$  \hspace{1cm} (3.28)

Since $S(\varphi^\omega) = 0$ and (3.28), one has

$$\omega_1^{-\frac{4N+2\alpha}{N+2+\alpha}} \int |\nabla \varphi|^2 dx + \frac{p}{p-1} \omega_1^{-\frac{4N+4+2\alpha}{N+2+\alpha}} \int \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx \hspace{1cm} (3.29)$$

$$+ \omega_1^{-\frac{2N}{N+2+\alpha}} \int |\varphi|^2 dx < \frac{2p}{p-1} \omega_1^{-\frac{4N+4+2\alpha}{N+2+\alpha}} d.$$ (3.29)

It follows from (3.29) that

$$\int |\nabla \varphi|^2 dx < C.$$

Thus according to Proposition 2.1, we obtain that the solution $\varphi(t, x)$ is global in time.

When $1 + \frac{2+\alpha}{N} < p < \frac{N+\alpha}{N-2}$, by (3.25), we also have

$$\int |\nabla \varphi|^2 + \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx < C.$$

By Proposition 2.1, the solution $\varphi(t, x)$ of Eq.(1.1) exists globally. Thus, the solution $\varphi(t, x)$ of Eq.(1.1) with initial data $\varphi_0 \in K_+$ exists globally on $t \in [0, +\infty)$.

Now we consider $\varphi_0 \in \mathbb{R}_+$. In view of Proposition 3.7, this gives immediately that the solution $\varphi(t, x)$ of Eq.(1.1) satisfies that $\varphi(t, x) \in \mathbb{R}_+$ for $t \in [0, T)$. That is, $I(\varphi) < d$ and $J(\varphi) > 0$ for $t \in [0, T)$. By (3.10) and (3.11), we get

$$\left(\frac{1}{2} - \frac{1}{2p}\right) \int (|\nabla \varphi|^2 + |\varphi|^2 + \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2) dx < d.$$ (3.30)

Thus, the solution of $\varphi(t, x)$ of Eq.(1.1) exists globally. This completes the proof. \hspace{1cm} \square

**Theorem 3.9.** Let $\max\{2, 1 + \frac{2+\alpha}{N}\} \leq p < \frac{N+\alpha}{N-2}$. If $\int |x|^2 |\varphi_0|^2 dx < \infty$ and $\varphi_0 \in \mathbb{K}$, then the solution $\varphi(t, x)$ of Cauchy problem (1.1) blows up in finite time.

**Proof.** For $\varphi_0 \in \mathbb{K}$, we know from Proposition 3.7 that the solution $\varphi(t, x)$ of Eq.(1.1) satisfies: $\varphi(t, x) \in \mathbb{K}$ for $t \in [0, T)$. For $V(t) = \int |x|^2 |\varphi|^2 dx$, it follows from (2.10) and (3.12) that

$$V''(t) < 8S(\varphi(t, \cdot)),$$ for $t \in [0, T)$.

Thus for $t \in [0, T)$, $\varphi$ satisfies that $S(\varphi) < 0$, $J(\varphi) < 0$. For $\mu > 0$, we take $\varphi_\mu = \mu^{\frac{N+\alpha}{2p}} \varphi(\mu x)$. Thus

$$J(\varphi_\mu) = \mu^{\frac{N+\alpha+2p-2p-Np}{p}} \int |\nabla \varphi|^2 dx + \mu^{\frac{N+\alpha-Np}{p}} \int |\varphi|^2 dx - \int (I_\alpha * |\varphi|^p)|\varphi|^p dx,$$

$$S(\varphi_\mu) = \mu^{\frac{N+\alpha+2p-2p-Np}{p}} \int |\nabla \varphi|^2 dx - \frac{Np - N - \alpha}{2p} \int (I_\alpha * |\varphi|^p)|\varphi|^p dx.$$

Since $1 + \frac{2+\alpha}{N} \leq p < \frac{N+\alpha}{N-2}$, $S(\varphi) < 0$, then there exists $\mu_1 > 1$ such that $S(\varphi_\mu_1) = 0$, and when $\mu \in [1, \mu_1)$, $S(\varphi_\mu) < 0$. For $\mu \in [1, \mu_1]$, since $J(\varphi) < 0$, $J(\varphi_\mu)$ has the following two cases:

(i) $J(\varphi_\mu) < 0$ for $\mu \in [1, \mu_1]$;
(ii) There exists $1 < \mu_2 \leq \mu_1$ such that $J(\varphi_{\mu_2}) = 0$.

For the case (i), we have $S(\varphi_{\mu_1}) = 0$ and $J(\varphi_{\mu_1}) < 0$. It follows from (3.14) and (3.23) that

$$I(\varphi_{\mu_1}) \geq d_2 \geq d.$$
Furthermore, one has

\[
I(\varphi) - I(\varphi_{\mu_1}) = \frac{1}{2} \left( 1 - \mu_1 \frac{N + \alpha - 2p - Np}{p} \right) \int |\nabla \varphi|^2 dx + \frac{1}{2} \left( 1 - \mu_1 \frac{N + \alpha - 2p - Np}{p} \right) \int |\varphi|^2 dx \\
+ \frac{1}{2} \left( 1 - \mu_1 \frac{N + \alpha - 2p - Np}{p} \right) \int \sum_{i=1}^{k} \nu_i^2 x_i^2 |\varphi|^2 dx, \tag{3.30}
\]

and

\[
S(\varphi) - S(\varphi_{\mu_1}) = \frac{1}{2} \left( 1 - \mu_1 \frac{N + \alpha - 2p - Np}{p} \right) \int |\nabla \varphi|^2 dx. \tag{3.31}
\]

Take into account that \(\mu_1 > 1\) and \(1 + \frac{2 + \alpha}{N} \leq p < \frac{N + \alpha}{N - 2}\), we infer from (3.30) and (3.31) that

\[
I(\varphi) - I(\varphi_{\mu_1}) \geq S(\varphi) - S(\varphi_{\mu_1}) = \frac{1}{2} S(\varphi). \tag{3.32}
\]

For the case (ii), we have \(J(\varphi_{\mu_2}) = 0\) and \(S(\varphi_{\mu_2}) \leq 0\). Thus (3.13) and (3.23) yield that

\[
I(\varphi_{\mu_2}) \geq d_1 \geq d. \tag{3.33}
\]

It follows from (3.30) and (3.31) that

\[
I(\varphi) - I(\varphi_{\mu_2}) \geq S(\varphi) - S(\varphi_{\mu_2}) = \frac{1}{2} S(\varphi). \tag{3.34}
\]

Since \(I(\varphi_{\mu_1}) \geq d, I(\varphi_{\mu_2}) \geq d\), from (3.32) and (3.33), we obtain

\[
S(\varphi) < 2[I(\varphi) - d].
\]

From \(I(\varphi) = I(\varphi_0), \varphi_0 \in K\) and (3.34), one can estimate as follows

\[
V''(t) < 8S(\varphi) < 16[I(\varphi_0) - d] < 0.
\]

Then by the convexity method introduced in [27], there must exist time \(0 < T < \infty\) such that \(V(T) = 0\). Then from Proposition 2.1 or Lemma 2.5, we have

\[
\lim_{t \to T} ||\varphi||_\Sigma = \infty.
\]

Thus, the proof is completed. \(\square\)

**Remark 3.10.** When \(k = N\) and \(\nu_1 = \nu_2 = \cdots = \nu_k = 1\) in Eq.(1.1), Feng [7] derived the sharp threshold for global existence and blow-up to the solutions of Eq.(1.1) (see Theorem 3.10 and Theorem 3.11 in [7]). Our results in Theorems 3.8 and 3.9 extend and compensate for the ones of [7] for Eq.(1.1) with anisotropic partial/whole harmonic confinement by constructing some new cross-invariant sets and minimization problems.

**Remark 3.11.** It is obvious that

\[
\{\varphi \in \Sigma \setminus \{0\}, I(\varphi) < d\} = \mathbb{R}_+ \cup \mathbb{K}_+ \cup \mathbb{K}.
\]

In this sense, Theorem 3.9 implies that Theorem 3.8 is sharp when \(\int |x|^2|\varphi_0|^2 dx < \infty\).

By the above corollary, we immediately have

**Corollary 3.12.** Let \(\max\{2, 1 + \frac{2 + \alpha}{N}\} \leq p < \frac{N + \alpha}{N - 2}\) and \(\varphi_0\) satisfy \(\int |x|^2|\varphi_0|^2 dx < \infty\) and \(I(\varphi) < d\). Then the solution \(\varphi(t, x)\) of Eq.(1.1) blows up in finite time if and only if \(\varphi_0 \in \mathbb{K}\).

By Theorem 3.8, we can get another sufficient condition of global existence of Eq.(1.1).
Corollary 3.13. If $\varphi_0 \in \Sigma$ and $\|\varphi_0\|_{L^2}^2 < 2d$, then the corresponding solution $\varphi(t, x)$ of Eq.(1.1) exists globally.

Proof. Since $\|\varphi_0\|_{L^2}^2 < 2d$, we have $I(\varphi_0) < d$. Thus, we only need to prove $J(\varphi_0) > 0$. If otherwise, there exists $\gamma$ with $0 < \gamma \leq 1$, such that $J(\gamma \varphi_0) = 0$. From (3.12), (3.22) and $J(\gamma \varphi_0) = 0$, we have

$$I(\gamma \varphi_0) \geq d.$$ 

On the other hand,

$$\|\gamma \varphi_0\|_{L^2}^2 = \gamma^2\|\varphi_0\|_{L^2}^2 < 2\gamma^2d \leq 2d.$$ 

Therefore, we have $I(\gamma \varphi_0) < d$, which gives a contradiction. Thus one has $\varphi_0 \in \mathbb{R}_+$. It follows from Theorem 3.8 that the corollary holds true. \hfill \Box

4. Mass concentration and dynamics of the $L^2$-minimal blow-up solutions

In this section, we are devoted to the dynamical properties of blow-up solutions to Eq.(1.1) with partial/whole harmonic confinement. We first study the mass concentration phenomenon and then the dynamics of the $L^2$-minimal blow-up solutions, including the precise mass-concentration and blow-up rate to the blow-up solutions with minimal mass.

In order to study the dynamical properties for the blow-up solutions of Eq.(1.1), we recall the refined compactness lemma established by Feng and Yuan [21].

Lemma 4.1. Let $p = 1 + \frac{2+\alpha}{N}$, $\{v_n\}_{n=1}^{\infty}$ be a bounded sequence in $H^1(\mathbb{R}^N)$ and satisfy

$$\limsup_{n \to \infty} \|\nabla v_n\|_2^2 \leq M, \quad \limsup_{n \to \infty} \int (I_\alpha |v_n|^p)v_n |dx| \geq m.$$ 

Then, there exists $\{x_n\}_{n=1}^{\infty} \subset \mathbb{R}^N$ such that, up to a subsequence,

$$v_n(x + x_n) \rightharpoonup U \text{ weakly in } H^1(\mathbb{R}^N),$$

with $\|U\|_2 \geq (\frac{m}{pM})^{\frac{1}{p-2}}\|Q(x)\|_2$.

Using the refined compactness lemma, we can establish the following concentration property to the blow-up solutions of Eq.(1.1).

Theorem 4.2. ($L^2$-concentration) Assume $N - 2 \leq \alpha < N$ and $p = 1 + \frac{2+\alpha}{N}$. Let $\varphi(t, x)$ be a solution of Eq.(1.1) that blows up in finite time $T$, and $s(t)$ be a real-valued nonnegative function on $[0, T)$ such that $s(t)\|\nabla \varphi\|_2 \to \infty$ as $t \to T$. Then there exists a function $x(t) \in \mathbb{R}^N$ for $t < T$ such that

$$\liminf_{t \to T} \int_{|x-x(t)| \leq s(t)} |\varphi(t, x)|^2 \, dx \geq \int Q(x)^2 \, dx,$$ 

where $Q(x)$ is the ground state solution of Eq.(2.6).

Proof. Set

$$\rho(t) = \frac{\|\nabla Q(x)\|_2^2}{\|\nabla \varphi\|_2^2}, \quad v(t, x) = \rho(t)^{\frac{N}{2}} \varphi(t, \rho(t)x).$$

Let $\{t_n\}_{n=1}^{\infty}$ be an arbitrary time sequence such that $t_n \to T$ as $n \to \infty$, and denote $\rho_n = \rho(t_n)$ and $v_n(x) = v(t_n, x)$. By (2.2), (2.3) and (4.2), we obtain

$$\|v_n\|_2 = \|\varphi(t_n)\|_2 = \|\varphi_0\|_2, \quad \|\nabla v_n\|_2 = \rho_n \|\nabla \varphi(t_n)\|_2 = \|\nabla Q(x)\|_2.$$

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For \( f(x) \in H^1(\mathbb{R}^N) \), we define the functional

\[
H(f(x)) = \frac{1}{2} \int \left( |\nabla f(x)|^2 - \frac{1}{p} (I_\alpha * |f(x)|^p) |f(x)|^p \right) dx.
\]

From (4.3), (2.1) and (4.2), one has that

\[
H(v_n) = \frac{1}{2} \int \left( |\nabla v_n|^2 - \frac{1}{p} (I_\alpha * |v_n|^p) |v_n|^p \right) dx
\]

\[
= \rho_n^2 \int \left( |\nabla \varphi(t_n)|^2 - \frac{1}{p} (I_\alpha * |\varphi(t_n)|^p) |\varphi(t_n)|^p \right) dx
\]

\[
\leq \rho_n^2 E(\varphi(t_n)) - \frac{1}{2} \int \sum_{i=1}^k \nu_i^2 x_i^2 |\varphi(t_n)|^2 dx
\]

which yields, in particular,

\[
\int (I_\alpha * |\varphi(t_n)|^p) |\varphi(t_n)|^p dx \to p \|\nabla Q(x)\|^2_2 \text{ as } n \to \infty.
\]

Take \( M = \|\nabla Q(x)\|^2_2 \) and \( m = p \|\nabla Q(x)\|^2_2 \). Then by Lemma 4.1, there exists \( U(x) \in H^1(\mathbb{R}^N) \) and \( \{x_n\}_{n=1}^\infty \subset \mathbb{R}^N \) such that, up to a subsequence,

\[
v_n(\cdot + x_n) = \rho_n^N \varphi(t_n, \rho_n \cdot + x_n) \to U \text{ weakly in } H^1(\mathbb{R}^N),
\]

(4.4)

with \( \|U\|_2 \geq \|Q(x)\|_2 \). From (4.4), it follows that

\[
v_n(\cdot + x_n) \to U \text{ weakly in } L^2(\mathbb{R}^N).
\]

(4.5)

Then, from (4.5) and the weakly lower semi-continuous of the \( L^2 \)-norm, it ensues that for any \( A > 0 \),

\[
\lim_{n \to \infty} \int_{|x| \leq A} \rho_n^N |\varphi(t_n, \rho_n x + x_n)|^2 dx \geq \int_{|x| \leq A} |U|^2 dx.
\]

(4.6)

Since

\[
\lim_{n \to \infty} s(t_n) / \rho_n = \infty,
\]

then there exists \( n_0 > 0 \) such that for any \( n > n_0 \), we obtain that \( A \rho_n < s(t_n) \). It follows from \( A \rho_n < s(t_n) \) and (4.6) that

\[
\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{|x-y| \leq s(t_n)} |\varphi(t_n, x)|^2 dx \geq \liminf_{n \to \infty} \int_{|x-x_n| \leq A \rho_n} |\varphi(t_n, x)|^2 dx
\]

\[
= \liminf_{n \to \infty} \int_{|x| \leq A} \rho_n^N |\varphi(t_n, \rho_n x + x_n)|^2 dx
\]

\[
\geq \int_{|x| \leq A} |U|^2 dx, \text{ for any } A > 0,
\]

which implies that

\[
\liminf_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{|x-y| \leq s(t_n)} |\varphi(t_n, x)|^2 dx \geq \int |U|^2 dx = \|U\|^2_2.
\]
Due to the arbitrariness of the sequence \( \{ t_n \}_{n=1}^{\infty} \), from \( \|U\|_2 \geq \|Q(x)\|_2 \), we get that

\[
\liminf_{t \to T} \sup_{y \in \mathbb{R}^N} \int_{|x-y| \leq s(t)} |\varphi(t,x)|^2 dx \geq \|Q\|_2^2. \tag{4.7}
\]

For every \( t \in [0,T) \), one can easily see that the function \( g(y) := \int_{|x-y| \leq s(t)} |\varphi(t,x)|^2 dx \) is continuous on \( y \in \mathbb{R}^N \) and \( \lim_{|y| \to \infty} g(y) = 0 \). Therefore, for every \( t \in [0,T) \), there exists a function \( x(t) \in \mathbb{R}^N \) such that

\[
\sup_{y \in \mathbb{R}^N} \int_{|x-y| \leq s(t)} |\varphi(t,x)|^2 dx = \int_{|x-x(t)| \leq s(t)} |\varphi(t,x)|^2 dx. \tag{4.8}
\]

Thus, it follows from (4.7) and (4.8) that (4.1) holds true.

\begin{remark}
According to Theorem 4.2, we know that the blow-up solutions of Eq.(1.1) must have a lower \( L^2 \)-bound, i.e. \( \|\varphi_0\|_2 \geq \|Q(x)\|_2 \), which on the contrary, indicates that Theorem 3.1 holds true.

By Theorem 4.2, we can immediately obtain the conclusion below.

\begin{corollary}
Let \( \varphi(t,x) \) be a solution of Eq.(1.1) that blows up in finite time \( T \). Then for all \( l > 0 \), there exists \( x(t) \in \mathbb{R}^N \) for \( t < T \) such that

\[
\liminf_{t \to T} \int_{B(x(t),l)} |\varphi(t,x)|^2 dx \geq \int Q^2 dx.
\]

where \( Q(x) \) is the ground state solution of Eq.(2.6) and \( B(x(t),l) = \{ x \in \mathbb{R}^N | |x-x(t)| \leq l \} \).
\end{corollary}

\begin{theorem}
Assume that \( N - 2 \leq \alpha < N \) and \( p = 1 + \frac{2 + \alpha}{N} \). Let \( \varphi_0 \in \Sigma \) and \( \varphi(t,x) \) be the corresponding solution of problem (1.1) that blows up in finite time \( T \) with \( \|\varphi_0\|_2 = \|Q(x)\|_2 \). Then

(i) (Location of \( L^2 \)-concentration point) there exists \( x_0 \in \mathbb{R}^N \) such that

\[
\lim_{t \to T} x(t) = x_0, \text{ and } |\varphi(t,x)|^2 \to 2Q^2_0 \delta_{x=x_0} \text{ in the distribution sense as } t \to T, \tag{4.9}
\]

where \( Q(x) \) is the ground state solution of Eq.(2.6).

(ii) (Blow-up rate) There exists a positive constant \( C > 0 \) such that

\[
\|\nabla \varphi(t)\|_2 \geq \frac{C}{T-t}, \quad \forall t \in [0,T).
\]

\end{theorem}

\begin{proof}
(i) According to (2.2) and \( \|\varphi_0\|_2 = \|Q(x)\|_2 \), for \( t < T \), we have

\[
\|\varphi\|_2 = \|\varphi_0\|_2 = \|Q(x)\|_2. \tag{4.10}
\]

On the other hand, from Theorem 4.2 and Corollary 4.4, for all \( l > 0 \), one has that

\[
\|Q(x)\|_2^2 \leq \liminf_{t \to T} \int_{|x-x(t)| \leq l} |\varphi(t,x)|^2 dx \leq \liminf_{t \to T} \int |\varphi(t,x)|^2 dx \leq \|\varphi_0\|_2^2. \tag{4.11}
\]

It is distinct that (4.10) and (4.11) deduce

\[
\liminf_{t \to T} \int_{|x-x(t)| < l} |\varphi(t,x)|^2 dx = \|Q(x)\|_2^2,
\]

which implies that

\[
|\varphi(t,x(t))|^2 \to \|Q(x)\|_2^2 \delta_{x=0}, \text{ in the distribution sense as } t \to T. \tag{4.12}
\]

\end{proof}
Next, we will prove that there exists \( x_0 \in \mathbb{R}^N \) such that

\[
|\varphi(t, x)|^2 \to \|Q(x)\|_{2\delta=0}^2 \text{ in the distribution sense as } t \to T.
\]

In fact, for any real-valued function \( \theta(x) \) defined on \( \mathbb{R}^N \) and any real number \( \beta \), from (2.5) and (2.2), one can estimate

\[
E(e^{i\beta \theta(x)} \varphi) = \frac{1}{2} \int \left( (\nabla(e^{i\beta \theta(x)} \varphi))^2 + \sum_{i=1}^{k} \nu_i^2 e^{i\beta \theta(x)} \varphi \right)^2 \left( I_\alpha * |e^{i\beta \theta(x)} \varphi|^p \right) dx \geq \frac{1}{2} \int |\nabla(e^{i\beta \theta(x)} \varphi)|^2 dx = 0.
\]

Therefore, for any \( \beta \in \mathbb{R} \), we infer from (2.3) that

\[
0 \leq E(e^{i\beta \theta(x)} \varphi) = \frac{1}{2} \int \left( (\nabla(e^{i\beta \theta(x)} \varphi))^2 + \sum_{i=1}^{k} \nu_i^2 e^{i\beta \theta(x)} \varphi \right)^2 \left( I_\alpha * |e^{i\beta \theta(x)} \varphi|^p \right) dx \leq \frac{1}{2} \int |\nabla \theta(x) \cdot \varphi|^2 dx + \beta \int |\nabla \theta(x) \cdot \varphi|^2 dx + E(\varphi),
\]

which implies that

\[
|\int |\nabla \theta(x) \cdot \varphi|^2 dx | \leq \left( 2E(\varphi_0) \int |\nabla \theta|^2 |\varphi|^2 dx \right)^{\frac{1}{2}} (4.13)
\]

Then, choosing \( \theta_j(x) = x_j \) for \( j = 1, 2, \cdots, N \) in (4.13), using (1.1), (4.13) and (2.3), we derive

\[
\left| \frac{d}{dt} \int |\varphi(t, x)|^2 x_j dx \right| = |2I \int i\varphi_t \cdot \varphi \cdot x_j dx |
\]

\[
= |2I \int (-\Delta \varphi + \sum_{i=1}^{k} \nu_i^2 x_i^2 \varphi - \lambda(I_\alpha * |\varphi|^p) |\varphi|^{p-2}) \varphi \cdot x_j dx |
\]

\[
= |2I \int -\Delta \varphi \cdot \varphi \cdot x_j dx |
\]

\[
= 2 \left( 2E(\varphi_0) \int |\varphi|^2 dx \right)^{\frac{1}{2}} = C. \quad (4.14)
\]

Taking any two sequences \( \{t_n\}_{n=1}^{\infty}, \{t_m\}_{m=1}^{\infty} \subset [0, T) \) such that \( \lim_{n \to \infty} t_n = \lim_{m \to \infty} t_m = T \). Therefore, for all \( j = 1, 2, \cdots, N \), we deduce from the inequality (4.14) that

\[
\left| \int |\varphi(t_n, x)|^2 x_j dx - \int |\varphi(t_m, x)|^2 x_j dx \right| \leq \int_{t_m}^{t_n} \left| \frac{d}{dt} \int |\varphi(t, x)|^2 x_j dx \right| dt \leq C|t_n - t_m| \to 0 \quad \text{as } n, m \to \infty,
\]

which implies that

\[
\lim_{t \to T} \int |\varphi(t, x)|^2 x_j dx \text{ exists for any } j = 1, 2, \cdots, N.
\]
In other words,
\[ \lim_{t \to T} \int |\varphi(t, x)|^2 dx \text{ exists.} \]

Set \( x_0 = \frac{\lim_{t \to T} \int |\varphi(t, x)|^2 dx}{\|Q(x)\|_2^2} \), then \( x_0 \in \mathbb{R}^N \) and we obtain
\[ \lim_{t \to T} \int |\varphi(t, x)|^2 dx = x_0\|Q(x)\|_2^2. \quad (4.15) \]

On the other hand, we infer from (2.10) that
\[ \frac{d^2}{dt^2} \int |x|^2 |\varphi(t, x)|^2 dx = 16E(\varphi_0) - 16 \int \sum_{i=1}^k \nu_i^2 x_i^2 |\varphi|^2 dx < 16E(\varphi_0). \]

Thus, for any \( t \in [0, T) \), there exists a constant \( c_1 > 0 \) such that
\[ \int |x|^2 |\varphi(t, x)|^2 dx \leq c_1. \]

Hence we deduce that
\[
\int |x|^2 |\varphi(t, x + x(t))|^2 dx \leq 2 \int |x + x(t)|^2 |\varphi(t, x + x(t))|^2 dx \\
+ 2 \int |x(t)|^2 |\varphi(t, x + x(t))|^2 dx \\
\leq 2c_1 + 2|x(t)|^2 \|\varphi_0\|_2^2 \\
= 2c_1 + 2|x(t)|^2 \|Q(x)\|_2^2. \quad (4.16)
\]

From (4.12), it follows that
\[
\limsup_{t \to T} |x(t)|^2 \|Q(x)\|_2^2 = \limsup_{t \to T} \int_{|x| < 1} |x + x(t)|^2 |\varphi(t, x + x(t))|^2 dx \\
\leq \int |x|^2 |\varphi(t, x)|^2 dx \leq c_0. \quad (4.17)
\]

From (4.17), one can estimate
\[ \limsup_{t \to T} |x(t)| \leq \frac{\sqrt{c_1}}{\|Q(x)\|_2}. \quad (4.18) \]

Combined (4.16) with (4.18), we have
\[ \limsup_{t \to T} \int |x|^2 |\varphi(t, x + x(t))|^2 dx \leq C, \]
where \( C = 4c_1 \). Thus, for any \( l_0 > 0 \), one has
\[
\limsup_{t \to T} \int_{|x| \geq l_0} l_0 |x| |\varphi(t, x + x(t))|^2 dx \leq \limsup_{t \to T} \int_{|x| \geq l_0} |x|^2 |\varphi(t, x + x(t))|^2 dx \leq C.
\]

Therefore, for any \( \varepsilon > 0 \), there exists a large enough \( l_0 = l_0(\varepsilon) > 0 \) such that
\[ \limsup_{t \to T} \left| \int_{|x| \geq l_0} x |\varphi(t, x + x(t))|^2 dx \right| \leq \frac{C}{l_0} < \varepsilon. \quad (4.19) \]

Then, using (4.19) and (4.12), we infer that for any \( \varepsilon > 0 \)
\[ \limsup_{t \to T} \left| \int x |\varphi(t, x)|^2 dx - x(t) \|Q(x)\|_2^2 \right| = \limsup_{t \to T} \left| \int x |\varphi(t, x)|^2 dx - x(t) \int |\varphi(t, x)|^2 dx \right| \]
\[
\begin{align*}
&= \limsup_{t \to T} \left| \int \varphi(t, x)|^2(x - x(t))dx \right| \\
&= \limsup_{t \to T} \left| \int \varphi(t, x + x(t))|^2x dx \right| \\
&\leq \limsup_{t \to T} \left| \int \varphi(t, x + x(t))|^2x dx \right| + \varepsilon \\
&= \varepsilon \\
\end{align*}
\] (4.20)

It follows from (4.15) and (4.20) that

\[
\lim_{t \to T} x(t) = x_0, \quad \text{and} \quad \limsup_{t \to T} \int x|\varphi(t, x)|^2dx = x_0\|Q(x)\|^2_2.
\]

Therefore, there exists \(x_0 \in \mathbb{R}^N\) (see (4.15)) such that

\[
|\varphi(t, x)|^2 \to \|Q(x)\|^2_{2\delta_{x=x_0}} \text{ in the distribution sense as } t \to T. \tag{4.21}
\]

Thus, we know that (4.9) holds true.

(ii) Taking \(z(x) \in C^\infty_0(\mathbb{R}^N)\) is a nonnegative radial function such that

\[
z(x) = z(|x|) = |x|^2, \quad \text{if } |x| < 1 \quad \text{and} \quad |\nabla z(x)|^2 \leq Cz(x).
\]

For \(h > 0\), we define that \(z_h(x) = h^2z(\frac{x}{h})\) and \(f_h(t) = \int z_h(x - x_0)|\varphi(t, x)|^2dx\) with \(x_0\) define by (4.15) (see also (4.21)). From (4.13), for every \(t \in [0, T)\), we derive

\[
\frac{d}{dt}|f_h(t)| = \frac{d}{dt}\left| \int |\varphi(t, x)|^2z_h(x - x_0)dx \right| \\
= 2\text{Im} \int \nabla \varphi \cdot \overline{\varphi} \cdot \nabla z_h(x - x_0)dx \\
\leq 2\left(2E(\varphi_0) \int |\varphi(t, x)|^2|\nabla z_h(x - x_0)|^2dx \right)^{\frac{1}{2}} \\
\leq C\sqrt{f_h(t)},
\]

which implies that

\[
\left| \frac{d}{dt}\sqrt{f_h(t)} \right| \leq C.
\]

By integrating on both sides, one has that

\[
|\sqrt{f_h(t)} - \sqrt{f_h(t^*)}| \leq C|t - t^*|. \tag{4.22}
\]

It is clear that (4.9) implies

\[
f_h(t^*) \to \|Q(x)\|_{2z_h(0)} \quad \text{as } t^* \to T,
\]

where \(\|Q(x)\|_{2z_h(0)} = 0\). Thus, by letting \(t^* \to T\) in (4.22), we deduce that

\[
f_h(t) \leq C(T - t)^2.
\]

Fix \(t \in [0, T)\) and let \(h \to \infty\), we obtain

\[
\int |\varphi(t, x)|^2|x - x_0|^2dx \leq C(T - t)^2.
\]
It follows from uncertainty principle and the above inequality that
\[
\|\varphi_0\|_2^2 = \int |\varphi(t,x)|^2 dx = -\frac{2}{N} \text{Re} \int \nabla \varphi \cdot (x-x_0) dx \\
\leq C \left( \int |\varphi(t,x)|^2 |x-x_0|^2 dx \right)^{\frac{1}{2}} \left( \int |\nabla \varphi|^2 dx \right)^{\frac{1}{2}} \\
\leq \bar{C} (T-t) \|\nabla \varphi(t)\|_2,
\]
which means that
\[
\|\nabla \varphi(t)\|_2 \geq \frac{\bar{C}}{T-t}, \quad \text{for} \quad \forall t \in [0,T).
\]
Therefore, the whole proof is completed.

**Remark 4.6.** (i) For Eq. (1.1) without harmonic confinement, Feng and Yuan [21] derived the similar mass concentration properties of blow-up solutions and dynamical properties of the $L^2$-minimal blow-up solutions in the $L^2$-critical case (see Theorems 1.4 and 1.5 in [21]). Theorems 4.2 and 4.5 in our present paper extend the corresponding conclusions of [21] to the Schrödinger-Hartree equation with anisotropic partial/whole harmonic confinement.

(ii) As we know, the characterization of the blow-up solutions with minimal mass depends strongly on the uniqueness of the ground state of Eq. (2.6). However, in the general case $2 \leq p < \frac{N+\alpha}{N-2}$ and $0 < \alpha < N$, the uniqueness of the ground state of Eq. (2.6) is still open, so we cannot obtain the limiting profile of the minimal mass blow-up solutions to initial-value problem (1.1) at the moment, except for some special cases discussed in [19, 20].

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**References**


