A Systematic Rotation Method to Color the Historic Heawood Map by Four Colors

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November 24, 2022
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Abstract

The four-color problem was first posed by Francis Guthrie in 1852. Over a century, many researchers tried many ways and obtained some useful results. One proposed proof was given by Alfred Kempe using Kempe Chain in 1879, but Percy Heawood found counterexample of Kempe’s proof in 1890 [3]. This historic Heawood map has 25 regions. It can be very challenging to just use trial and error method to make it four-colored. In this paper, a systematic way to color the map with four colors using a novel method of rotation [4] inspired by a rotation principle from Zhuan Falun book [2] of Falun Dafa will be demonstrated. It shows that the novel method of rotation is very powerful and can provide a systematic approach to make maps four-colored.

Keywords

A Systematic Rotation Method; Heawood Map; Four Color Theorem

1. Introduction

The four-color problem was first posed by Francis Guthrie in 1852 [3], it indicate that no more than four colors are required to color the regions of any map so that no two adjacent regions have the same color. The two adjacent regions need to have a shared boundary line not just shared discretized points. However, the four-color problem is very challenging to be proved mathematically. Over a century, many researchers tried many ways and obtained some useful results.

It can be proved that every map has at least one region with five or fewer neighbors by using Euler's formula [1]. One proposed proof was given by Alfred Kempe using Kempe Chain in 1879, but Percy Heawood found counterexample of Kempe’s proof in 1890 as shown in Fig.1 [3]. This historic Heawood map has 25 regions. It can be very challenging to just use trial and error method to make it four-colored because the full combinations for every region possible to have one of four colors will be a very large number as $4^{25}$. Even people with some expertise of graph coloring may still need considerable time to make the four-colored of this historic Heawood map because they do not have a systematical way to achieve it.

Luckily, the author of this paper, Weiguo Xie, has developed a novel method of rotation [4] inspired by a rotation principle from Zhuan Falun book [2] of Falun Dafa to prove the Four Color Theorem [4]. This novel method of rotation will be applied in this paper to systematically color the historic Heawood map with four colors.
This paper will be organized in a few sections: firstly to introduce the features of the historic Heawood map; secondly to introduce the novel method of rotation [4]; thirdly to apply the novel method of rotation to systematically color the historic Heawood map by Four Colors.

2. The features of the historic Heawood map

In Fig. 1, the middle uncolored region surrounding with anti-clockwise b(blue)-r(red)-y(yellow)-g(green)-r(red) and all regions in the map have at least five neighbors. In Fig. 2, After following Kempe’s method, Percy Heawood found that the originally green region A and the originally yellow region B both become red, which is not permissible [3]. It was written by Percy Heawood, “But, unfortunately, it is conceivable that though either transposition would remove a red, both may not remove both reds. My map is an actual exemplification of this possibility, where either transposition prevents the other from being of any avail, by bringing the red and the other divisions into the same region; so that Mr. Kempe’s proof does not hold, unless some modifications can be introduced into it to meet this case of failure.” [3]
Fig. 2. After following Kempe’s method, the originally green region A and the originally yellow region B both become red, which is not permissible [3].

Because this historic Heawood map has 25 regions, the full combinations for every region possible to have one of four colors will be a very large number as $4^{25}$. It can be very challenging to just use trial and error method to make it four-colored, even people with some expertise of graph coloring may still need considerable time to do it. Therefore, it is necessary to develop a systematic way to achieve it, and the systematic way will have potential wider applications in coloring graphs, especially with a much larger number of regions in a map.

3. The novel method of rotation

Xie has developed a novel method of rotation [4] inspired by a rotation principle from Zhuan Falun book [2] of Falun Dafa to prove the Four Color Theorem [4]. This novel method of rotation will be applied in this paper to systematically color the historic Heawood map with four colors. The rotation table is shown in Table 1 [4]. Every rotation will include: a) Two chain groups with features of shared r and double (the same color) inside the 5-neighbors of the uncolored region; b) 2-Color linked chain(s) interchange inside the corresponding section in the map; c) After the interchange, if the new chain (which can be just a single region for certain special cases) can be formed, they will be the last column in the table, otherwise, if new chain cannot be formed, then the color of uncolored region can be determined.

Table 1. The rotation table with the first 17 rotates [4]. (*Note: after 16 rotates, the relative color and location of the 5-neighbor regions back to the initial one, however, we can just keep increasing the number of the subscript for more rotates.)
<table>
<thead>
<tr>
<th>Rotates</th>
<th>Chain 1</th>
<th>Chain 2</th>
<th>Share</th>
<th>Doubles</th>
<th>interchange</th>
<th>Form new chains</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>GR</td>
<td>RY</td>
<td>R</td>
<td>B, B</td>
<td>Inside GR, B→Y₁</td>
<td>BG</td>
</tr>
<tr>
<td>2</td>
<td>BG</td>
<td>GR</td>
<td>G</td>
<td>Y, Y₁</td>
<td>Inside BG, Y→R₁</td>
<td>Y₁B</td>
</tr>
<tr>
<td>3</td>
<td>Y₁B</td>
<td>BG</td>
<td>B</td>
<td>R, R₁</td>
<td>Inside Y₁B, R→G₁</td>
<td>R₁Y₁</td>
</tr>
<tr>
<td>4</td>
<td>R₁Y₁</td>
<td>Y₁B</td>
<td>Y₁</td>
<td>G, G₁</td>
<td>Inside R₁Y₁, G→B₁</td>
<td>G₁R₁</td>
</tr>
<tr>
<td>5</td>
<td>G₁R₁</td>
<td>R₁Y₁</td>
<td>R₁</td>
<td>B, B₁</td>
<td>Inside G₁R₁, B→Y₂</td>
<td>B₁G₁</td>
</tr>
<tr>
<td>6</td>
<td>B₁G₁</td>
<td>G₁R₁</td>
<td>G₁</td>
<td>Y₁, Y₂</td>
<td>Inside B₁G₁, Y₁→R₂</td>
<td>Y₂B₁</td>
</tr>
<tr>
<td>7</td>
<td>Y₂B₁</td>
<td>B₁G₁</td>
<td>B₁</td>
<td>R₁, R₂</td>
<td>Inside Y₂B₁, R₁→G₂</td>
<td>R₂Y₂</td>
</tr>
<tr>
<td>8</td>
<td>R₂Y₂</td>
<td>Y₂B₁</td>
<td>Y₂</td>
<td>G₂, G₂</td>
<td>Inside R₂Y₂, G₁→B₂</td>
<td>G₂R₂</td>
</tr>
<tr>
<td>9</td>
<td>G₂R₂</td>
<td>R₂Y₂</td>
<td>R₂</td>
<td>B₁, B₂</td>
<td>Inside G₂R₂, B₁→Y₃</td>
<td>B₂G₂</td>
</tr>
<tr>
<td>10</td>
<td>B₂G₂</td>
<td>G₂R₂</td>
<td>G₂</td>
<td>Y₂, Y₃</td>
<td>Inside B₂G₂, Y₂→R₃</td>
<td>Y₃B₂</td>
</tr>
<tr>
<td>11</td>
<td>Y₃B₂</td>
<td>B₂G₂</td>
<td>B₂</td>
<td>R₂, R₃</td>
<td>Inside Y₃B₂, R₂→G₃</td>
<td>R₃Y₃</td>
</tr>
<tr>
<td>12</td>
<td>R₃Y₃</td>
<td>Y₃B₂</td>
<td>Y₃</td>
<td>G₂, G₃</td>
<td>Inside R₃Y₃, G₂→B₃</td>
<td>G₃R₃</td>
</tr>
<tr>
<td>13</td>
<td>G₃R₃</td>
<td>R₃Y₃</td>
<td>R₃</td>
<td>B₂, B₃</td>
<td>Inside G₃R₃, B₂→Y₄</td>
<td>B₃G₃</td>
</tr>
<tr>
<td>14</td>
<td>B₃G₃</td>
<td>G₃R₃</td>
<td>G₃</td>
<td>Y₃, Y₄</td>
<td>Inside B₃G₃, Y₃→R₄</td>
<td>Y₄B₃</td>
</tr>
<tr>
<td>15</td>
<td>Y₄B₃</td>
<td>B₃G₃</td>
<td>B₃</td>
<td>R₃, R₄</td>
<td>Inside Y₄B₃, R₃→G₄</td>
<td>R₄Y₄</td>
</tr>
<tr>
<td>16</td>
<td>R₄Y₄</td>
<td>Y₄B₃</td>
<td>Y₄</td>
<td>G₃, G₄</td>
<td>Inside R₄Y₄, G₃→B₄</td>
<td>G₄R₄</td>
</tr>
<tr>
<td>17</td>
<td>G₄R₄</td>
<td>R₄Y₄</td>
<td>R₄</td>
<td>B₃, B₄</td>
<td>Inside G₄R₄, B₃→Y₅</td>
<td>B₄G₄</td>
</tr>
</tbody>
</table>

4. To apply the novel method of rotation to systematically color the historic Heawood map

In order to use Xie’s method, it will need to convert colors of the 5-neighbor regions of the middle uncolored region by exchanging colors becoming anticlockwise R(Red) – B(Blue) – G(Green) – Y(Yellow) –
B(Blue). All color r-b interchange will be performed in Fig. 1 of the historic Heawood map, and then Fig. 1 will become Fig. 3.

![Fig. 3](image)

**Fig. 3.** The new map after all color r-b interchange in Fig. 1 of the historic Heawood map.

All color y-g interchange will be performed in Fig. 3, and then it will become Fig. 4. Now, the colors of the 5-neighbor regions of the middle uncolored region in the map become anticlockwise R(Red) – B(Blue) – G(Green) – Y(Yellow) – B(Blue).
Fig. 4. The new map after all color y-g interchange in Fig. 3.

To open the left-side edge of r (red) region in Fig. 4 to make the region fully cover a sphere surface, and then it becomes the Fig. 5. By doing this operation, r (red) region will become a neighbor to all edge regions.

After the above three preparation steps, it is now ready to perform the novel rotation method.
**Fig. 5.** To open the left-side edge of r (red) region in Fig. 4 to make the region fully cover a sphere surface.

In order to make it clearer, it is to mark the RY chain and GR chain of the Fig. 5, which becomes the Fig. 6.

**Fig. 6.** Based on Fig. 5, to mark the RY chain and GR chain in the map.

Then, to follow the Step 1 rotation in Table 1 to perform B→Y₁ color interchange inside GR (*note: the meaning inside GR is the side of g(green)-b(blue)-r(red) of the five neighbor regions in Fig. 6, rather than the side of g(green)-y(yellow)-b(blue)–r(red) of the five neighbor regions). After the color B→Y₁ interchange, Fig. 6 becomes Fig. 7.
Fig. 7. Based on Fig. 6, to follow the Step 1 rotation in Table 1 to perform B→Y₁ color interchange inside GR.

After the B→Y₁ color interchange, to check whether the new chain BG (in the last column of the row 1 in the Table 1, which can be just a single region for certain special cases) can be formed. The answer is “Yes” for BG chain being formed for the 5-neighbor regions of the middle uncolored region in Fig. 7. Based on Fig.7, the BG chain has been marked and shown in Fig. 8.
Fig. 8. Based on Fig. 7, the BG chain has been marked.

For the Step 1 rotation, because the BG chain (the element in the last column of row 1 in Table 1) can be naturally formed in Fig. 8, it will need to perform Step 2 rotation. Then, to follow the Step 2 rotation in Table 1 to perform $Y\rightarrow R_1$ color interchange inside BG (note: the meaning inside BG is the side of $b$(blue) – $y$(yellow) - $g$(green) of the five neighbor regions in Fig. 8, rather than the side of $b$(blue) – $r$(red) – $y_1$(yellow) - $g$(green) of the five neighbor regions). After the color $Y\rightarrow R_1$ interchange, Fig. 8 becomes Fig. 9.

Fig. 9. Based on Fig. 8, to follow the Step 2 rotation in Table 1 to perform $Y\rightarrow R_1$ color interchange inside BG.

After the $Y\rightarrow R_1$ color interchange, to check whether the new chain $Y_1B$ (in the last column of the row 2 in the Table 1, which can be just a single region for certain special cases) can be formed. The answer is “No” for $Y_1B$ chain being formed for the 5-neighbor regions of the middle uncolored region in Fig. 9. Based on Fig. 9, the $Y_1B$ chain (and YB chain because $Y_1$ and $Y$ are just the same yellow color) starting from $Y_1$ of the 5-neighbor regions of the middle uncolored region has been marked and shown in Fig. 10. It can be seen that the $Y_1B$ chain (and YB chain) cannot reach the b (blue) region of the 5-neighbor regions of the middle uncolored region.
Fig. 10. Based on Fig. 9, the Y₁B chain (and YB chain) starting from Y₁ of the 5-neighbor regions of the middle uncolored region has been marked.

Because the Y₁B chain cannot be naturally formed for the 5-neighbor regions of the middle uncolored region, then B-Y₁ color interchange starting from b (blue) the b (blue) region of the 5-neighbor regions of the middle uncolored region will not reach Y₁ of the 5-neighbor regions of the middle uncolored region. After B-Y₁ color interchange, the left-side edge can be closed and back to the original edge of the planar map. Now the uncolored pentagon region can be colored as B (blue). The final result of the historic Heawood map that can be successfully four-colored by this novel rotation method is shown in Fig. 11.
Fig. 11. The historic Heawood map is successfully four-colored by this novel rotation method.

5. Conclusions

A systematic way to color the historic Heawood map with four colors by the novel method of rotation [4] inspired by a rotation principle from Zhuan Falun book [2] of Falun Dafa is shown in this article. Firstly, it can be easily converted by exchanging colors to make 5-neighbor regions being R(Red) – B(Blue) – G(Green) – Y(Yellow) – B(Blue), and open the left-side edge to make the map fully cover a sphere surface. After these preparations, it can be seen that only two rotation steps from Table 1 are needed to color the map with four colors. It shows that the novel method of rotation is very powerful and can provide a systematic approach to make maps four-colored.

The normal trial and error method can be avoided, especially for very challenging maps with a much larger number of regions. It can be envisaged that the systematic rotation method shown in this article will have potential wider applications in coloring graphs.

Data Availability Statements

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest.
References:


