The Earth’s rotational modes revisited

Behnam Seyed-Mahmoud\textsuperscript{1}

\textsuperscript{1}University of Lethbridge

November 24, 2022

Abstract

In the conventional treatment of the Earth’s rotational dynamics using the Earth’s angular momentum description (AMD) it is customary to assume that the velocity/displacement of a mass element in the liquid core (LC) has a rigid rotation and a displacement components in addition to the uniform (solid-body) rotation. This makes for a very complex set of differential equations in the treatment of the dynamics of this body. In this work I will use a simple three-layer Earth model with rigid mantle (MT) and inner core (IC) and an incompressible and homogeneous LC to show that it is redundant to assign a rigid rotation component to the motion in the LC. Further, in order to study another shortcoming of the conventional approach above, I will assume that the MT rotates uniformly, and use an approximation commonly used in dealing with the Earth’s rotational dynamics, and find identical analytical results to those in the literature for the periods of the inner-core wobble (ICW) and the free inner-core nutation (FICN). This shows that the aforementioned approximation is not suitable in computing these modes, especially FICN.
The Earth’s rotational modes revisited

B. Seyed-Mahmoud
†Department of Physics and Astronomy,
University of Lethbridge, Lethbridge, Alberta T1K 3M4, Canada
(Received 00 Month 20xx; final version received 00 Month 20xx)

In the conventional treatment of the Earth’s rotational dynamics using the Earth’s angular momentum description (AMD) it is customary to assume that the velocity/displacement of a mass element in the liquid core (LC) has a rigid rotation and a displacement component in addition to the uniform (solid-body) rotation. This makes for a very complex set of differential equations in the treatment of the dynamics of this body. In this work I will use a simple three-layer Earth model with rigid mantle (MT) and inner core (IC) and an incompressible and homogeneous LC to show that it is redundant to assign a rigid rotation component to the motion in the LC. Further, in order to study another shortcoming of the conventional approach above, I will assume that the MT rotates uniformly, and use an approximation commonly used in dealing with the Earth’s rotational dynamics, and find identical analytical results to those in the literature for the periods of the inner-core wobble (ICW) and the free-inner core nutation (FICN). This shows that the aforementioned approximation is not suitable in computing these modes, especially FICN.

1. Introduction

The Earth’s known free wobble/nutation modes are the tilt-over mode (TOM), the Chandler wobble (CW), the free-core nutation (FCN), the inner-core wobble (ICW) and the free-inner core nutation (FICN). The periods of these modes are predicted by applying the conservation laws, which are in the form of partial differential equations, to the rotating and deformable Earth. The presence of a large LC bounded by the MT and the IC complicates dealing with these equations. The second order partial differential equation describing the dynamics of a uniformly rotating, homogeneous and incompressible liquid spherical shell with rigid boundaries is hyperbolic. Hyperbolic equations normally describe initial value problems while in the LC the equation is subjected to boundary conditions (BC), a condition which make the problem ill-posed. In core dynamics where the objective is to computed the long period (inertial modes, for example) of a uniformly rotating spherical shell (Rieutord 1995, Rieutord and Valdettaro 1997, Rieutord et al. 2001), or deal with elliptical (tidal) instability of the liquid shell (Cébron et al. 2012, for example), liquid of small viscosity is considered in order to avoid the ill-posedness problem. Estimates of the Ekman number in the liquid core range from the order of $10^{-16}$ up to $10^{-4}$ (Lumb and Aldridge 1991). The displacement of a mass element in the liquid core during the wobble is predominantly rigid rotation and involves negligible shear, therefore, viscosity is ignored.

To compute the periods of the Earth’s wobble modes either the linear momentum description (LMD) (e.g., Smith 1977, Rogister 2001, Huang et al. 2001, Rochester et al. 2014) or the angular momentum description (AMD) (e.g., Wahr 1981, de Vries and Wahr 1991, Mathews et al. 1991, Rochester and Crossley 2009, Seyed-Mahmoud et al. 2017, Rekier et al. 2020, Seyed-Mahmoud and Rogister 2021) of the dynamics of the Earth’s interiors is used. In the AMD approach it is customary to use a reference frame attached to the wobbling MT (e.g.,
where $\Omega$ is the Earth’s mean rotation rate, $\hat{e}_3$ is the unit vector along the mean rotation axis and

$$m_3 = \alpha_3 (\hat{e}_1 - i \hat{e}_2)e^{i\omega t},$$

(2)

$\alpha_3$, being the amplitude the MT wobble, $\omega$ the frequency of the wobble, $i = \sqrt{-1}$, $\hat{e}_1$ and $\hat{e}_2$ are unit vectors which, along with $\hat{e}_3$, define a mutually perpendicular and right handed coordinate system. The IC (when included in the Earth model) and the LC are assigned their own rotation vectors (e.g., de Vries and Wahr 1991, Mathews et al. 1991, Dumberry 2009, Rekier et al. 2020, Triana et al. 2021) and are given respectively as

$$\Omega_1 = \Omega_3 + \omega_1 = \Omega(\hat{e}_3 + m_3 + m_1)$$

(3)

$$\Omega_2 = \Omega_3 + \omega_2 = \Omega(\hat{e}_3 + m_3 + m_2).$$

(4)

As in (2)

$$m_1 = \alpha_1 (\hat{e}_1 - i \hat{e}_2)e^{i\omega t},$$

(5)

$$m_2 = \alpha_2 (\hat{e}_1 - i \hat{e}_2)e^{i\omega t}$$

(6)

where $\alpha_1$ and $\alpha_2$ are the amplitudes of the IC and LC wobble with respect to the reference frame respectively. Accordingly, relative to the reference frame, the velocity of a mass element in the LC has a rigid-rotation and a small departure from this state

$$v_f = \omega_2 \times r + v.$$  

(7)

This leads to a very complicated analysis of the Earth’s wobble problem and requires tedious algebra to solve for the periods of the Earth’s wobble modes (see, for example, de Vries and Wahr 1991, Mathews et al. 1991). In section 2 I will use a uniformly rotating reference frame and show that it is redundant to to assign a wobbling (rigid rotation) motion to the LC. That is, it is sufficient to assume that with respect to the uniformly rotating reference frame a mass element in the liquid ore has a velocity only due to departure from solid-body-rotation

$$v_f = v.$$  

(8)

In section 3, to demonstrate that the conventional treatment of the LC dynamics using AMD is not adequate in dealing with the IC dynamics, I will first set $\alpha_3$ to zero, that is, I will assume that the mantle rotates uniformly, then ignore the LC pressure on the IC, and will show that the periods of the ICW and FICN so computed are identical, to first order in the ellipticity, to those cited in the literature.

2. Conservation laws

As mentioned in previous section, we consider an Earth model with a homogeneous and incompressible LC and rigid MT and IC. We choose a reference frame which rotates uniformly,

$$\Omega_r = \Omega \hat{e}_3,$$

(9)

$\Omega$ being the Earth’s average rotation rate about the rotation axis and $\hat{e}_3$ the unit vector fixed in space along that axis. The rotation rates of the IC, the LC and the MT are given by equations (3), (4) and (1) respectively.
Relative to the reference frame, the velocity of a mass element in the IC, the LC and the MT are given respectively as

\[ v_1 = \Omega m_1 \times r \]  
\[ v_2 = \Omega m_2 \times r + w, \]  
\[ v_3 = \Omega m_3 \times r; \]  

\( r \) is the position vector of a mass element and \( w \) the velocity in the LC due to departure from rigid rotation. \( |\Omega_k \times r|, k = 1, 2, 3 \), and \( w \) are small and of the same order. Since we are dealing with the Earth’s oscillations, we assume that a dynamical quantity \( \psi(r, t) \) has time dependent of the form

\[ \psi(r, t) = \psi(r)e^{i\omega t}. \]  

The displacement vectors in the IC, LC and MT are then given as

\[ q_1 = \Omega \frac{i\omega}{i\omega} m_1 \times r, \]  
\[ q_2 = \Omega \frac{i\omega}{i\omega} m_2 \times r + u \]  
\[ q_3 = \Omega \frac{i\omega}{i\omega} m_3 \times r, \]  

where \( u \) is the displacement superimposed on rigid rotation in the LC. Not that now \( \alpha_k, k = 1, 2, 3 \) are the amplitudes of the IC, LC and MT with respect to the uniformly rotating reference frame.

For the Earth model considered, the free oscillations of the LC are given by governing equations describing the conservation laws for mass, momentum and gravitational flux

\[ \nabla \cdot v_2 = 0 \]  
\[ \frac{\partial v_2}{\partial t} + 2\Omega \mathbf{e}_3 \times v_2 = -\nabla \chi, \]  
\[ \nabla^2 V_1 = 0 \]  

with

\[ \chi = \frac{p_1}{\rho} - V_1 \]  

where \( V_1 \) is due to reorientation of the IC, the LC and the MT. \( p_1 \) and \( V_1 \) are, respectively, the Eulerian departure of the pressure and the gravitational potential in the LC from their equilibrium values \( P_0 \) and \( V_0 \),

\[ \nabla P_0 = \rho g_0, \]  
\[ g_0 = \nabla W_0, \]  
\[ \nabla^2 W_0 = -4\pi G \rho + 2\Omega^2, \]  
\[ V_0 = W_0 - \frac{1}{2} |\Omega \mathbf{e}_3 \times r| \]  

where \( \rho \) is the constant density of the LC, \( W_0 \) is the gravity potential and \( G \) is the gravitational constant. Substituting from (13) and (15), (17) and (18) become

\[ \nabla \cdot u = 0 \]  

\[-\omega^2 \mathbf{u} + 2i\Omega \omega \dot{\mathbf{e}}_3 \times \mathbf{u} + i\omega \Omega \mathbf{m}_2 \times \mathbf{r} + 2\Omega^2 \dot{\mathbf{e}}_3 \cdot \mathbf{r} \mathbf{m}_2 = -\nabla \chi \]  
(26)

after using a triple vector product identity. Note again that we consider \( \mathbf{w} \) and \( \Omega \mathbf{m}_2 \times \mathbf{r} \) to be of the same order. We render equation (26) dimensionless by dividing it by \( 4R^2\Omega^2 \), \( R \) being the mean outer radius of the Earth,

\[-\sigma^2 \mathbf{u}' + i\sigma \dot{\mathbf{e}}_3 \times \mathbf{u}' = -\nabla' \chi' - \frac{1}{2} (i\sigma \mathbf{m}_2 \times \mathbf{e}_3 \cdot \mathbf{e}_3 \cdot \mathbf{m}_2) \]  
(27)

with \( \sigma = \omega/2\Omega \), \( \mathbf{u}' = \mathbf{u}/R \), \( \chi' = \chi/(4R^2\Omega^2) \), \( \xi = \mathbf{r}/R \) and \( \nabla' = R \nabla \). Henceforth we drop the primes for convenience. The displacement \( \mathbf{u} \) is then given as

\[\sigma^2(\sigma^2 - 1) \mathbf{u} = \tilde{\Gamma}_p \cdot \left[ \nabla \chi + \frac{1}{2} (i\sigma \mathbf{m}_2 \times \mathbf{e}_3 + \dot{\mathbf{e}}_3 \cdot \mathbf{m}_2) \right] \]  
(28)

\[\tilde{\Gamma}_p = \sigma^2 \hat{\mathbf{1}} - \dot{\mathbf{e}}_3 \hat{\mathbf{e}}_3 + i\sigma \hat{\mathbf{e}}_3 \times \hat{\mathbf{1}} \]  
(29)

(Seyed-Mahmoud and Rochester 2006) with \( \hat{\mathbf{1}} \) being the unit dyadic.

Taking the divergence of (28) and using (25) we get

\[\sigma^2 \nabla^2 \chi + (\dot{\mathbf{e}}_3 \cdot \nabla)^2 \chi = 0 \]  
(30)

which is the Poincaré equation.

The no-penetration boundary condition require that

\[\hat{n}_k \cdot (\mathbf{v}_2 - \mathbf{v}_k) = 0 \]  
(31)

where the subscript \( k = 1 \) or 3 denotes whether this condition is applied at the inner-core boundary ICB \( (k = 1) \) or core-mantle boundary CMB \( (k = 3) \), with \( \hat{n}_k \) being the unit vector normal to the boundary,

\[\hat{n}_k = (-1)^{(k+1)/2} \left[ \hat{\mathbf{r}} + \frac{2}{3} \varepsilon_k P_2^1(\theta) \hat{\mathbf{\theta}} \right], \]  
(32)

where \( \varepsilon_k \) are the ellipticities of the respective boundaries and \( P_n^m(\theta) \) are the associated Legendre polynomials of degree \( n \) and order \( m \),

\[P_2^1(\theta) = -3 \cos \theta \sin \theta. \]  
(33)

Using equation (28) these boundary conditions become

\[-\frac{2i\sigma}{\sigma^2(\sigma^2 - 1)} \hat{n}_k \cdot \tilde{\Gamma}_p \left( \nabla \chi(\xi_k) + \frac{1}{2} (i\sigma \mathbf{m}_2 \times \xi_k + \dot{\mathbf{e}}_3 \cdot \xi_3 \mathbf{m}_2) \right) - \hat{n}_k \cdot \left( (\mathbf{m}_k - \mathbf{m}_2) \times \xi_k \right) = 0 \]  
(34)

for \( k = 1 \) or 2. Now,

\[\tilde{\Gamma}_p \cdot (i\sigma \mathbf{m}_2 \times \xi + \dot{\mathbf{e}}_3 \cdot \xi \mathbf{m}_2) = i\sigma^3 \mathbf{m}_2 \times \xi + \sigma(\xi \cdot \mathbf{m}_2 \dot{\mathbf{e}}_3 - \dot{\mathbf{e}}_3 \cdot \xi \mathbf{m}_2). \]  
(35)

Therefore,

\[\hat{n}_k \cdot \tilde{\Gamma}_p \cdot (i\sigma \mathbf{m}_2 \times \xi + \dot{\mathbf{e}}_3 \cdot \xi \mathbf{m}_2) = -4\sigma(\sigma^2 - 1) \varepsilon_k \xi_k \alpha_2 Y_2^{-1}(\theta, \phi). \]  
(36)

Substituting equation (36) into equation (34) we get

\[\hat{n}_k \cdot \tilde{\Gamma}_p \cdot \nabla \chi(\xi_k) - 2\sigma(\sigma^2 - 1) \xi_k \varepsilon_k \alpha_k Y_2^{-1}(\theta, \phi) = 0 \]  
(37)

which is independent of \( \alpha_2 \), the amplitude of LC wobble, or, equivalently, equation (34) is independent of \( \mathbf{m}_2 \).

\[Y_n^m(\theta, \phi) = P_n^m(\theta)e^{im\phi} \]  
(38)

are the spherical harmonics of degree \( n \) and azimuthal order \( m \). Since \( \rho \) is a constant in the LC, \( V_1 \) is due to rigid rotation and (19) ensures that it is continuous across the ICB and CMB.
The continuity of that gravitational flux across the ICB and CMB requires that
\[ \hat{n}_k \cdot [\nabla V_1 - 4\pi G \rho q_2]_{k-} = \hat{n}_k \cdot [\nabla V_1 - 4\pi G \rho q_k]_{k+} \]  
(39)
k = 1 or k = 3, and a − or + refers to the region just inside or just outside the LC at the boundaries. However, application of (37) reduces (39) to
\[ \hat{n}_k \cdot [\nabla V_1 - 4\pi G \rho u]_{k-} = \hat{n}_k \cdot [\nabla V_1 - 4\pi G \rho q_k]_{k+} \]  
(40)
which is, of course, independent of \( m_2 \).

We use the conservation of the Earth’s angular moment, in place of the continuity of the normal component of the stress tensor across the LC boundaries, as the alternative equation to solve along with equations (37) for the periods of the wobble modes. The Earth’s angular momentum \( H \) about the canter of mass is given as
\[ H = H_{LC} + H_{MT} + H_{IC}. \]  
(41)
The angular momentum of the LC is written as
\[ H_{LC} = \int_{lc} (r + \Omega i \omega m_2 \times r) \times [\hat{\omega} \hat{e}_3 \times (r + u + \Omega i \omega m_2 \times r) + i \omega u + \Omega m_2 \times r] dm \]
\[ + \Omega \int_{lc} (r \times \hat{e}_3 \times u + u \times \hat{e}_3 \times r + 2i \sigma r \times u)] dm \]  
(42)
The angular momenta of the IC (\( k = 1 \)) and the MT (\( k = 3 \)) are given as
\[ H_k = \int_{\Delta k} (r + \Omega i \omega m_k \times r) \times [\hat{\omega} \hat{e}_3 \times (r + u + \Omega i \omega m_k \times r) + \Omega m_k \times r] dm \]
\[ = \Omega \int_{\Delta k} (r + \frac{1}{2i} m_k \times r) \times [\hat{e}_3 \times (r + \frac{1}{2i} m_k \times r) + m_k \times r)] dm \]  
(43)
after linearization; \( \Delta_k \) refers to the mass of the \( kth \) region. We can now write the angular momentum as
\[ H = H_0 + \delta H \]  
(44)
where \( H_0 \) is the Earth’s angular momentum before any perturbation or deformation and \( \delta H \) is due to wobble and, when applicable, deformation.

The conservation is of the Earth’s angular momentum then becomes
\[ \frac{dH}{dt} = \frac{d(\delta H)}{dt} = \frac{\partial(\delta H)}{\partial t} + \hat{\omega} \hat{e}_3 \times (\delta H) = 0. \]  
(45)
Consistent with \( m \), we write \( \delta H \) as
\[ \delta H = (\hat{e}_1 - i\hat{e}_2)\hat{h}, \]  
(46)
\[ \hat{h} = \frac{1}{2}(\hat{e}_1 + i\hat{e}_2) : (\delta H) \]  
(47)
as in Seyed-Mahmoud and Rogister (2021). Using this equation, equation (45) becomes
\[ i(\omega + \Omega)\hat{h} = 0 \]  
(48)
which immediately yields the solution for the frequency of the TOM,
\[ \omega = -\Omega. \]  
(49)
Now,
\[ \hat{h} = h_1 + h_2 + h_3 \]  
(50)
For the IC, the LC and the MT respectively. We now proceed to find the components of \( h \) for the other solutions of equations (48). Expanding equation (43) we get

\[
H_k = \Omega \left\{ \int_{\Delta_k} \left[ r^2 \hat{e}_3 - r \cdot \hat{e}_3 r \right] \, dm + \frac{1}{2\sigma} \int_{\Delta_k} \hat{e}_3 \cdot \bar{r} \times m_k \, dm \right. \\
- \frac{1}{2\sigma} \int_{\Delta_k} (m_k \times r) \cdot \hat{e}_3 r \, dm + \int_{\Delta_k} (r^2 m_k - \bar{r} \cdot m_k) \, dm \right\} \\
= \Omega \left\{ \tilde{I}_k \cdot \hat{e}_3 + \frac{1}{2\sigma} \frac{A_k}{2} \left[ \hat{e}_3 \cdot [(1 + e_k)I - 2e_k \hat{e}_3 \hat{e}_3] \times m_k \right. \\
- i[(1 + e_k)I - 2e_k \hat{e}_3 \hat{e}_3] \cdot m_k \left. \right\} + A_k m_k \right\} \tag{51}
\]

or

\[
\frac{\delta H_k}{\Omega} = \frac{1}{2\sigma} \frac{A_k}{2} \left[ (1 + e_k) - 2e_i - (1 + e_k) \right] m_k + A_k m_k
\]

\[
= A_k \left( 1 - \frac{e_k}{2\sigma} \right) m_k \tag{52}
\]

where \( A_k \) and \( e_k \) are the moment of inertia about an equatorial axis and the dynamical ellipticity for the Earth’s \( k \)th component respectively,

\[
e_k = \frac{A_k - C_k}{A_k} \tag{53}
\]

and \( C_k \) is the moment of inertia of the same component about the rotation axis.

Similarly, the first term in equation (42) reduces to

\[
\frac{\delta H_{lc1}}{\Omega} = A_2 \left( 1 - \frac{e_2}{2\sigma} \right) m_2 \tag{54}
\]

We rewrite the second term on the right-hand-side of equation (42) as

\[
\frac{\delta H_{lc2}}{\Omega} = \int_{lc} [2i\sigma \times u + 2r \cdot u \hat{e}_3 - (ru + ur) \cdot \hat{e}_3] \, dm. \tag{55}
\]

\( u \) is given by rewriting equation (28) as

\[
u = \frac{1}{\sigma^2 (\sigma^2 - 1)} \tilde{G}_p \cdot \left[ \nabla \chi \times \frac{1}{2} (i\sigma m_2 \times r + \hat{e}_3 \cdot rm_2) \right]. \tag{56}
\]

Using equations (56) and (35), equation (55) reduces to

\[
\frac{\delta H_{lc2}}{\Omega} = \delta H_\chi - A_2 \left( 1 - \frac{e_2}{2\sigma} \right) m_2, \tag{57}
\]

where \( \delta H_\chi \) is due to the first term on the right-hand-side of (56). The change in the Earth’s angular momentum is then given as

\[
\delta H = \delta H_1 + \delta H_3 + \delta H_{lc1} + \delta H_{lc2}
\]

\[
= \Omega \left\{ \delta H_\chi + A_1 \left( 1 - \frac{e_1}{2\sigma} \right) m_1 + A_3 \left( 1 - \frac{e_3}{2\sigma} \right) m_3 \right\} \tag{58}
\]

which is also independent of \( m_2 \), hence \( \alpha_2 \). This shows that when dealing with the rotational dynamics of the LC it is sufficient to assume that the liquid core undergoes uniform rotation plus small deformation, that is, \( m_2 = 0 \).
3. Frequencies of the wobble modes

Equations (58) and (37) are a set of three equations in three unknowns $\alpha_1, \alpha_3$ and $\chi$. In theory we should be able to solve these equations for the frequencies $\sigma$ of the rotational modes. In practice, however, we know that the problem is ill-posed and, unless the eigenfunctions are purely toroidal, the presence of the IC prevents smooth solutions. If we ignore the presence of the IC, the problem reduces to that considered by Seyed-Mahmoud and Rogister (2021) and the analytical solutions for the frequencies of the CW and FCN are recovered.

Seyed-Mahmoud and Rogister (2021) also show that if the wobbling rotation of the MT is ignored (i.e., $\alpha_3=0$) and Sasao’s approximation (Sasao et al. 1980) is implemented, the numerical frequencies of the ICW and FICN so computed are nearly identical to those in the literature. Here we follow the same procedure as that in Seyed-Mahmoud and Rogister (2021) and show that the analytical solutions found for the frequencies of these modes are identical to those in the literature. The Liouville equation for the IC, equation (29) of Seyed-Mahmoud et al. (2017) by setting $k=1$, becomes

$$-2i(\omega - \Omega)(\omega + e_1\Omega)A_1\alpha_1 = \Pi_1 + \Gamma_1$$  \hspace{1cm} (59)

where $\Pi_1$ and $\Gamma_1$ are the pressure and gravitational torques on the IC respectively. Now, setting $\alpha_3=0$ and ignoring the LC pressure $\chi$, see equation (88) of Seyed-Mahmoud and Rogister (2021), as given by equation (34) of Seyed-Mahmoud et al. (2017), equations (30) and (42) of the same paper become

$$\Pi_1 = -i\frac{16\pi}{5}Ge_1A_1[J_3 + \rho_0e(b)]\alpha_1,$$  \hspace{1cm} (60)

$$\Gamma_1 = -i\frac{16\pi}{5}e(a)a^3\rho_0\left\{\frac{2}{3}e(a)g_0(a)a - G\left[e_1A_1 - \frac{8\pi}{15}\rho_0a^5e(a)\right]\right\}\alpha_1,$$  \hspace{1cm} (61)

$a$ being the mean radius of the ICB. Substituting from (60) and (61) into (59) we get

$$(\omega - \Omega)(\omega + e_1\Omega)A_1 - \frac{8\pi}{5}Ge_1A_1(J_3 + \rho_0e(b))$$

$$+ \frac{8\pi}{5}e(a)\rho_0\left[\frac{2}{3}e(a)g_0a^4 - G\left(e_1A_1 - \frac{8\pi}{15}\rho_0a^5e(a)\right)\right] = 0.$$  \hspace{1cm} (62)

Since the LC is considered incompressible and homogeneous, it is reasonable that we set the dynamical ellipticities $e_1$ and $e_2$ equal to their geometric counterparts $e(a)$ and $e(b)$. Further, following Rochester and Crossley (2009) and letting

$$\delta = \frac{8\pi\rho_0a^5}{15A_1}, \quad \Lambda = \frac{5\Omega^2}{8\pi G},$$  \hspace{1cm} (63)

and noting that

$$g_0(a) = \frac{4}{3}\pi G < \rho_{IC} > a,$$  \hspace{1cm} (64)

$< \rho_{IC} >$ being the mean density of the IC, and that

$$\frac{8\pi}{15}e_1^2 < \rho_{IC} > a^5 \approx e_1(e_1A_1),$$  \hspace{1cm} (65)

(62) becomes

$$\Lambda(2\sigma - 1)(2\sigma + e_1) - e_1\left\{J_3 + \rho_0e_2 - e_1\rho_0\left[\frac{5}{3} - 1 + \delta\right]\right\} = 0.$$  \hspace{1cm} (66)
or

\[ 4\sigma^2 - (1 - e_1)2\sigma - \frac{e_1}{\Lambda} \left\{ \Lambda + J_3 + J_2 + \rho_0 e_1 - e_1 \rho_0 \left[ \frac{2}{3} + \delta \right] \right\} = 0, \]  

(67)

with

\[ J_2 = \rho_0 e_2 - \rho_0 e_1 \]  

(68)

for the Earth model considered. Also, noting that

\[ \Lambda + J_3 + J_2 = \frac{2}{3} \rho_0 e_1 / \delta, \]  

(69)

(67) is written as

\[ 4\sigma^2 - (1 - e_1)2\sigma - e_1 (1 - \delta) (1 + \alpha_g), \]  

(70)

which simplifies to

\[ 4\sigma^2 - (1 - e_1)2\sigma - e_1 (1 - \delta)(1 + \alpha_g), \]  

(71)

\[ \alpha_g = \frac{1}{\Lambda} (J_3 + J_2 + \rho_0 e_1) \]  

(72)

as in Mathews et al. (1991). Solutions of (71) are

\[ 2\sigma = \frac{(1 - e_1) \pm \sqrt{(1 - e_1)^2 + 4e_1(1 - \delta)(1 + \alpha_g)}}{2}. \]  

(73)

The first solution of (73) is

\[ 2\sigma = \frac{1}{2} \{ (1 - e_1) - (1 - e_1) [1 + 2(1 - \delta)(1 + \alpha_g)e_1] \} = -(1 - \delta)(1 + \alpha_g)e_1 \]  

(74)

or

\[ \omega = -(1 - \delta)(1 + \alpha_g)e_1 \Omega \]  

(75)

which is the same as equation (126) of Rochester and Crossley (2009) which, in turn, is identical to equation (30) of Mathews et al. (1991) for the frequency of the ICW of the simple Earth model considered. Rochester and Crossley (2009) show that this equation is valid in the limit of low-frequency approximation, \( \sigma = O\varepsilon \), i.e., ignoring \( \sigma^2 \) is the governing equations.

The second solution of (73) is

\[ 2\sigma = \frac{1}{2} \{ (1 - e_1) + (1 - e_1) [1 + 2(1 - \delta)(1 + \alpha_g)e_1] \} = 1 - (\delta - \alpha_g + \delta \alpha_g)e_1 \]  

(76)

or

\[ \omega = [1 - (\delta - \alpha_g + \delta \alpha_g)e_1] \Omega \]  

(77)

which is identical to the second equation in equations (29) of Mathews et al. (1991) for a an Earth model with a rigid IC \( [\nu = 0 in their (29)] \), except for a small term which is of the order of

\[ \frac{A_1 e_1}{A_3}. \]  

(78)

Now

\[ \frac{A_1}{A_3} \approx 8.2 \times 10^{-4} < O\varepsilon \]  

(79)
Therefore, (78) is at best of second order in the ellipticity, $e_1 \approx 2.42 \times 10^{-3}$, and its effect on the (dimensionless) frequency of the FICN, (76), which is of the order of unity, is insignificant.

Recall that solutions (75) and (77) are found assuming that the mantle rotates uniformly. Solution (75) may be reasonable, as Rochester and Crossley (2009) show, because the dimensionless frequency of the ICW is of the order of the ellipticity and the displacement eigenfunctions in the LC are nearly geostrophic (Seyed-Mahmoud and Rogister 2021). The dimensionless frequency of the FICN, on the other hand, is of the order of unity and the aforementioned approximation means that there are no gravitational interactions (couplings) due to wobble between the MT and IC during the excitation of this mode.

4. Discussions

In section 2 we show that it is redundant to assign an extra rigid rotation component, in addition to the displacement component, to the velocity of a mass element in the LC. This extra velocity component makes dealing with the already very complicated dynamics of the LC even more complicated. Even though we consider a simple Earth model with an incompressible and homogeneous liquid core, our analysis applies equally to more realistic core models as the redundant velocity component is due only to the LC rigid rotation. Further, we consider a uniformly rotating MT and ignore the effects of the LC pressure on the torque experienced by the IC, yet our analytical results so computed for the frequencies of the ICW and FICN are identical to those in the literature using the commonly used Sasa’s (Sasao et al. 1980) approximation. Seyed-Mahmoud and Rogister (2021) used a similar approach and computed the frequencies of the ICW and FICN which were numerically nearly identical to those in the literature. Therefore, our results in section 3 may be considered as complimentary to those in Seyed-Mahmoud and Rogister (2021). Accordingly, we conclude that this approximation is not suitable for computing the frequencies of these mode.

5. Acknowledgments

I would like to thank M.G. Rochester for his many comments which helped improve the quality of the work.

References


REFERENCES